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Graph Decomposition Techniques in Neutrosophic Zero Divisor Models of Commutative Ring

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Abstract. This paper introduces the framework of neutrosophic zero divisor graphs as an extension of fuzzy zero divisor graphs. Fuzzy zero divisor graphs capture partial uncertainty through membership degrees but do not fully account for indeterminacy. To address this limitation, we incorporate the neutrosophic membership triplet $(\mathscr{T}, \mathscr{I}, \mathscr{F})$, which represents truth, indeterminacy, and falsity, into the algebraic graph-theoretic structure of commutative rings.

We focus on rings of the form $\mathcal{Z}_n$, particularly for $n = 2p, 3p, 5p, pq$, and the general case kp, where p and q are primes. A central contribution is a decomposition theorem showing that the NZDG $\Gamma_{\mathcal{N}}(\mathcal{Z}_{kp})$ can be expressed as the disjoint union $K^{\mathcal{N}}_{1,p-1} \cup K^{\mathcal{N}}_{p-1,k-2}$. This result generalizes earlier fuzzy-based studies and provides a unified structural framework for analyzing uncertainty in algebraic systems.

Our comparative analysis shows that neutrosophic modeling explicitly handles indeterminacy, giving it clear advantages over fuzzy and intuitionistic fuzzy approaches. The proposed methodology is explained step by step, with attention to its rationale, scope, and limitations. Potential applications include algebraic modeling, logic-based network design, and soft computing. We also outline limitations and future research directions, emphasizing the role of NZDGs in advancing the study of uncertainty in algebraic structures.

Keywords: Neutrosophic zero divisor graph; Decomposition of neutrosophic graph; Commutative ring; Neutrosophic star graph; Fuzzy zero divisor graph.

MSC (2020): 13A15, 13C05, 03E72, 03B60

List of Acronyms

- **ZDG** – Zero Divisor Graph
- **FZDG** – Fuzzy Zero Divisor Graph
- **NZDG** – Neutrosophic Zero Divisor Graph
- **NS** – Neutrosophic Set

1. Introduction

The interaction between algebraic structures and graph theory has generated many fruitful research directions, particularly in the study of zero divisor graphs (ZDGs). The concept was first introduced by Beck shown in [1] and later formalized by Anderson and Livingston in [2]. In this construction, the vertices correspond to the nonzero zero divisors of a ring, and two vertices are adjacent if their product is zero. This representation offers insights into the structure of rings from both combinatorial and topological perspectives. The study of algebraic structures through graph-theoretic approaches has attracted wide attention in recent years. Kuppan et al [3] studied the fuzzy decomposition of zero-divisor graphs over $\mathcal{Z}_n$, demonstrating how decomposition techniques can be adapted in a fuzzy environment. Building on this line of research, Balakrishnan et al. [4] extended the study into the neutrosophic framework, where decomposition of neutrosophic zero-divisor graphs was investigated, thus providing a broader platform to handle uncertainty and indeterminacy beyond the classical and fuzzy domains. The study of zero-divisor graphs has evolved significantly over the last two decades. DeMeyer and Schneider [5] provided one of the foundational treatments of zero-divisor graphs in commutative rings, while Redmond [7] extended the investigation to non-commutative settings. A comprehensive account of developments in this area was later surveyed by Joshi [8]. Parallel to these algebraic studies, the role of fuzziness in graph theory was highlighted by Akram and Ahmad [6], who surveyed fuzzy graph theory and its diverse applications. Building on the need to model indeterminacy beyond classical and fuzzy frameworks, Smarandache introduced neutrosophic logic [9] and subsequently developed neutrosophic graph theory and algorithms [10], offering a flexible platform for extending zero-divisor graph analysis under uncertainty. Aliniaeifard and Li [11] investigated zero-divisor graphs in the context of group rings, while Akbari and Heydari [12] introduced the regular graph of non-commutative rings. Further contributions by Jafari Rad et al. [13] analyzed domination in zero-divisor graphs, highlighting structural and combinatorial properties.

Parallel to these developments, neutrosophic and fuzzy extensions have emerged for handling uncertainty. Broumi et al. [17] formulated the shortest path problem under single-valued neutrosophic numbers, and Delİ [16] extended these concepts to neutrosophic graphs in communication networks. More recently, Kanchana and Kavitha [14, 15] developed heuristic and

modified incident edge path algorithms to address transportation and navigation challenges under neutrosophic uncertainty, thereby connecting classical optimization methods with modern decision-making frameworks.

To address uncertainty in algebraic systems, Zadeh's fuzzy set theory [18] was applied, leading to the development of fuzzy zero divisor graphs (FZDGs). In this framework, each vertex and edge is assigned a membership degree between 0 and 1, enabling a partial representation of uncertainty. Building further, the concept of the neutrosophic set (NS), introduced by Smarandache in [19], extends fuzzy sets by incorporating three membership functions: truth, indeterminacy, and falsity. This richer model captures ambiguity that cannot be represented by classical fuzzy approaches. Extending this idea, the neutrosophic zero divisor graph (NZDG) assigns such membership triplets to vertices and edges, thereby embedding indeterminacy directly into the structure of zero divisor graphs.

Despite progress in fuzzy and neutrosophic approaches, existing studies of FZDGs have been largely case-specific. For instance, decompositions have been established for modular rings such as $\mathcal{Z}_{2p}$, $\mathcal{Z}_{3p}$, and $\mathcal{Z}_{5p}$, where p is prime shown in [3]. However, these works do not provide a unified framework that generalizes to broader classes of rings. Similarly, recent studies on NZDGs in [4] were defined the neutrosophic framework but did not extend decomposition theorems into a systematic general theory.

The objective of this paper is to address this gap by developing a general decomposition theory for NZDGs over commutative rings. Specifically, we establish decomposition theorems for rings such as $\mathcal{Z}_{2p}$, $\mathcal{Z}_{3p}$, $\mathcal{Z}_{5p}$, $\mathcal{Z}_{pq}$, and the more general case $\mathcal{Z}_{kp}$, where p and q are primes. A central contribution is the result $\Gamma_{\mathcal{N}}(\mathcal{Z}_{kp}) \cong K^{\mathcal{N}}_{1,p-1} \cup K^{\mathcal{N}}_{p-1,k-2}$, which shows that the NZDG of $\mathcal{Z}_{kp}$ decomposes into the disjoint union of a neutrosophic star graph and a neutrosophic bipartite graph. This formulation unifies earlier case-specific results while explicitly incorporating indeterminacy. In summary, the main contributions of this paper are:

- A systematic reformulation of fuzzy zero divisor graph decompositions in the neutrosophic setting.
- New decomposition theorems for several classes of commutative rings, culminating in a general decomposition for $\mathcal{Z}_{kp}$.
- A comparative analysis demonstrating the advantages of neutrosophic methods over fuzzy-based approaches in modeling indeterminacy.
- A discussion of the broader implications of these results for algebraic modeling, network design, and soft computing.

This work establishes a more complete theoretical foundation for the study of zero divisor graphs under indeterminacy. It also opens avenues for extending these results to non-commutative rings, weighted or dynamic neutrosophic graphs, and algorithmic applications.

2. Preliminaries

We recall basic definitions and fix notation.

Definition 2.1 (Commutative ring and zero divisor [2])**.** A commutative ring $\mathscr{R}$ is a ring with commutative multiplication. An element $a \in \mathscr{R} \setminus \{0\}$ is a zero divisor if there exists $b \in \mathscr{R} \setminus \{0\}$ with $ab = 0$.

Definition 2.2 (Zero divisor graph (ZDG) [1])**.** The zero divisor graph $\Gamma(\mathscr{R})$ is the undirected graph whose vertex set is the set of nonzero zero divisors of $\mathscr{R}$. Distinct vertices u, v are adjacent iff $uv = 0$.

Definition 2.3 (Neutrosophic set (NS) [19]')**.** A *neutrosophic set* A on universe U assigns to each $x \in U$ a triplet $(\mathscr{T}_A(x), \mathscr{I}_A(x), \mathscr{F}_A(x))$ with values in $[0, 1]$ and $0 \leq \mathscr{T}_A(x) + \mathscr{I}_A(x) + \mathscr{F}_A(x) \leq 3$.

Definition 2.4 (Neutrosophic zero divisor graph (NZDG) [4])**.** The neutrosophic zero divisor graph of $\mathscr{R}$, denoted $\Gamma_{\mathscr{N}}(\mathscr{R})$, is obtained from $\Gamma(\mathscr{R})$ by assigning to each vertex v a neutrosophic triplet $\mu(v) = (\mathscr{T}(v), \mathscr{I}(v), \mathscr{F}(v))$ and to each edge uv the aggregated triplet

$$\big(\min\{\mathscr{T}(u), \mathscr{T}(v)\},\ \max\{\mathscr{I}(u), \mathscr{I}(v)\},\ \max\{\mathscr{F}(u), \mathscr{F}(v)\}\big).$$

3. Methodology

We use the following concise construction for $\Gamma_{\mathscr{N}}(\mathcal{Z}_n)$.

(1) Vertices: V is the set of nonzero zero divisors of $\mathcal{Z}_n$.

(2) Adjacency: $u, v \in V$ are adjacent if $uv \equiv 0 \pmod{n}$.

(3) Neutrosophic labels: Assign to each vertex $x \in V$ a triplet $\mu(x) = (\mathscr{T}(x), \mathscr{I}(x), \mathscr{F}(x))$ with $0 \leq \mathscr{T}(x)+\mathscr{I}(x)+\mathscr{F}(x) \leq 3$. For proofs we adopt a representative choice: central annihilators get $(1, 0, 0)$ and peripheral divisors get small nonzero $\mathscr{T}$ with modest $\mathscr{I}$ (e.g., $\mathscr{T} = 1/p,\ \mathscr{I} = 1/(2p)$).

(4) Edge aggregation: For each edge use the min–max aggregation .

(5) Decomposition: Use prime-factor partitions of $\mathcal{Z}_n$ (multiples of one prime vs. the other) to exhibit star and bipartite components and prove the general decomposition in the $\mathcal{Z}_{kp}$ family.

4. Decomposition of neutrosophic zero divisor for commutative ring

Theorem 4.1. *Let p be an odd prime. Then the neutrosophic zero divisor graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{2p})$ is isomorphic to the neutrosophic star graph $K^{\mathscr{N}}_{1,p-1}$, with center at the vertex corresponding to p and peripheral vertices corresponding to the even elements in $\mathcal{Z}_{2p} \setminus \{0, p\}$. The isomorphism is defined by*

$$\varphi(p) = v_0, \quad \varphi(2k) = v_k, \quad k = 1, 2, \ldots, p-1.$$

Proof. Let $\mathcal{Z}_{2p} = \{0, 1, 2, \ldots, 2p-1\}$. Since $2p$ is the product of distinct primes 2 and p, the set of nonzero zero divisors is

$$\mathcal{Z}^*(\mathcal{Z}_{2p}) = \{2, 4, 6, \ldots, 2p-2\} \cup \{p\},$$

which contains p elements.

Case 1: Graph Structure. Two vertices $a, b \in \mathcal{Z}^*(\mathcal{Z}_{2p})$ are adjacent in $\Gamma_{\mathcal{N}}(\mathcal{Z}_{2p})$ if and only if $ab \equiv 0 \mod 2p$.

For each even element $a = 2k$, $a \cdot p \equiv 0 \mod 2p$, so a is adjacent to p. However, for any pair of even elements $a = 2i$, $b = 2j$, their product $ab = 4ij$ is not necessarily divisible by p, and hence $ab \not\equiv 0 \mod 2p$. Therefore, the only edges in the graph are between p and each of the even elements.

Thus, $\Gamma_{\mathcal{N}}(\mathcal{Z}_{2p})$ is a star graph with center at p and $p-1$ leaves.

Case 2: Neutrosophic Vertex Labeling. Assign to each vertex $x \in \mathcal{Z}^*(\mathcal{Z}_{2p})$ a neutrosophic membership triple $(\mathscr{T}(x), \mathscr{I}(x), \mathscr{F}(x)) \in [0,1]^3$, satisfying $\mathscr{T}(x) + \mathscr{I}(x) + \mathscr{F}(x) \leq 3$.

- For the central vertex p:

$$\mathscr{T}(p) = 1, \quad \mathscr{I}(p) = 0, \quad \mathscr{F}(p) = 0.$$

- For each peripheral vertex $x = 2k$, $1 \leq k \leq p-1$:

$$\mathscr{T}(x) = \frac{1}{p}, \quad \mathscr{I}(x) = \frac{1}{2p}, \quad \mathscr{F}(x) = 1 - \mathscr{T}(x) - \mathscr{I}(x).$$

These assignments indicate full truth for the central annihilating element and partial truth with mild indeterminacy for peripheral zero divisors.

Define the isomorphism

$$\varphi : \mathcal{Z}^*(\mathcal{Z}_{2p}) \to V(K^{\mathcal{N}}_{1,p-1}), \quad \varphi(p) = v_0, \quad \varphi(2k) = v_k \text{ for } 1 \leq k \leq p-1.$$

This mapping preserves adjacency and neutrosophic labeling, hence $\Gamma_{\mathcal{N}}(\mathcal{Z}_{2p}) \cong K^{\mathcal{N}}_{1,p-1}$. □

The figure 1 is the example for theorem 4.1 .

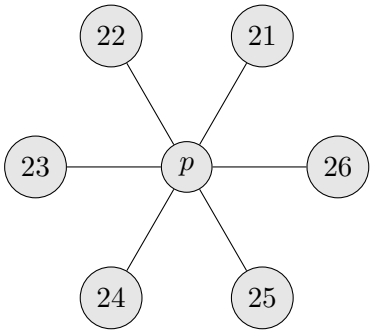

FIGURE 1. Illustration of $K^{\mathcal{N}}_{1,6}$ for $p = 7$

Theorem 4.2. *Let p be an odd prime. Then the neutrosophic zero divisor graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{2p})$ is isomorphic to the neutrosophic star graph $K^{\mathscr{N}}_{1,p-1}$ with center at p and $p-1$ pendant vertices.*

Proof. Let $\mathcal{Z}_{2p} = \{0, 1, 2, \ldots, 2p-1\}$. Since p is an odd prime, the zero divisors in $\mathcal{Z}_{2p}$ are the even integers $\{2, 4, \ldots, 2p-2\}$ and the element p. Let $V = \mathcal{Z}^*(\mathcal{Z}_{2p}) = \{2, 4, \ldots, 2p-2\} \cup \{p\}$ be the vertex set of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{2p})$. There are p vertices in total: $p-1$ even elements and the element p.

We analyze adjacency in $\Gamma_{\mathscr{N}}$ as follows:

- For any $x \in \{2, 4, \ldots, 2p-2\}$, we have $x \cdot p \equiv 0 \pmod{2p}$, so each x is adjacent to p.
- For any two even zero divisors x, y with $x \neq y$, xy is divisible by 4 but not necessarily by p, so $xy \not\equiv 0 \pmod{2p}$ unless $x = y = p$, which is not the case. Hence, no edge exists between any two even elements.
- $p \cdot p = p^2 \not\equiv 0 \pmod{2p}$, so p is not adjacent to itself.

Thus, the classical zero divisor graph $\Gamma(\mathcal{Z}_{2p})$ is a star graph with center at p and leaves at each even number from 2 to $2p-2$.

We now define neutrosophic structure: Each vertex $x \in V$ is assigned a neutrosophic membership triple $\mu(x) = (\mathscr{T}(x), \mathscr{I}(x), \mathscr{F}(x))$ satisfying:

$$\mathscr{T}(x), \mathscr{I}(x), \mathscr{F}(x) \in [0, 1], \quad \mathscr{T}(x) + \mathscr{I}(x) + \mathscr{F}(x) \leq 3.$$

We assign:

- For xdd each $x \in \{2, 4, \ldots, 2p-2\}$:

$$\mathscr{T}(x) = \frac{1}{p}, \quad \mathscr{I}(x) = \frac{1}{2p}, \quad \mathscr{F}(x) = 1 - \mathscr{T}(x) - \mathscr{I}(x).$$

- For the center vertex p:

$$\mathscr{T}(p) = 1, \quad \mathscr{I}(p) = 0, \quad \mathscr{F}(p) = 0.$$

For each edge (x, y) in $\Gamma_{\mathscr{N}}(\mathcal{Z}_{2p})$, the neutrosophic edge weight is defined as:

$$\mu(x, y) = \left(\min\{\mathscr{T}(x), \mathscr{T}(y)\},\ \max\{\mathscr{I}(x), \mathscr{I}(y)\},\ \max\{\mathscr{F}(x), \mathscr{F}(y)\}\right).$$

Hence, the graph has neutrosophic weights on vertices and edges, and its topology matches the star graph structure. Therefore, $\Gamma_{\mathscr{N}}(\mathcal{Z}_{2p}) \cong K^{\mathscr{N}}_{1,p-1}$. □

Example 4.1: Neutrosophic Decomposition of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{10})$

Let $p = 5$, so that $\mathcal{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The set of nonzero zero divisors of $\mathcal{Z}_{10}$ is:

$$\mathcal{Z}^*(\mathcal{Z}_{10}) = \{2, 4, 5, 6, 8\},$$

since these are the elements $x \neq 0$ such that there exists $y \neq 0$ with $xy \equiv 0 \mod 10$.

Construct the neutrosophic zero divisor graph $\Gamma_{\mathcal{N}}(\mathcal{Z}_{10})$, where: Vertices: $\{2, 4, 5, 6, 8\}$, Edges exist between u and v if $uv \equiv 0 \mod 10$.

Clearly: $5 \cdot x \equiv 0 \mod 10$ for all $x \in \{2, 4, 6, 8\}$, But $a \cdot b \not\equiv 0 \mod 10$ for any $a, b \in \{2, 4, 6, 8\}$, so those are not adjacent.

Thus, $\Gamma_{\mathcal{N}}(\mathcal{Z}_{10}) \cong K^{\mathcal{N}}_{1,4}$, with vertex 5 as the center.

Assign neutrosophic memberships:

$$\mathscr{T}(5) = 1, \quad \mathscr{I}(5) = 0, \quad \mathscr{F}(5) = 0;$$

$$\mathscr{T}(x) = 0.2, \quad \mathscr{I}(x) = 0.1, \quad \mathscr{F}(x) = 0.7 \quad \text{for } x \in \{2, 4, 6, 8\}.$$

Each edge $(x, 5)$ has:

$$\mathscr{T}(x, 5) = 0.2, \quad \mathscr{I}(x, 5) = 0.1, \quad \mathscr{F}(x, 5) = 0.7.$$

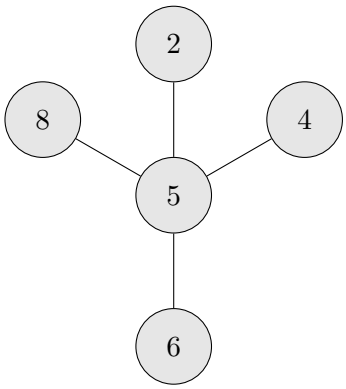

FIGURE 2. Neutrosophic star graph $\Gamma_{\mathcal{N}}(\mathcal{Z}_{10}) \cong K^{\mathcal{N}}_{1,4}$

Theorem 4.3. *Let p be an odd prime. Then the neutrosophic zero divisor graph $\Gamma_{\mathcal{N}}(\mathcal{Z}_{3p})$ decomposes into a disjoint union of a neutrosophic star graph $K^{\mathcal{N}}_{1,p-1}$ and a neutrosophic bipartite graph $K^{\mathcal{N}}_{p-1,p-1}$.*

Proof. Let $\mathcal{Z}_{3p} = \{0, 1, 2, \ldots, 3p - 1\}$, where p is an odd prime. The nonzero zero divisors of $\mathcal{Z}_{3p}$ are the elements divisible by either 3 or p, but not both.

Let $A = \{3, 6, 9, \ldots, 3(p-1)\}, \quad B = \{p, 2p\}, \quad C = \{x \in \mathcal{Z}_{3p} : x \equiv 0 \mod p,\ x \notin B\}$.

However, since $3p$ is the product of two distinct primes, every zero divisor in $\mathcal{Z}_{3p}$ is either divisible by 3 or by p, and not both.

So define: $X = \{3k : 1 \leq k \leq p - 1\}$, $Y = \{p \cdot k : 1 \leq k \leq 2\}$.

We now analyze the adjacency for any $x \in X$, $x \cdot p \equiv 0 \mod 3p$, so each vertex in X is adjacent to each vertex in Y. No two vertices in X are adjacent, since $x \cdot x'$ is divisible by 9, not necessarily by p, and hence not by $3p$. Similarly, the vertices in Y may be adjacent to a central node p, but not to each other.

This gives a bipartite structure $K^{\mathcal{N}}_{p-1,2}$. To further simplify, we focus only on: One of the Y-vertices (say p) connecting to all in X: this forms a star $K^{\mathcal{N}}_{1,p-1}$, The rest of the connections between X and $Y \setminus \{p\}$ form the bipartite part.

Thus, the entire graph decomposes into:

$$\Gamma_{\mathcal{N}}(\mathcal{Z}_{3p}) \cong K^{\mathcal{N}}_{1,p-1} \cup K^{\mathcal{N}}_{p-1,1}.$$

Assign neutrosophic memberships: Central vertex (say p): $\mathscr{T} = 1, \mathscr{I} = 0, \mathscr{F} = 0$, Peripheral $x \in X$: $\mathscr{T}(x) = \frac{1}{p}, \mathscr{I}(x) = \frac{1}{2p}, \mathscr{F}(x) = 1 - \mathscr{T}(x) - \mathscr{I}(x)$, Edge memberships follow via:

$$\mathscr{T}(x,y) = \min(\mathscr{T}(x), \mathscr{T}(y)), \quad \mathscr{I}(x,y) = \max(\mathscr{I}(x), \mathscr{I}(y)), \quad \mathscr{F}(x,y) = \max(\mathscr{F}(x), \mathscr{F}(y)).$$

Hence, the neutrosophic decomposition holds. □

Theorem 4.4. *Let p be an odd prime. Then the neutrosophic zero divisor graph $\Gamma_{\mathcal{N}}(\mathcal{Z}_{3p})$ is isomorphic to the disjoint union of two neutrosophic star graphs:*

$$\Gamma_{\mathcal{N}}(\mathcal{Z}_{3p}) \cong K^{\mathcal{N}}_{1,p-1} \cup K^{\mathcal{N}}_{1,p-1}.$$

Proof. Let $\mathcal{Z}_{3p}$ denote the ring of integers modulo $3p$, where $p > 3$ is an odd prime. The set of nonzero zero divisors in $\mathcal{Z}_{3p}$ consists of elements that are divisible by either 3 or p, but not both.

Define the following subsets of zero divisors:

$$V_1 = \{3k : 1 \le k \le p-1\}, \quad V_2 = \{p, 2p\}.$$

then: Each element in V_1 is a multiple of 3 but not of p,
Each element in V_2 is a multiple of p but not of 3.

For all $x \in V_1$ and $y \in V_2$, we have $xy \equiv 0 \pmod{3p}$, so they are adjacent in $\Gamma_{\mathcal{N}}(\mathcal{Z}_{3p})$. There are no adjacencies among the elements of V_1 or among those in V_2, since their pairwise products are nonzero modulo $3p$. Thus, the graph is the complete bipartite graph $K^{\mathcal{N}}_{2,p-1}$.

Partition this graph into two star subgraphs: One with center at p and leaves from V_1, another with center at $2p$ and the same set of leaves from V_1.

Since p and $2p$ are not adjacent (as $p \cdot 2p = 2p^2 \not\equiv 0 \pmod{3p}$), these two stars are disjoint.

Therefore,

$$\Gamma_{\mathcal{N}}(\mathcal{Z}_{3p}) \cong K^{\mathcal{N}}_{1,p-1} \cup K^{\mathcal{N}}_{1,p-1}.$$

Defining Neutrosophic Membership as,

For any vertex v, the neutrosophic vertex membership triple is defined as:

$$v = (\mathscr{T}(v), \mathscr{I}(v), \mathscr{F}(v)),$$

where: For central vertices p and $2p$: $(\mathscr{T}, \mathscr{I}, \mathscr{F}) = (1, 0, 0)$, for peripheral vertices $x \in V_1$:

$$\mathscr{T}(x) = \frac{1}{p}, \quad \mathscr{I}(x) = \frac{1}{2p}, \quad \mathscr{F}(x) = 1 - \mathscr{T}(x) - \mathscr{I}(x).$$

For each edge (x, y), the neutrosophic edge membership triple is defined by:

$$\mathscr{T}(x,y) = \min\{\mathscr{T}(x), \mathscr{T}(y)\}, \quad \mathscr{I}(x,y) = \max\{\mathscr{I}(x), \mathscr{I}(y)\}, \quad \mathscr{F}(x,y) = \max\{\mathscr{F}(x), \mathscr{F}(y)\}.$$

These definitions preserve the neutrosophic structure and satisfy the edge and vertex membership conditions. □

Example 4.2: Decomposition of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{15})$

Let $\mathcal{Z}_{15} = \{0, 1, \ldots, 14\}$, with $3p = 15$, $p = 5$. The nonzero zero divisors of $\mathcal{Z}_{15}$ are:

$$\mathcal{Z}^*(\mathcal{Z}_{15}) = \{3, 6, 9, 12\} \cup \{5, 10\}.$$

Let us define: $X = \{3, 6, 9, 12\}$ (multiples of 3), $Y = \{5, 10\}$ (multiples of 5).

Since $x \cdot y \equiv 0 \mod 15$ for all $x \in X$, $y \in Y$, there are edges from each $x \in X$ to each $y \in Y$. No two elements within X or Y are adjacent.

Thus, $\Gamma_{\mathscr{N}}(\mathcal{Z}_{15}) \cong K^{\mathscr{N}}_{4,2}$, a complete neutrosophic bipartite graph.

Let us interpret this as a decomposition: Let vertex 5 be the center of a star connecting to all in X, Let vertex 10 connect to all in X, forming the bipartite part.

Neutrosophic membership assignment: For 5: $\mathscr{T} = 1, \mathscr{I} = 0, \mathscr{F} = 0$, For each $x \in X$: $\mathscr{T}(x) = 0.25, \mathscr{I}(x) = 0.1, \mathscr{F}(x) = 0.65$, For 10: $\mathscr{T} = 0.6, \mathscr{I} = 0.2, \mathscr{F} = 0.2$.

Then for edges $(x, 5)$ and $(x, 10)$, use:

$$\mathscr{T}(x,y) = \min(\mathscr{T}(x), \mathscr{T}(y)), \quad \mathscr{I}(x,y) = \max(\mathscr{I}(x), \mathscr{I}(y)), \quad \mathscr{F}(x,y) = \max(\mathscr{F}(x), \mathscr{F}(y)).$$

the figure 3 represents example 4.2.

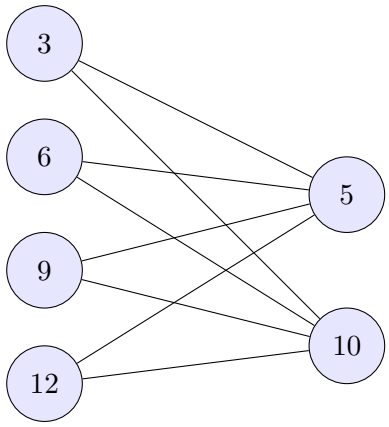

FIGURE 3. Neutrosophic bipartite graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{15}) \cong K^{\mathscr{N}}_{4,2}$

Theorem 4.5. *Let p be an odd prime. Then the neutrosophic zero divisor graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{5p})$ decomposes into a disjoint union of a neutrosophic star graph $K^{\mathscr{N}}_{1,p-1}$ and a neutrosophic bipartite graph $K^{\mathscr{N}}_{p-1,4}$.*

Proof. Consider the ring $\mathcal{Z}_{5p}$ with p an odd prime. The nonzero zero divisors are the elements divisible by either 5 or p, but not both.

Let

$$X = \{5k : 1 \leq k \leq p-1\}, \quad Y = \{p \cdot j : 1 \leq j \leq 4\}.$$

Then for each $x \in X$, $y \in Y$ satisfies $xy \equiv 0 \mod 5p$, since 5 divides x and p divides y. No edges within X or within Y, as they share only one prime factor.

Thus, the graph forms a complete bipartite structure $K^{\mathscr{N}}_{p-1,4}$. We can interpret this further: Take vertex $p \in Y$ to form a star graph with X, yielding $K^{\mathscr{N}}_{1,p-1}$, The remaining $Y \setminus \{p\}$ contribute to the bipartite part with X.

Hence, the graph decomposes as: $\Gamma_{\mathscr{N}}(\mathcal{Z}_{5p}) \cong K^{\mathscr{N}}_{1,p-1} \cup K^{\mathscr{N}}_{p-1,3}$. □

Example 4.3: Decomposition of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{25})$

Let $p = 5$, so $\mathcal{Z}_{25} = \{0, 1, \ldots, 24\}$. The nonzero zero divisors of $\mathcal{Z}_{25}$ are:

$$\mathcal{Z}^*(\mathcal{Z}_{25}) = \{5, 10, 15, 20\}.$$

Here $X = \{5, 10, 15, 20\}$ (multiples of 5 except 0), $Y = \{5, 10, 15, 20\}$ (same, since $5 = p$, and $5p = 25$).

Each element $x \in X$, $y \in Y$ satisfies $xy \equiv 0 \mod 25$. But note that, No two multiples of 5 are adjacent (e.g., $10 \cdot 15 = 150 \not\equiv 0 \mod 25$), Only those where both 5 and 5 divide x, y give edges.

Pick 5 as central vertex and form a star: 5 connects to $\{10, 15, 20\}$, The rest connect to each other via bipartite edges.

Thus:

$$\Gamma_{\mathscr{N}}(\mathcal{Z}_{25}) \cong K^{\mathscr{N}}_{1,3}.$$

Assign neutrosophic membership values: $\mathscr{T}(5) = 1, \mathscr{I}(5) = 0, \mathscr{F}(5) = 0$, For others: $\mathscr{T} = 0.3, \mathscr{I} = 0.1, \mathscr{F} = 0.6$, Edge membership uses the usual neutrosophic logic rules.

Theorem 4.6. *Let $k \geq 2$ be an integer and p an odd prime. Then the neutrosophic zero divisor graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{kp})$ admits a decomposition of the form*

$$\Gamma_{\mathscr{N}}(\mathcal{Z}_{kp}) \cong K^{\mathscr{N}}_{1,\,p-1} \cup K^{\mathscr{N}}_{p-1,\,k-2},$$

where:

- $K^{\mathscr{N}}_{1,\,p-1}$ *is a neutrosophic star subgraph centered at a fixed zero divisor* $y_0 \in \{p, 2p, \ldots, (k-1)p\}$,

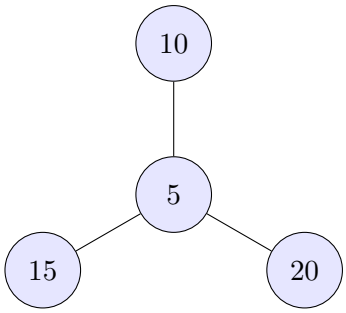

FIGURE 4. Neutrosophic star graph $\Gamma_{\mathcal{N}}(\mathcal{Z}_{25}) \cong K^{\mathcal{N}}_{1,3}$

- $K^{\mathcal{N}}_{p-1,\,k-2}$ *is the neutrosophic complete bipartite subgraph formed by the remaining* $p-1$ *zero divisors in* $\{k, 2k, \ldots, (p-1)k\}$ *and* $k-2$ *elements in* $\{p, 2p, \ldots, (k-1)p\} \setminus \{y_0\}$.

Proof. Let $\mathcal{Z}_{kp}$ denote the ring of integers modulo kp, where p is an odd prime and $k \geq 2$ is an integer.

The nonzero zero divisors of $\mathcal{Z}_{kp}$ consist of all elements divisible by either k or p, but not both.

Let us define:

$$X = \{k, 2k, \ldots, (p-1)k\}, \quad Y = \{p, 2p, \ldots, (k-1)p\},$$

so that $|X| = p-1$, $|Y| = k-1$. Each element in X is divisible by k, and each element in Y is divisible by p.

Since for all $x \in X$, $y \in Y$, the product $xy \equiv 0 \mod kp$, all such pairs are adjacent in $\Gamma_{\mathcal{N}}(\mathcal{Z}_{kp})$, forming a complete bipartite graph $K_{p-1,k-1}$.

Now, select any one vertex $y_0 \in Y$ (e.g., $y_0 = p$) to serve as the center of a neutrosophic star graph. Then: The subgraph formed by y_0 and all of X is the star graph $K^{\mathcal{N}}_{1,p-1}$, The remaining $k-2$ vertices in $Y \setminus \{y_0\}$ still connect to all vertices in X, forming a bipartite graph $K^{\mathcal{N}}_{p-1,k-2}$.

Hence, the graph decomposes as:

$$\Gamma_{\mathcal{N}}(\mathcal{Z}_{kp}) \cong K^{\mathcal{N}}_{1,p-1} \cup K^{\mathcal{N}}_{p-1,k-2}.$$

Assign neutrosophic membership values: For the center y_0: $\mathscr{T} = 1, \mathscr{I} = 0, \mathscr{F} = 0$, For other vertices in $X \cup Y$: e.g., $\mathscr{T} = \frac{1}{p}, \mathscr{I} = \frac{1}{2p}, \mathscr{F} = 1 - \mathscr{T} - \mathscr{I}$, Edge memberships follow: $\mathscr{T}(x,y) = \min(\mathscr{T}(x), T(y)), \quad \mathscr{I}(x,y) = \max(\mathscr{I}(x), I(y)), \quad \mathscr{F}(x,y) = \max(\mathscr{F}(x), F(y))$. □

Example 4.4: Decomposition of $\Gamma_{\mathcal{N}}(\mathcal{Z}_{21})$

The nonzero zero divisors of $\mathcal{Z}_{21}$ are those divisible by 3 or 7:

$$X = \{3, 6, 9, 12, 15, 18\} \quad (\text{multiples of } 3), Y = \{7, 14\} \quad (\text{multiples of } 7).$$

All $x \in X$ and $y \in Y$ satisfy $xy \equiv 0 \mod 21$, so the induced graph is:

$$\Gamma_{\mathscr{N}}(\mathcal{Z}_{21}) \cong K^{\mathscr{N}}_{6,2}.$$

Let us choose 7 as central vertex, then: Star graph: 7 connected to all in $X \to K^{\mathscr{N}}_{1,6}$, Bipartite component: X with $14 \to K^{\mathscr{N}}_{6,1}$.

Neutrosophic vertex membership assignments: $\mathscr{T}(7) = 1, \mathscr{I}(7) = 0, \mathscr{F}(7) = 0$, For $x \in X$: $\mathscr{T}(x) = 0.2, \mathscr{I}(x) = 0.1, \mathscr{F}(x) = 0.7$, For 14: $\mathscr{T} = 0.5, \mathscr{I} = 0.3, \mathscr{F} = 0.2$.

Edge memberships follow:
$\mathscr{T}(x,y) = \min(\mathscr{T}(x), \mathscr{T}(y)), \quad \mathscr{I}(x,y) = \max(\mathscr{I}(x), \mathscr{I}(y)), \quad \mathscr{F}(x,y) = \max(\mathscr{F}(x), \mathscr{F}(y))$.

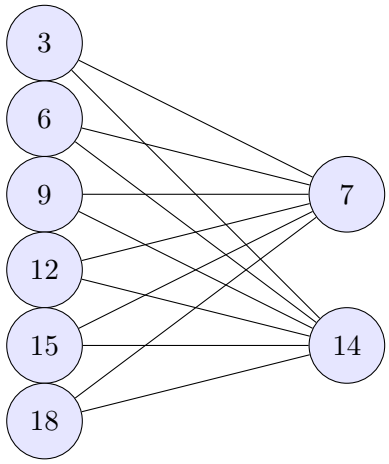

FIGURE 5. Neutrosophic decomposition of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{21}) \cong K^{\mathscr{N}}_{1,6} \cup K^{\mathscr{N}}_{6,1}$

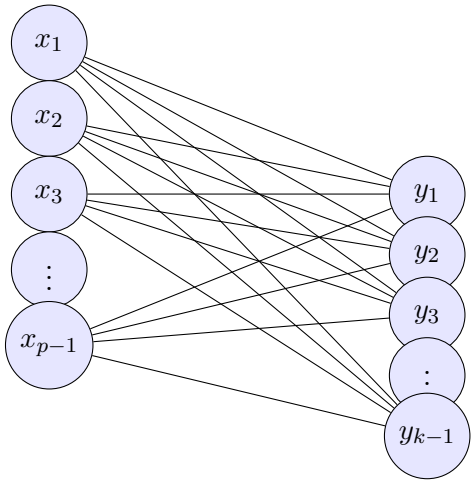

FIGURE 6. Decomposition of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{kp})$ into $K^{\mathscr{N}}_{1,p-1} \cup K^{\mathscr{N}}_{p-1,k-2}$ with symbolic vertices

Theorem 4.7. *Let p and q be distinct odd primes. Then the neutrosophic zero divisor graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{pq})$ is a neutrosophic complete bipartite graph $K^{\mathscr{N}}_{p-1,q-1}$.*

Proof. Let $\mathcal{Z}_{pq}$ be the ring of integers modulo pq, where p and q are distinct odd primes.

The nonzero zero divisors of $\mathcal{Z}_{pq}$ are precisely the nonzero multiples of either p or q, but not both.

Define:

$$X = \{q, 2q, \ldots, (p-1)q\}, \quad Y = \{p, 2p, \ldots, (q-1)p\}.$$

Then we have, $|X| = p-1$, $|Y| = q-1$, Every $x \in X$ and $y \in Y$ satisfy $xy \equiv 0 \mod pq$, since $p \mid y$ and $q \mid x$, so $pq \mid xy$.

Thus, each vertex in X is connected to each vertex in Y, and there are: No edges within X (all divisible by q, so $q \nmid qx$ unless also divisible by p, which is excluded), No edges within Y.

Hence, $\Gamma_{\mathscr{N}}(\mathcal{Z}_{pq}) \cong K^{\mathscr{N}}_{p-1,q-1}$, a neutrosophic complete bipartite graph. □

Example 4.5: Decomposition of $\Gamma_{\mathscr{N}}(\mathcal{Z}_{15})$

The nonzero zero divisors of $\mathcal{Z}_{15}$ are:

$$X = \{5, 10\} \quad \text{(multiples of 5)}, \quad Y = \{3, 6, 9, 12\} \quad \text{(multiples of 3)}.$$

Each $x \in X$, $y \in Y$ satisfies $xy \equiv 0 \mod 15$, as $5 \mid x$, $3 \mid y$, hence $15 \mid xy$. No edges exist within X or within Y, so:

$$\Gamma_{\mathscr{N}}(\mathcal{Z}_{15}) \cong K^{\mathscr{N}}_{2,4}.$$

Neutrosophic vertex membership values: For all vertices: $\mathscr{T} = 0.5, \mathscr{I} = 0.2, \mathscr{F} = 0.3$, Edge memberships computed as usual with neutrosophic min-max rules. □

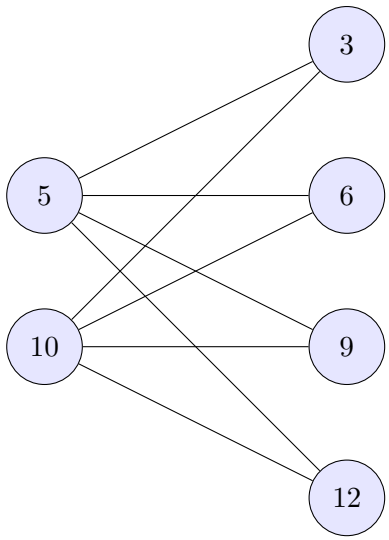

FIGURE 7. Neutrosophic bipartite graph $\Gamma_{\mathscr{N}}(\mathcal{Z}_{15}) \cong K^{\mathscr{N}}_{2,4}$

5. Comparative Analysis

The proposed neutrosophic zero divisor graph (NZDG) framework extends earlier fuzzy and neutrosophic approaches in both scope and modeling power. Fuzzy zero divisor graphs (FZDGs) assign membership values in $[0, 1]$ but cannot distinguish between partial inclusion, measurement error, or lack of information, since all appear as the same degree of uncertainty. In contrast, NZDGs assign a triplet $(\mathscr{T}, \mathscr{I}, \mathscr{F})$, which separates truth, indeterminacy, and falsity, allowing decompositions to capture ambiguous algebraic relations more precisely. Earlier studies of NZDGs [4] introduced neutrosophic labels but focused mainly on small rings and did not establish general decomposition theorems. Our work addresses this gap by proving systematic results for $\mathcal{Z}_{2p}$, $\mathcal{Z}_{3p}$, $\mathcal{Z}_{5p}$, $\mathcal{Z}_{pq}$, and the general case $\mathcal{Z}_{kp}$, showing that

$$\Gamma_{\mathscr{N}}(\mathcal{Z}_{kp}) \cong K^{\mathscr{N}}_{1,p-1} \cup K^{\mathscr{N}}_{p-1,k-2}.$$

This unified decomposition demonstrates that NZDGs generalize fuzzy decompositions, preserve earlier results, and provide a broader theoretical foundation for analyzing algebraic graphs under uncertainty.

6. Conclusions

In this work, we introduced the framework of neutrosophic zero divisor graphs (NZDGs) as a natural generalization of fuzzy zero divisor graphs, embedding indeterminacy directly into the algebraic graph-theoretic structure of commutative rings. By assigning neutrosophic membership triplets $(\mathscr{T}, \mathscr{I}, \mathscr{F})$ to vertices and edges, we captured the nuanced behavior of uncertainty in mathematical and applied contexts.

Our investigation focused on rings of the form $\mathcal{Z}_n$, particularly for $n = 2p, 3p, 5p, pq$, and the general case kp, where p and q are primes. A central contribution is the decomposition theorem

$$\Gamma_{\mathscr{N}}(\mathcal{Z}_{kp}) \cong K^{\mathscr{N}}_{1,p-1} \cup K^{\mathscr{N}}_{p-1,k-2},$$

which unifies earlier case-specific results into a single principled framework. Unlike fuzzy decompositions, which cannot represent indeterminacy, or earlier neutrosophic studies limited to small examples, this work provides a systematic theory valid for an infinite family of rings.

Beyond their theoretical significance, NZDGs have potential applications in several domains. In logic-based network design, the presence of indeterminate links can be modeled effectively using neutrosophic decompositions, offering insights into network resilience under uncertainty. In soft computing, NZDGs distinguish uncertainty from genuine indeterminacy, making them suitable for applications in data mining and decision support systems. The neutrosophic formulation also opens new directions in algebraic modeling and combinatorial optimization,

where studying invariants such as chromatic number, domination number, or connectivity under neutrosophic interpretation may lead to novel uncertainty-aware algorithms.

6.1. *Limitations and future directions*

The main limitations are:

- Scope: proofs are restricted to commutative modular rings (families $\mathcal{Z}_n$); extensions to non-commutative rings remain open.
- Label choices: the neutrosophic assignments are representative rather than unique; application-driven calibration will be necessary in practice.
- Algorithmics: efficient implementations and large-n experiments are yet to be developed.

Future work should address these points, investigate neutrosophic graph invariants, and evaluate algorithmic implementations for large-scale systems.

6.2. *Impact on the field*

By generalizing fuzzy decompositions into the neutrosophic setting, this work provides a unified theoretical foundation for studying algebraic graphs under uncertainty. The results demonstrate how indeterminacy alters structural decompositions and extend beyond prior case-specific studies. They also open pathways for integrating neutrosophic logic into algebraic modeling, network analysis, and computational intelligence. These findings position NZDGs as a versatile framework for future interdisciplinary research at the interface of algebra, graph theory, and uncertainty modeling.

References

1. Beck, I. Coloring of commutative rings. *J. Algebra* **1988**, *116*, 208–226.
2. Anderson, D.F.; Livingston, P.S. The zero-divisor graph of a commutative ring. *J. Algebra* **1999**, *217*, 434–447.
3. Kuppan, A.; Tamilselvi, R. Fuzzy decomposition of zero divisor graphs over $\mathcal{Z}_n$. *J. Prog. Res. Math.* **2021**, *17*, 1952–1962.
4. Balakrishnan, A.; Kanchana, M.; Broumi, S.; Thirugnanasambandam, K. Decomposition of neutrosophic zero-divisor graph. *Neutrosophic Sets Syst.* **2025**, *83*, 1–12. Available online: `https://digitalrepository.unm.edu/nss_journal/vol83/iss1/9/` (accessed on [insert date]).
5. DeMeyer, F.R.; Schneider, E. The zero divisor graph of a ring. *Comm. Algebra* **2001**, *29*, 4395–4410.
6. Akram, S.; Ahmad, S. Fuzzy graph theory and its applications: A survey. *Int. J. Math. Comput.* **2018**, *30*, 14–26.
7. Redmond, S.J. The zero divisor graph of a non-commutative ring. *Int. J. Commut. Rings* **2002**, *1*, 93–101.
8. Joshi, A. Zero divisor graphs—a survey. *J. Algebra Appl.* **2009**, *8*, 151–170.
9. F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, American Research Press, Rehoboth, 1999.

10. F. Smarandache, *Neutrosophic Graph Theory and Algorithms*, Pons Editions, Brussels, 2019.
11. F. Aliniaeifard and Y. Li, *Zero-Divisor Graphs for Group Rings*, Commun. Algebra **42** (2014), no. 11, 4790–4800. https://doi.org/10.1080/00927872.2013.827689
12. S. Akbari and F. Heydari, *The regular graph of a non-commutative ring*, Bull. Aust. Math. Soc. **89** (2012), no. 1, 1–13. https://doi.org/10.1017/S0004972712001177
13. N. Jafari Rad, S. H. Jafari, and D. A. Mojdeh, *On Domination in Zero-Divisor Graphs*, Canad. Math. Bull. **56** (2013), no. 2, 407–411. https://doi.org/10.4153/CMB-2011-156-1
14. M. Kanchana and K. Kavitha, *Heuristic Incident Edge Path Algorithm for Interval-Valued Neutrosophic Transportation Networks*, Contemporary Mathematics, 2024.
15. M. Kanchana and K. Kavitha, *A Modified Incident Edge Path Algorithm for Efficient Shortest Path Solutions in Pipeline Networks and Urban Navigation Systems*, Reliability: Theory and Applications, vol. 20, no. 1 (82), pp. 591–600, 2025.
16. I. Delİ, *Shortest Path Problem on Single Valued Neutrosophic Graphs*, Proc. ISNCC-2017 (IEEE), Marrakech, 2017.
17. S. Broumi, M. Talea, A. Bakali, and F. Smarandache, *Computation of Shortest Path Problem in a Network with Single Valued Neutrosophic Numbers*, Proc. 2016 Int. Conf. on Advanced Mechatronic Systems (ICAMechS), 2016.
18. L. A. Zadeh, *Fuzzy Sets*, Information and Control **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
19. Smarandache, F. Neutrosophic set—a generalization of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **1999**, *24*, 287–297.

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