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Fermatean Quadripartitioned Neutrosophic Ideals in Subtraction Algebra

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Abstract. In this paper, the notions of a Fermatean Quadripartitioned Neutrosophic Fuzzy subalgebra and a Fermatean Quadripartitioned Neutrosophic Fuzzy ideal of a subtraction algebra are introduced and its characterizations of above are investigated.

Keywords: Subtraction algebra (Fermatean Quadripartitioned Neutrosophic), subalgebra (Fermatean Quadripartitioned Neutrosophic) ideal.

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1. **Introduction**

The system of the form $(\phi; \circ,)$ was studied by B. M. Schein[7], where ϕ is a set of functions closed under the composition ”$\circ$” of functions (so $(\phi; \circ)$ is a semigroup of functions) and the set theoretic subtraction ”̈(so $(\phi;)$ is a subtraction algebra). B. Zelinka[10] addressed a problem pertaining to the structure of multiplication in a subtraction semigroup that was put forth by B. M. Schein. For subtraction algebras of a particular kind known as the atomic subtraction algebras, he found a solution to the problem. The concept of ideals in subtraction algebras and their characterization were presented by Y. B. Jun et al[4,5]. The concepts of an intersectional soft subalgebra and an intersectional soft ideal of a subtraction algebra were presented by S. S. Ahn and Y. H. Kim[1], who also looked into some of their connected characteristics.

In 1965, Zadeh[9] defined the fuzzy set and introduced the degree of membership/truth (t). In 1986, Atanassov[2] defined the intuitionistic fuzzy set and introduced the degree of non-membership/falsehood (f) as a generalization of fuzzy sets. In 1995 (published in 1998), Smarandache[8] defined the Neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsity) and introduced the degree of indeterminacy/neutrality (i) as an independent component. M. Ramya[6] outlined a brand new mathematical approach of Fermatean Quadripartitioned Neutrosophic Sets (FQNS) in 2022.

In this paper, we introduce the notions of a Fermatean Quadripartitioned Neutrosophic Fuzzy subalgebra and a Fermatean Quadripartitioned Neutrosophic of a subtraction algebra. Characterizations of Fermatean Quadripartitioned Neutrosophic subalgebra and a Fermatean Quadripartitioned Neutrosophic fuzzy ideal are investigated.

2. PRELIMINARIES

Definition 2.1. [3] An algebra $(X;-)$ is called a subtraction algebra if a single binary operation $-$ satisfies the following identities: for any $x, y, z \in X$,
$x-(y-x)=x$
$x-(x-y)=y-(y-x)$
$(x-y)-z=(x-z)-y$ We introduced an order relation X on a subtraction algebras:
$a \leq b \iff a-b=0$ where $0=a-a$ is an element that does not depend on the choice of $a \in X$.

Definition 2.2. [3] A neutrosophic set A of X is called a neutrosophic ideal of X if the following conditions are true $\forall x, y \in X$
$N_A(x-y) \geq N_A(x)$ i.e., $t_A(x-y) \geq t_A(x); i_A(x-y) \geq i_A(x)$ and $f_A(x-y) \leq f_A(x)$;
$\exists\, x \vee y \Rightarrow N_A(x \vee y) \geq N_A(x) \wedge N_A(y)$ i.e., $t_A(x \vee y) \geq t_A(x) \wedge t_A(y); i_A(x \vee y) \geq i_A(x) \wedge i_A(y)$ and $f_A(x \vee y) \leq f_A(x) \vee f_A(y)$ whenever there exists $x \vee y$.

Definition 2.3. [6] Let X be a universe. A Fermatean Quadripartitioned Neutrosophic Set(FQN) $\hat{\mathbb{L}}$ on X is an object of the form
$\hat{\mathbb{L}} = \{\langle(\mathsf{x}, \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}, \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{L}}}}, \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{L}}}}, \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}) : \mathsf{x} \in \mathsf{X}\rangle\}$.
$(\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}})^3 + (\hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{L}}}})^3 + (\hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{L}}}})^3 + (\hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}})^3 \leq 2$
Here $\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the truth membership,
$\hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the contradiction membership,
$\hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the ignorance membership,
$\hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the falsity membership.

3. **FQN Ideals in Subtraction Algebra**

Definition 3.1.

A Fermatean Quadripartitioned Neutrosophic set A in R is called a Fermatean Quadripartitioned Neutrosophic subalgebra of R if it satisfies:

(5.1)$(\forall x, y \in R)(TR_A(x-y) \geq min\{TR_A(x), TR_A(y)\}, CR_A(x-y) \geq min\{CR_A(x), CR_A(y)\},$ $UR_A(x-y) \leq max\{UR_A(x), UR_A(y)\}$ and $FR_A\ (x-y) \leq max\{FR_A(x), FR_A(y)\})$.

Proposition 3.2.

Every Fermatean Quadripartitioned Neutrosophic subalgebra of R satisfies the following conditions:

(5.2)$(\forall x \in R)(TR_A(0) \geq TR_A(x), CR_A(0) \geq CR_A(x), UR_A(0) \leq UR_A(x)$ and $FR_A(0) \leq FR_A(x))$.

Example 3.3.

Let X = {0,1,2,3} be a subraction algebra with the following table:

-	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	1
3	3	2	1	0

Define a Fermatean Quadripartitioned Neutrosophic set A in R as follows:

$$TR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} 0.83 & \text{if x} \in \{0,3\} \\ 0.13 & \text{if x} \in \{1,2\} \end{cases},$$

$$CR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} 0.83 & \text{if x} \in \{0,3\} \\ 0.13 & \text{if x} \in \{1,2\} \end{cases},$$

$$UR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} 0.11 & \text{if x} \in \{0,3\} \\ 0.81 & \text{if x} \in \{1,2\} \end{cases},$$

and

$$FR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} 0.11 & \text{if x} \in \{0,3\} \\ 0.81 & \text{if x} \in \{1,2\} \end{cases},$$

It is easy to check that A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R.

Theorem 3.4.

Let A be a Fermatean Quadripartitioned Neutrosophic set in R and let α,β,δ,ξ $\in$ [0,1] with $0 \leq \alpha^2+\beta^2+\delta^2+\xi^2 \leq 3$. Then A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R if and only if all of (α,β,δ,ξ)-level set $A^{(\alpha,\beta,\delta,\xi)}$ are subalgebras of R when $A^{(\alpha,\beta,\delta,\xi)} \neq \varnothing$.

Proof.

Assume that A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R. Let α,β,δ,ξ $\in$ [0,1] be such that $0 \leq \alpha^2+\beta^2+\delta^2+\xi^2 \leq 3$ and $A^{(\alpha,\beta,\gamma,\delta,\xi)} \neq \varnothing$. Let $x, y \in A^{(\alpha,\beta,\delta,\xi)}$. Then $TR_A(x) \geq \alpha, TR_A(y) \geq \alpha$, $CR_A(x) \geq \beta, CR_A(y) \geq \beta$, $UR_A(x) \leq \delta, UR_A(y) \leq \delta$ and $FR_A(x) \leq \xi, FR_A(y) \leq \xi$. Using (5.1), we have $TR_A(x-y) \geq min\{TR_A(x), TR_A(y)\} \geq \alpha$, $CR_A(x-y) \geq min\{CR_A(x), CR_A(y)\} \geq \beta$, $UR_A(x-y) \leq max\{UR_A(x), UR_A(y)\} \leq \delta$ and $FR_A(x-y) \leq max\{FR_A(x), FR_A(y)\} \leq \xi$. Hence $x-y \in A^{(\alpha,\beta,\delta,\xi)}$. Therefore $A^{(\alpha,\beta,\delta,\xi)}$ is a subalgebra of R.

Conversely, all of $(\alpha,\beta,\delta,\xi)$-level set $A^{(\alpha,\beta,\delta,\xi)}$ are subalgebras of R when $A^{(\alpha,\beta,\delta,\xi)} \neq \varnothing$. Assume that there exist $a_t, b_t, a_i, b_i \in R$ and $a_f, b_f \in R$ such that $TR_A(a_t - b_t) < min\{TR_A(a_t), TR_A(b_t)\}$, $CR_A(a_i - b_i) < min\{CR_A(a_i), CR_A(b_i)\}$, $UR_A(a_f - b_f) > max\{UR_A(a_f), UR_A(b_f)\}$ and $FR_A(a_f - b_f) > max\{ FR_A(a_f), FR_A(b_f)\}$. Then $TR_A(a_t - b_t) < \alpha_1 \leq min\{TR_A(a_t),\ TR_A(b_t)\}$, $CR_A(a_i - b_i) < \beta_1 \leq min\{CR_A(a_i), CR_A(b_i)\}$, $UR_A(a_f - b_f) > \delta_1 \geq max\{UR_A(a_f), UR_A(b_f)\}$, and $FR_A(a_f - b_f) > \xi_1 \geq max\{FR_A(a_f), FR_A(b_f)\}$ for some $\alpha_1, \beta_1 \in (0,1]$ and $\delta_1, \xi_1 \in [0,1)$. Hence $a_t, b_t, a_i, b_i \in A(\alpha_1, \beta_1, \delta_1, \xi_1)$, and $a_f, b_f \in A(\alpha_1, \beta_1, \delta_1,\ \xi_1)$. But $a_t - b_t, a_i - b_i \notin A(\alpha_1, \beta_1, \delta_1,\ \xi_1)$ and $a_f - b_f \notin A(\alpha_1, \beta_1, \delta_1,\ \xi_1)$, which is a contradiction. Hence $TR_A(x-y) \geq min\{TR_A(x), TR_A(y)\}$, $CR_A(x-y) \geq min\{CR_A(x), CR_A(y)\}$, $UR_A(x-y) \leq max\{UR_A(x), UR_A(y)\}$, $FR_A(x-y) \leq max\{FR_A(x), FR_A(y)\}$, for any $x, y, z \in R$. Therefore A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R.

Since [0,1] is a completely distributive lattice with respect to the usual ordering, we have the following theorem. □

Theorem 3.5.

If $\{A_i | i \in N\}$ is a family of Fermatean Quadripartitioned Neutrosophic subalgebra of R, then $(\{A_i | i \in N\}, \subseteq)$ forms a complete distributive lattice.

Theorem 3.6.

Let A be a Fermatean Quadripartitioned Neutrosophic subalgebra of R. If there exists a sequence $\{a_n\}$ in R such that $lim_{n \longrightarrow \infty} TR_A(a_n) = 1$, $lim_{n \longrightarrow \infty} CR_A(a_n) = 1$, $lim_{n \longrightarrow \infty} UR_A(a_n) = 0$, and $lim_{n \longrightarrow \infty} FR_A(a_n) = 0$, then $TR_A(0) = 1$, $CR_A(0) = 1$, $UR_A(0) = 0$ and $FR_A(0) = 0$.

Proof.

By Proposition 5.3.2, we have $TR_A(0) \geq TR_A(x)$, $CR_A(0) \geq CR_A(x)$, $UR_A(0) \leq UR_A(x)$ and $FR_A(0) \leq FR_A(x)$ for all $x \in R$. Hence we have $TR_A(0) \geq TR_A(a_n)$, $CR_A(0) \geq CR_A(a_n)$, $UR_A(0) \leq UR_A(a_n)$ and $FR_A(0) \leq FR_A(a_n)$ for every positive integer n. Therefore $1 = lim_{n \longrightarrow \infty} TR_A(a_n) \leq TR_A(0) \leq 1$, $1 = lim_{n \longrightarrow \infty} CR_A(a_n) \leq CR_A(0) \leq 1$, $0 \leq UR_A(0) \leq lim_{n \longrightarrow \infty} UR_A(a_n) = 0$ and $0 \leq FR_A(0) \leq lim_{n \longrightarrow \infty} FR_A(a_n) = 0$. Thus we have $TR_A(0) = 1$, $CR_A(0) = 1$, $UR_A(0) = 0$ and $FR_A(0) = 0$. □

Proposition 3.7.

If every Fermatean Quadripartitioned Neutrosophic subalgebra A of R satisfies the conditions (5.3.3)$(\forall x, y \in R)(TR_A(x-y) \geq TR_A(y), CR_A(x-y) \geq CR_A(y), UR_A(x-y) \leq UR_A(y)$ and $FR_A(x-y) \leq FR_A(y))$, then TR_A, CR_A, UR_A and FR_A are constant functions.

Proof.

It follows from (5.3.3) that $TR_A(x) = TR_A(x-0) \geq TR_A(0)$, $CR_A(x) = CR_A(x-0) \geq CR_A(0)$, $UR_A(x) = UR_A(x-0) \leq UR_A(0)$ and $FR_A(x) = FR_A(x-0) \leq FR_A(0)$ for any $x \in R$. By proposition (5.3.2), we have $TR_A(x) = TR_A(0)$, $CR_A(x) = CR_A(0)$, $UR_A(x) = UR_A(0)$ and $FR_A(x) = FR_A(0)$ for any $x \in R$. Hence TR_A, CR_A, UR_A and FR_A are constant functions. □

Theorem 3.8.

Every subalgebra of R can be represented as an $(\alpha, \beta, \delta, \xi)$-level set of a Fermatean Quadripartitioned Neutrosophic subalgebra A of R.

Proof.

Let S be a subalgebra of R and let A be a Fermatean Quadripartitioned Neutrosophic subalgebra of R. Define a Fermatean Quadripartitioned Neutrosophic set A in R as follows:

$$TR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} \alpha_1 & \text{if x} \in \text{S} \\ \alpha_2 & \text{otherwise} \end{cases},$$

$$CR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} \beta_1 & \text{if x} \in \text{S} \\ \beta_2 & \text{otherwise} \end{cases},$$

$$UR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} \delta_1 & \text{if x} \in \text{S} \\ \delta_2 & \text{otherwise} \end{cases},$$

and

$$FR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} \xi_1 & \text{if x} \in \text{S} \\ \xi_2 & \text{otherwise} \end{cases},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1]$ and $\delta_1, \xi_1 \in [0,1)$ with $\alpha_1 > \alpha_2, \beta_1 > \beta_2, \delta_1 < \delta_2, \xi_1 < \xi_2$ and $0 \leq \alpha_1^2 + \beta_1^2 + \delta_1^2 + \xi_1^2 \leq 3, 0 \leq \alpha_2^2 + \beta_2^2 + \delta_2^2 + \xi_2^2 \leq 3$. Obviously, $S = A^{(\alpha_1,\beta_1,\delta_1,\xi_1)}$. We now prove that A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R. Let x,y $\in$ R. If x,y $\in$ S,then x-y $\in$ S because S is a subalgebra of S. Hence $TR_A(x) = TR_A(y) = TR_A(x-y) = \alpha_1, CR_A(x) = CR_A(y) = CR_A(x-y) = \beta_1, UR_A(x) = UR_A(y) = UR_A(x-y) = \delta_1, FR_A(x) = FR_A(y) = FR_A(x-y) = \xi_1$ and so $TR_A(x-y) \geq min\{TR_A(x), TR_A(y)\}$, $CR_A(x-y) \geq min\{CR_A(x), CR_A(y)\}$, $UR_A(x-y) \leq max\{UR_A(x), UR_A(y)\}$ and $FR_A(x-y) \leq max\{FR_A(x), FR_A(y)\}$. Obviously, if x $\notin$ S and y $\notin$ S, then $TR_A(x-y) \geq min\{TR_A(x), TR_A(y)\} = \alpha_2$, $CR_A(x-y) \geq min\{CR_A(x), CR_A(y)\} = \beta_2$, $UR_A(x-y) \leq max\{UR_A(x), UR_A(y)\} = \delta_2$ and $FR_A(x-y) \leq max\{FR_A(x), FR_A(y)\} = \xi_2$. Therefore A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R. □

Theorem 3.9.

Let A be a Fermatean Quadripartitioned Neutrosophic set of R and let $\alpha, \beta \in [0,1]$ with $0 \leq \alpha^2 + \beta^2 + \delta^2 + \xi^2 \leq 3$. Define a Fermatean Quadripartitioned Neutrosophic set A^* in R as follows:

$$TR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} TR_A(x) & \text{if x} \in A^{(\alpha,\beta,\gamma,\delta,\xi)} \\ 0 & \text{otherwise} \end{cases},$$

$$CR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} CR_A(x) & \text{if x} \in A^{(\alpha,\beta,\gamma,\delta,\xi)} \\ 0 & \text{otherwise} \end{cases},$$

$$UR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} UR_A(x) & \text{if x} \in A^{(\alpha,\beta,\gamma,\delta,\xi)} \\ 1 & \text{otherwise} \end{cases},$$

and

$$FR_A : R \longrightarrow [0,1], \text{x} \mapsto \begin{cases} FR_A(x) & \text{if x} \in A^{(\alpha,\beta,\gamma,\delta,\xi)} \\ 1 & \text{otherwise} \end{cases}$$

If A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R, then so is A^*.

Proof.

Let A be a Fermatean Quadripartitioned Neutrosophic subalgebra of R. By theorem 5.3.8, all of $(\alpha, \beta, \delta, \xi)$-level set $x \in A^{(\alpha,\beta,\delta,\xi)}$ are subalgebras of R. If $x, y \in x \in A^{(\alpha,\beta,\delta,\xi)}$, then $x - y \in x \in A^{(\alpha,\beta,\delta,\xi)}$ If $x \notin A^{(\alpha,\beta,)}$ or $y \notin A^{(\alpha,\beta)}$, then $TR_{A^*}(x) = 0, IR_{A^*}(x) = 0, FR_{A^*}(x) = 1$ or $TR_{A^*}(y) = 0, IR_{A^*}(y) = 0, FR_{A^*}(y) = 1$. Therefore we get $TR_{A^*}(x-y) \geq min\{TR_{A^*}(x), TR_{A^*}(y)\} = 0$, $CR_{A^*}(x-y) \geq min\{CR_{A^*}(x), CR_{A^*}(y)\} = 0$, $UR_{A^*}(x-y) \leq max\{UR_{A^*}(x), UR_{A^*}(y)\} = 1$ and $FR_{A^*}(x-y) \leq max\{FR_{A^*}(x),\ FR_{A^*}(y)\} = 1$ for any $x, y \in R$. Thus A^* is a Fermatean Quadripartitioned Neutrosophic subalgebra of R. □

Definition 3.10.

A Fermatean Quadripartitioned Neutrosophic set A in R is called a Fermatean Quadripartitioned Neutrosophic ideal of R if it satisfies (5.3.2) and

(5.3.4)$(\forall x, y \in R)(TR_A(x) \geq min\{TR_A(x-y), TR_A(y)\},\ CR_A(x) \geq min\{CR_A(x-y), CR_A(y)\},\ UR_A(x) \leq max\{UR_A(x-y), UR_A\ (y)\}$ and $FR_A(x) \leq max\{FR_A(x-y), FR_A(y)\})$.

Proposition 3.11.

Every Fermatean Quadripartitioned Neutrosophic ideal of R is a Fermatean Quadripartitioned Neutrosophic subalgebra of R.

Proof.

Let A be a Fermatean Quadripartitioned Neutrosophic ideal of R. Put x:= x-y and y:=x in (5.3.4). Then we have $TR_A(x-y) \geq min\{TR_A((x-y)-x), TR_A(x)\}$, $IR_A(x-y) \geq min\{IR_A((x-y)-x), IR_A(x)\}$ and $FR_A(x-y) \leq max\{FR_A((x-y)-x), FR_A(x)\}$. It follows from (5.3.3) and (5.3.2) that

$TR_A(x-y) \geq min\{TR_A((x-x)-y), TR_A(x)\} = min\{TR_A(0), TR_A(x)\} \geq min\ \{TR_A(x), TR_A(y)\}$,

$CR_A(x-y) \geq min\{CR_A((x-y)-x), CR_A(x)\} = min\{CR_A(0), CR_A(x)\} \geq min\{CR_A(x), CR_A(y)\}$,

$UR_A\ (x-y) \leq max\{UR_A((x-y)-x), UR_A(x)\} = max\{UR_A(0), UR_A(x)\} \leq max\{UR_A(x), UR_A(y)\}$ and

$FR_A(x-y) \leq max\{FR_A((x-y)-x), FR_A(x)\} = max\{FR_A(0), FR_A(x)\} \leq max\{FR_A(x), FR_A(y)\}$, for any $x, y \in R$. Thus A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R.

The converse of Proposition 5.3.11 may not be true in general(see Example 5.3.12.) □

Example 3.12.

(a) Let R = {0,a,b,c} be a subtraction algebra with the following table:

-	0	a	b	C
0	0	0	0	0
a	a	0	a	A
b	b	b	0	B
c	c	c	c	0

Define a Fermatean Quadripartitioned Neutrosophic set A in R as follows:

$$TR_A : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.72 & \text{if x} \in \{0,\text{a}\} \\ 0.11 & \text{if x} \in \{\text{b,c}\} \end{cases},$$

$$CR_A : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.72 & \text{if x} \in \{0,\text{a}\} \\ 0.11 & \text{if x} \in \{\text{b,c}\} \end{cases},$$

$$UR_A : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.13 & \text{if x} \in \{0,\text{a}\} \\ 0.71 & \text{if x} \in \{\text{b,c}\} \end{cases},$$

and

$$FR_A : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.13 & \text{if x} \in \{0,\text{a}\} \\ 0.71 & \text{if x} \in \{\text{b,c}\} \end{cases},$$

It is easy to check that A is a Fermatean Quadripartitioned Neutrosophic ideal of R.

(b) Let R = {0,1,2,3} be a subtraction algebra as in Example 5.3.3. Define a Fermatean Quadripartitioned Neutrosophic set B in R as follows:

$$TR_B : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.33 & \text{if x} = 0 \\ 0.12 & \text{if x} \in \{1,2\} \\ 0.03 & \text{if x} = 3 \end{cases},$$

$$CR_B : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.33 & \text{if x} = 0 \\ 0.12 & \text{if x} \in \{1,2\} \\ 0.03 & \text{if x} = 3 \end{cases},$$

$$UR_B : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.11 & \text{if x} = 0 \\ 0.15 & \text{if x} \in \{1{,}2\} \\ 0.36 & \text{if x} = 3 \end{cases},$$

and

$$FR_B : R \longrightarrow [0,1], \mathrm{x} \mapsto \begin{cases} 0.11 & \text{if x} = 0 \\ 0.15 & \text{if x} \in \{1{,}2\} \\ 0.36 & \text{if x} = 3 \end{cases}.$$

It is easy to check that B is a Fermatean Quadripartitioned Neutrosophic subalgebra of R. But it is not a Fermatean Quadripartitioned Neutrosophic ideal of R, since $TR_B(3) = 0.03 \ngeq min\{TR_B(3-1), TR_B(1)\} = max\{TR_B(2), TR_B(1)\} = 0.12$

Theorem 3.13.

Let A be a Fermatean Quadripartitioned Neutrosophic set in R and let α,β,δ,ξ $\in$ [0,1] with $0 \leq \alpha^2 + \beta^2 + \delta^2 + \xi^2 \leq 3$. Then A is a Fermatean Quadripartitioned Neutrosophic ideal of R if and only if all of (α,β,δ,ξ)-level set $A^{(\alpha,\beta,\delta,\xi)}$ are ideals of R when $A^{(\alpha,\beta,\delta,\xi)} \neq \varnothing$.

Proof.

Assume that A is a Fermatean Quadripartitioned Neutrosophic ideal of R. Let α,β,δ,ξ $\in$ [0,1] be such that $0 \leq \alpha^2 + \beta^2 + \delta^2 + \xi^2 \leq 3$ and $A^{(\alpha,\beta,\delta,\xi)} \neq \varnothing$. Let $x, y \in R$ be such that $x - y, y \in A^{(\alpha,\beta,\delta,\xi)}$. Then $TR_A(x-y) \geq \alpha, TR_A(y) \geq \alpha$, $CR_A(x-y) \geq \beta, CR_A(y) \geq \beta$, $UR_A(x-y) \leq \delta, UR_A(y) \leq \delta$ and $FR_A(x-y) \leq \xi, FR_A(y) \leq \xi$. By Definition (5.3.10), we have $TR_A(0) \geq TR_A(x) \geq min\{TR_A(x-y), TR_A(y)\} \geq \alpha$, $CR_A(0) \geq CR_A(x) \geq min\{CR_A(x-y), CR_A(y)\} \geq \beta$, $UR_A(0) \leq UR_A(x) \leq max\{UR_A(x-y), UR_A(y)\} \leq \delta$ and $FR_A(0) \leq FR_A(x) \leq max\{FR_A(x-y), FR_A(y)\} \leq \xi$. Hence $0, x \in A^{(\alpha,\beta,\delta,\xi)}$. Therefore $A^{(\alpha,\beta,\delta,\xi)}$ is an ideal of R.

Conversely, suppose that there exist $a, b, c \in R$ such that $TR_A(0) < TR_A(a)$, $CR_A(0) < CR_A(b)$, $UR_A(0) > UR_A(d)$ and $FR_A(0) > FR_A(e)$. Then there exist, $a_t, b_t \in (0,1]$ and $c_t, d_t, e_t \in [0,1)$ such that $TR_A(0) < a_t < TR_A(a)$, $CR_A(0) < b_t < CR_A(b)$, $UR_A(0) > d_t > UR_A(d)$ and $FR_A(0) > e_t > FR_A(e)$. Hence $0 \notin A^{(a_t,b_t,c_t,d_t,e_t)}$, which is a contradiction. Therefore $TR_A(0) \geq TR_A(x), CR_A(0) \geq CR_A(x), UR_A(0) \leq UR_A(x)$ and $FR_A(0) \leq FR_A(x)$ for all $x \in R$. Assume that there exist $a_t, b_t, a_c, b_c, a_g,\ b_g, a_u, b_u, a_f, b_f \in R$ such that $TR_A(a_t) < min\{TR_A(a_t - b_t),\ TR_A(b_t)\}$, $CR_A(a_c) < min\{CR_A(a_c - b_c), CR_A(b_c)\}$, $UR_A(a_u) > max\{UR_A(a_u - b_u), UR_A(b_u)\}$ and $FR_A(a_f) > max\{FR_A(a_f - b_f), FR_A(b_f)\}$. Then there exist $s_t, s_c \in (0,1]$ and $s_g, s_u, s_f \in [0,1)$ such that $TR_A(a_t) < s_t \leq min\{TR_A(a_t - b_t),\ TR_A(b_t)\}$, $CR_A(a_c) < s_c \leq min\{CR_A(a_c - b_c), CR_A(b_c)\}$, $UR_A(a_u) >$

$s_u \geq max\{UR_A(a_u - b_u), UR_A(b_u)\}$ and $FR_A(a_f) > s_f \geq max\ \{FR_A\ (a_f - b_f), FR_A(b_f)\}$. Hence $a_t - b_t, b_t, a_c - b_c, a_g \in A^{(a_t,b_t,c_t)}$ and $b_t, b_i, b_f \in A^{(a_t,b_t,c_t)}$. But $a_t, a_i \notin A^{(a_t,b_t,c_t)}$ and $a_f \notin A^{(a_t,b_t,c_t)}$. This is a contradiction. Therefore $TR_A(x) \geq min\{TR_A(x-y), TR_A(y)\}$, $CR_A(x) \geq min\{CR_A(x-y), CR_A(y)\}$, $UR_A(x) \leq max\{UR_A(x-y), UR_A(y)\}$ and $FR_A(x) \leq max\{FR_A(x-y), FR_A(y)\}$ for any x,y $\in$ R. Therefore A is a Fermatean Quadripartitioned Neutrosophic ideal of R. □

Proposition 3.14.

Every Fermatean Quadripartitioned Neutrosophic ideal A of R satisfies the following properties:

(i) ($\forall$ x,y $\in$ R) ($x \leq y \Rightarrow TR_A(x) \geq TR_A(y)$, $CR_A(x) \geq CR_A(y)$, $UR_A(x) \leq UR_A(y)$, $FR_A(x) \leq FR_A(y)$),

(ii) ($\forall$ x,y,z $\in$ R) ($x - y \leq z \Rightarrow TR_A(x) \geq min\{TR_A(y), TR_A(z)\}$, $CR_A(x) \geq min\ \{CR_A(y), CR_A(z)\}$, $UR_A(x) \leq max\ \{UR_A(y), UR_A(z)\}$ and $FR_A(x) \leq max\{FR_A(y), FR_A(z)\}$).

Proof.

(i) Let x,y $\in$ R be such that $x \leq y$. Then x - y = 0. Using (1.4) and (1.2),we have $TR_A(x) \geq min\{TR_A(x-y), TR_A(y)\} = min\{TR_A(0), TR_A(y)\} = TR_A(y)$, $CR_A(y) \geq min\{CR_A(x-y), CR_A(y)\} = min\{CR_A(0), CR_A(y)\} = CR_A(y)$, $UR_A(x) \leq max\{UR_A\ (x-y), UR_A(y)\} = max\{UR_A(0), UR_A(y)\} = UR_A(y)$ and $FR_A(x) \leq max\{FR_A(x-y), FR_A(y)\} = max\{FR_A(0), FR_A(y)\} = FR_A(y)$.

(ii) Let x,y,z $\in$ R be such that $x - y \leq z$. By (5.3.4) and (5.3.2),we get $TR_A(x-y) \geq min\{TR_A((x-y)-z), TR_A(z)\} = min\{TR_A(0), TR_A(z)\} = TR_A(z)$, $CR_A(x-y) \geq min\{CR_A((x-y)-z), CR_A(z)\} = min\{CR_A(0), CR_A(z)\} = CR_A(z)$, $UR_A(x-y) \leq max\{UR_A\ ((x-y)-z), UR_A(z)\} = max\{UR_A(0), UR_A(z)\} = UR_A(z)$ and $FR_A(x-y) \leq max\{FR_A\ ((x-y)-z), FR_A(z)\} = max\{FR_A(0), FR_A(z)\} = FR_A(z)$.

Hence $TR_A(x) \geq min\{TR_A(x-y), TR_A(y)\} \geq min\{TR_A(y), TR_A(z)\}$, $CR_A\ (x) \geq min\ \{CR_A(x-y), CR_A(y)\} \geq min\{CR_A(y), CR_A(z)\}$, $UR_A(x) \leq max\{UR_A\ (x-y), UR_A(y)\} \leq max\{UR_A(y), UR_A\ (z)\}$ and $FR_A(x) \leq max\{FR_A\ (x-y), FR_A(y)\} \leq max\{FR_A(y), FR_A(z)\}$, for any x,y,z $\in$ R. The following corollary is easily proved by induction. □

Corollary 3.15.

Every Fermatean Quadripartitioned Neutrosophic ideal A of R satisfies the following property: (5.3.5) $(\forall x, a_1, ..., a_n \in R)((...(x - a_1) - ...) - a_n = 0 \Rightarrow TR_A(x) \geq \wedge_{k=1}^{n} TR_A(a_k)$, $CR_A(x) \geq \wedge_{k=1}^{n} CR_A(a_k)$, $UR_A(x) \leq \vee_{k=1}^{n} UR_A(a_k)$, $FR_A(x) \leq \vee_{k=1}^{n} FR_A\ (a_k)$.

Definition 3.16.

Let A and B be Fermatean Quadripartitioned Neutrosophic sets of a set X. The union of A and B is defined to be a Fermatean Quadripartitioned Neutrosophic set

$$A \cup^{\sim} B := \{\langle x, TR_{A\cup B}(x), CR_{A\cup B}(x), UR_{A\cup B}(x), FR_{A\cup B}(x)\rangle | x \in R\},$$

where $TR_{A\cup B}(x) = max\{TR_A(x), TR_B(x)\}$, $CR_{A\cup B}(x) = max\{CR_A(x), CR_B(x)\}$, $UR_{A\cup B}(x) = min\{UR_A(x), UR_B(x)\}$ and $FR_{A\cup B}(x) = min\{FR_A\ (x), FR_B(x)\}$, for all $x \in R$. The intersection of A and B is defined to be a Fermatean Quadripartitioned Neutrosophic set

$$A \cap^{\sim} B := \{\langle x, TR_{A\cap B}(x), CR_{A\cap B}(x), UR_{A\cap B}(x), FR_{A\cap B}(x)\rangle | x \in R\},$$

where $TR_{A\cap B}(x) = min\{TR_A(x), TR_B(x)\}$, $CR_{A\cap B}(x) = min\{CR_A(x), CR_B(x)\}$, $UR_{A\cap B}(x) = max\{UR_A(x), UR_B(x)\}$ and $FR_{A\cap B}(x) = max\{FR_A\ (x), FR_B(x)\}$, for all $x \in R$.

Theorem 3.17.

The intersection of two Fermatean Quadripartitioned Neutrosophic ideals of R is also a Fermatean Quadripartitioned Neutrosophic ideal of R.

Proof.

Let A and B be Fermatean Quadripartitioned Neutrosophic ideals of R.

For any $x \in R$, we have $TR_{A\cap B}(0) = min\{TR_A(0), TR_B(0)\} \geq min\{TR_A(x), TR_B(x)\} = TR_{A\cap B}(x)$, $CR_{A\cap B}(0) = min\{CR_A(0), CR_B(0)\} \geq min\{CR_A(x), CR_B\ (x)\} = CR_{A\cap B}(x)$, $UR_{A\cap B}(0) = max\{UR_A(0), UR_B(0)\} \leq max\{UR_A(x), UR_B(x)\} = UR_{A\cap B}(x)$ and $FR_{A\cap B}(0) = max\ \{FR_A\ (0), FR_B(0)\} \leq max\{FR_A(x), FR_B(x)\} = FR_{A\cap B}(x)$. Let $x, y \in R$. Then we have

$$\begin{aligned}
TR_{A\cap B}(x) &= min\{TR_A(x), TR_B(x)\}\\
&\geq min\{min\{TR_A(x-y), TR_A(y)\}, min\{TR_B(x-y), TR_B(y)\}\}\\
&= min\{min\{TR_A(x-y), TR_B(x-y)\}, min\{TR_A(y), TR_B(y)\}\}\\
&= min\{TR_{A\cap B}(x-y), TR_{A\cap B}(y)\}
\end{aligned}$$

$$\begin{aligned}
CR_{A\cap B}(x) &= min\{CR_A(x), CR_B(x)\}\\
&\geq min\{min\{CR_A(x-y), CR_A(y)\}, min\{CR_B(x-y), CR_B(y)\}\}\\
&= min\{min\{CR_A(x-y), CR_B(x-y)\}, min\{CR_A(y), CR_B(y)\}\}\\
&= min\{CR_{A\cap B}(x-y), CR_{A\cap B}(y)\}
\end{aligned}$$

$$\begin{aligned}
UR_{A\cap B}(x) &= max\{UR_A(x), UR_B(x)\}\\
&\leq max\{max\{UR_A(x-y), UR_A(y)\}, max\{UR_B(x-y), UR_B(y)\}\}\\
&= max\{max\{UR_A(x-y), UR_B(x-y)\}, max\{UR_A(y), UR_B(y)\}\}\\
&= max\{UR_{A\cap B}(x-y), UR_{A\cap B}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
FR_{A\cap B}(x) &= max\{FR_A(x), FR_B(x)\}\\
&\leq max\{max\{FR_A(x-y), FR_A(y)\}, max\{FR_B(x-y), FR_B(y)\}\}\\
&= max\{max\{FR_A(x-y), FR_B(x-y)\}, max\{FR_A(y), FR_B(y)\}\}\\
&= max\{FR_{A\cap B}(x-y), FR_{A\cap B}(y)\}
\end{aligned}$$

Hence $A \cap^{\sim} B$ is a Fermatean Quadripartitioned Neutrosophic ideal of R. □

Corollary 3.18.

If $\{A_i | i \in N\}$ is a family of Fermatean Quadripartitioned Neutrosophic ideals of R, then so is $\cap^{\sim}_{i\in N} A_i$.

Proposition 3.19.

Let A be a Fermatean Quadripartitioned Neutrosophic ideal of R. Then $R_{TR} := \{x \in R | TR_A(x) = TR_A(0)\}$, $R_{CR} := \{x \in R | CR_A(x) = CR_A(0)\}$, $R_{UR} := \{x \in R | UR_A(x) = UR_A(0)\}$ and $R_{FR} := \{x \in R | FR_A(x) = FR_A(0)\}$ are ideals of R.

Proof.

Clearly, $0 \in R_{TR}$. Let $x - y, y \in R_{TR}$. Then $TR_A(x-y) = TR_A(0)$ and $TR_A(y) = TR_A(0)$. It follows from (1.4) that $TR_A(x) \geq min\{TR_A(x-y), TR_A(y)\} = TR_A(0)$. By (1.2), we get $TR_A(x) = TR_A(0)$. Hence $x \in R_{TR}$. Therefore R_{TR} is an ideal of R. By a similar way, R_{CR}, R_{UR} and R_{FR} are ideals of R. Let F: R → S be a function of sets. If

$$M = \{\langle y, TR_M(y), CR_M(y), UR_M(y) FR_M(y)\rangle | y \in S\}$$

is a Fermatean Quadripartitioned Neutrosophic set of a set S, then the preimage of M under f is defined to be a Fermatean Quadripartitioned Neutrosophic set
$f^{-1}(M) = \{\langle x, f^{-1}(TR_M)(x), f^{-1}(CR_M)(x), f^{-1}(UR_M)(x), f^{-1}(FR_M)\ (x)\rangle\ | x \in R\}$
of R, where $f^{-1}(TR_M)(x) = TR_M(f(x))$, $f^{-1}(CR_M)(x) = CR_M(f(x))$, $f^{-1}(UR_M)(x) = UR_M(f(x))$ and $f^{-1}(FR_M)(x) = FR_M(f(x))$ for all $x \in R$. □

Theorem 3.20.

Let f : R → S be a homomorphism of subraction algebras. If $M = \{\langle y, TR_M(y), CR_(y), UR_M(y), FR_M(y)\rangle | y \in S\}$ is a Fermatean Quadripartitioned Neutrosophic subalgebra of S, then the preimage of M under f is a Fermatean Quadripartitioned Neutrosophic subalgebra of R.

Proof.
Let $f^{-1}(M)$ be the preimage of M under f. For any $x, y \in R$, we have

$$\begin{aligned} f^{-1}(TR_M(x-y)) &= TR_M(f(x-y)) = TR_M(f(x)-f(y)) \\ &\geq min\{TR_M(f(x)), TR_M(f(y))\} \\ &= min\{f^{-1}(TR_M)(x), f^{-1}(TR_M)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(CR_M(x-y)) &= CR_M(f(x-y)) = CR_M(f(x)-f(y)) \\ &\geq min\{CR_M(f(x)), CR_M(f(y))\} \\ &= min\{f^{-1}(CR_M)(x), f^{-1}(CR_M)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(UR_M(x-y)) &= UR_M(f(x-y)) = UR_M(f(x)-f(y)) \\ &\leq max\{UR_M(f(x)), UR_M(f(y))\} \\ &= max\{f^{-1}(UR_M)(x), f^{-1}(UR_M)(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(FR_M(x-y)) &= FR_M(f(x-y)) = FR_M(f(x)-f(y)) \\ &\leq max\{FR_M(f(x)), FR_M(f(y))\} \\ &= max\{f^{-1}(FR_M)(x), f^{-1}(FR_M)(y)\}, \end{aligned}$$

Hence $f^{-1}(M)$ is a Fermatean Quadripartitioned Neutrosophic subalgebra of R.
Let f : R → S be an onto function of sets. If A is a Fermatean Quadripartitioned Neutrosophic set

$$f(A) := \{\langle y, f(TR_A)(y), f(CR_A)(y), f(UR_A)(y), f(FR_A)(y)\rangle | y \in S\}$$

of S,where $f(TR_A)(y) = \vee_{x\in f^{-1}(y)} TR_A(x)$, $f(CR_A)(y) = \vee_{x\in f^{-1}(y)} CR_A(x)$, $f(UR_A)(y) = \wedge_{x\in f^{-1}(y)} UR_A(x)$ and $f(FR_A)(y) = \wedge_{x\in f^{-1}(y)} FR_A(x)$. □

Theorem 3.21.
For an onto homomorphism f : R → S of subtraction algebras, let A be a Fermatean Quadripartitioned Neutrosophic set of R such that
(1.6)$(\forall C \subseteq R)$ $(\exists x_0 \in C)$ $(TR_A(x_0) = \vee_{z\in C} TR_A(z)$, $CR_A(x_0) = \vee_{z\in C} CR_A(z)$, $UR_A(x_0) = \wedge_{z\in C} UR_A(z)$, $FR_A(x_0) = \wedge_{z\in C} FR_A(z)$.
If A is a Fermatean Quadripartitioned Neutrosophic subalgebra of R, then the image of A under f is a Fermatean Quadripartitioned Neutrosophic subalgebra of S.

Proof.

Let f(A) be the image of A under f. Let a,b $\in$ S. Then $f^{-1}(a) \neq \varnothing$ and $f^{-1}(b) \neq \varnothing$ in R. By (5.3.6), there exist $x_a \in f^{-1}(a)$ and $x_b \in f^{-1}(b)$ such that

$TR_A(x_a) = \vee_{z\in f^{-1}(a)}TR_A(z), CR_A(x_a) = \vee_{z\in f^{-1}(a)}CR_A(z)$,
$UR_A(x_a) = \wedge_{z\in f^{-1}(a)}UR_A(z), FR_A(x_a) = \wedge_{z\in f^{-1}(a)}FR_A(z)$,
$TR_A(x_b) = \vee_{w\in f^{-1}(b)}TR_A(w), CR_A(x_b) = \vee_{w\in f^{-1}(b)}CR_A(w)$,
$UR_A(x_b) = \wedge_{w\in f^{-1}(b)}UR_A(w), FR_A(x_b) = \wedge_{w\in f^{-1}(b)}FR_A(w)$,

$$\begin{aligned} f(TR_A)(a-b) &= \vee_{x\in f^{-1}(a-b)}TR_A(x) \geq TR_A(x_a - x_b) \geq min\{TR_A(x_a), TR_A(x_b)\} \\ &= min\{\vee_{z\in f^{-1}(a)}TR_A(z), \vee_{w\in f^{-1}(b)}TR_A(w)\} \\ &= min\{f(TR_A)(a), f(TR_A)(b)\}, \end{aligned}$$

$$\begin{aligned} f(CR_A)(a-b) &= \vee_{x\in f^{-1}(a-b)}CR_A(x) \geq CR_A(x_a - x_b) \geq min\{CR_A(x_a), CR_A(x_b)\} \\ &= min\{\vee_{z\in f^{-1}(a)}CR_A(z), \vee_{w\in f^{-1}(b)}CR_A(w)\} \\ &= min\{f(CR_A)(a), f(CR_A)(b)\}, \end{aligned}$$

$$\begin{aligned} f(UR_A)(a-b) &= \wedge_{x\in f^{-1}(a-b)}UR_A(x) \leq UR_A(x_a - x_b) \leq max\{UR_A(x_a), UR_A(x_b)\} \\ &= max\{\wedge_{z\in f^{-1}(a)}UR_A(z), \wedge_{w\in f^{-1}(b)}UR_A(w)\} \\ &= max\{f(UR_A)(a), f(UR_A)(b)\}, \end{aligned}$$

and

$$\begin{aligned} f(FR_A)(a-b) &= \wedge_{x\in f^{-1}(a-b)}FR_A(x) \leq FR_A(x_a - x_b) \leq max\{FR_A(x_a), FR_A(x_b)\} \\ &= max\{\wedge_{z\in f^{-1}(a)}FR_A(z), \wedge_{w\in f^{-1}(b)}FR_A(w)\} \\ &= max\{f(FR_A)(a), f(FR_A)(b)\}, \end{aligned}$$

Hence f(A) is a Fermatean Quadripartitioned Neutrosophic subalgebra of S. □

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