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e-open Maps and Homeomorphisms in Fermatean Neutrosophic Topological Spaces with Applications to Entropy Measures

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Abstract. In this article, we introduce the concepts of $\mathfrak{FN}e$-open and $\mathfrak{FN}e$-closed mappings in Fermatean neutrosophic topological spaces and study some of their related properties. Further, the work is extended to $\mathfrak{FN}e$-homeomorphism, $\mathfrak{FN}e$-C homeomorphism, and $\mathfrak{FN}eT_{\frac{1}{2}}$-space in Fermatean neutrosophic topological spaces, and some of their related attributes are established. Also we give an real life application for decision making using entropy measure on Fermatean Neutrosophic sets.

Keywords: $\mathfrak{FN}e$-open map, $\mathfrak{FN}e$-closed map, $\mathfrak{FN}eT_{\frac{1}{2}}$-space, $\mathfrak{FN}e$-homeomorphism, $\mathfrak{FN}e$-C homeomorphism, entropy measure.

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1. Introduction

In 1965, Zadeh [17] introduced the concept of fuzzy sets to systematically model ambiguity and vagueness that frequently arise in real-world problems. The fuzzy set framework provided a mathematical structure to handle degrees of membership, thereby moving beyond the rigid dichotomy of classical set theory. However, Zadeh's model was limited in that it only accounted for the membership degree of an element, leaving the non-membership aspect implicit.

To overcome this shortcoming, Atanassov [1] proposed the notion of intuitionistic fuzzy sets (IFS's) in 1986. IFS's extend Zadeh's framework by explicitly incorporating both membership and non-membership degrees, subject to the constraint that their sum does not exceed one. This development offered a more comprehensive means of capturing uncertainty. Nevertheless, in many real-life scenarios, the sum of membership and non-membership degrees may exceed unity, revealing an inherent limitation of the IFS model and suggesting the need for more adaptable approaches.

Yager [16] addressed this issue by introducing the Pythagorean fuzzy set (PFS), wherein the squares of the membership and non-membership degrees are required to sum to at most one. This quadratic condition allowed for a broader domain of admissible values, thus enabling greater expressive power in modeling uncertainty. Building further on this idea, Senapati and Yager [9] introduced the Fermatean fuzzy set ($\mathfrak{F}fs$), which generalized the concept by constraining the sum of the cubes of membership and non-membership degrees to be less than or equal to one. This cubic condition substantially enhances the flexibility of representation, thereby offering more nuanced handling of uncertain and imprecise information. As a result, Fermatean fuzzy sets often outperform both IFS's and PFS's in complex decision-making environments and other practical applications.

Parallel to these developments, Smarandache [10] introduced neutrosophic sets (NS's), which represent a significant leap in uncertainty modeling. Unlike fuzzy or intuitionistic fuzzy frameworks, neutrosophic sets are based on three independent components: truth-membership (T), indeterminacy-membership (I), and falsity-membership (F). This independence allows neutrosophic sets to capture not only incomplete and inconsistent information but also indeterminate and contradictory knowledge, making them highly effective in modeling complex and heterogeneous data. Consequently, neutrosophic theory has been successfully applied across diverse domains such as engineering, philosophy, economics, medical sciences, and information systems.

The inherent independence of T, I, and F makes neutrosophic sets particularly powerful in analyzing scenarios where information is simultaneously uncertain, incomplete, and inconsistent. This characteristic broadens the scope of application, ranging from scientific research to decision-support systems, where traditional fuzzy-based approaches are often insufficient.

Recognizing the complementary strengths of Fermatean fuzzy sets and neutrosophic sets, researchers proposed the concept of Fermatean neutrosophic sets (FNS's). This hybrid model inherits the enhanced representational capacity of Fermatean fuzzy logic while incorporating the tri component structure of neutrosophic sets. As a result, FNS's provide a more robust and flexible framework for handling uncertainty, imprecision, and indeterminacy in multi-criteria decision-making and related fields.

For instance, in evaluating a local restaurant, the Fermatean fuzzy framework can model the subjective importance of attributes such as quality, naturalness, freshness, taste, and presentation factors that are inherently ambiguous and prone to human hesitation. By integrating neutrosophic principles, the model is further enriched to capture the indeterminacy and inconsistency often present in customer perceptions and preferences. Thus, the Fermatean neutrosophic approach allows for a more accurate and comprehensive evaluation of consumer decision-making processes.

The choice of local food service as a case study is particularly appropriate due to the inherently uncertain nature of consumer behavior and preference formation. Traditional crisp or even fuzzy models fail to capture this complexity adequately, whereas Fermatean neutrosophic sets provide a refined and powerful tool to represent such uncertainties, thereby yielding more reliable outcomes for both analysis and strategic planning.

From a topological perspective, Vadivel et al. [12] introduced the notion of δ-open sets within neutrosophic topological spaces. Prior to this, in 2008, Ekici [4] introduced e-open sets in general topology. Building on this foundation, Seenivasan et al. [8] in 2014 developed the concept of fuzzy e-open sets and their corresponding fuzzy e-continuity. Later, Vadivel et al. [3] extended these ideas into the realm of intuitionistic fuzzy topological spaces. More recently, Vadivel and collaborators [13–15] have made notable contributions by exploring various types of open sets in Fermatean fuzzy topological spaces, advancing the study of fuzzy and neutrosophic topology.

The motivation for selecting Fermatean Neutrosophic ($\mathfrak{FN}$) e-open sets as the foundation of this work stems from their ability to generalize and extend the classical notions of openness in topological spaces while effectively addressing ambiguity, imprecision, and indeterminacy in real-world applications. Traditional open sets in topology are inadequate when dealing with multi-dimensional uncertainty that arises from human judgment, incomplete information, and inconsistent data. The incorporation of Fermatean Neutrosophic theory allows us to overcome these limitations by combining the expressive power of Fermatean fuzzy sets with the triadic structure of neutrosophic sets, thereby offering a more robust modeling tool.

In particular, Extension of classical openness: The concept of $\mathfrak{FN}e$-open sets generalizes the classical and fuzzy topological notions of openness by embedding Fermatean membership, non-membership, and neutrosophic indeterminacy. This extension enables a more flexible characterization of topological structures under uncertainty. Relevance of e-open mappings: By defining and studying $\mathfrak{FN}e$-open mappings, we extend the mapping theory from crisp and fuzzy topology into the Fermatean neutrosophic environment. These mappings preserve the e-open nature of sets under continuous transformations, which is essential in applications such as decision-making, optimization, and data analysis where transformations of uncertain data occur frequently. Importance of Homomorphisms: Homomorphisms between $\mathfrak{FN}$ topological spaces provide a natural way to establish structural relationships and preserve topological properties across spaces. Their study allows us to investigate equivalences, invariants, and symmetry-like properties under the Fermatean neutrosophic framework, which is vital for building consistent theoretical foundations and for practical applications in areas such as information systems and knowledge representation. Theoretical and Practical Impact: The joint study of $\mathfrak{F}\mathfrak{N}e$-open sets, open mappings, and homomorphisms enriches the existing literature in both fuzzy/neutrosophic topology and applied decision sciences. These concepts are not only theoretically significant but also practically relevant, as they provide new tools to analyze complex systems involving uncertainty and human subjectivity—ranging from decision support systems to social and behavioral sciences.

Therefore, the choice of Fermatean Neutrosophic e-open sets and their associated mappings and homomorphisms is both natural and essential. It represents a step toward developing a comprehensive topological framework that integrates flexibility, expressiveness, and applicability for uncertainty-driven problems.

Advantages of the Proposed Work: Although many research works have been carried out in fuzzy, intuitionistic fuzzy, Pythagorean fuzzy, and neutrosophic topological spaces, the proposed Fermatean Neutrosophic ($\mathfrak{F}\mathfrak{N}$) e-open sets, mappings, and homomorphisms offer several distinct advantages such as higher flexibility in modeling uncertainty, Generalization of existing theories, New structural insights via homomorphisms, Applicability in real-world decision-making and bridging the gap between theory and application.

Research Gap: To the best of our knowledge, no prior investigation has been carried out on the concepts of open, closed and homeomorphism maps with respect to Fermatean neutrosophic e-open sets within the framework of Fermatean fuzzy topological spaces. While various notions of open maps and homeomorphism mappings have been studied extensively in classical topology, fuzzy topology, and intuitionistic fuzzy topology, the specific treatment of these notions under the richer structure of Fermatean neutrosophic sets remains unexplored in the existing literature. This absence highlights a significant research gap, thereby motivating the present study to initiate and advance the theory of Fermatean neutrosophic e-open and e-homeomorphism maps.

The focus of this article is to introduce the concepts of $\mathfrak{F}\mathfrak{N}e$-open and $\mathfrak{F}\mathfrak{N}e$-closed mappings in Fermatean neutrosophic topological spaces. The study is further extended to $\mathfrak{F}\mathfrak{N}e$-homeomorphisms, $\mathfrak{F}\mathfrak{N}e$-C homeomorphisms, and $\mathfrak{F}\mathfrak{N}eT_{\frac{1}{2}}$-spaces in Fermatean neutrosophic topological spaces, and some of their fundamental properties are established. Additionally, we introduce an entropy measure for Fermatean neutrosophic sets and provide an example illustrating its application in decision-making for real-life problems.

2. Preliminaries

Definition 2.1. [9] Let $\mathfrak{C}$ be a universe of discourse. A Fermatean fuzzy set (FFS) F in $\mathfrak{C}$ is defined as

$$F = \{\langle \imath, \alpha_F(\imath), \beta_F(\imath)\rangle : \imath \in \mathfrak{C}\},$$

where

$$\alpha_F(\imath) : \mathfrak{C} \to [0,1] \quad \text{and} \quad \beta_F(\imath) : \mathfrak{C} \to [0,1],$$

satisfying the condition

$$0 \leq (\alpha_F(\imath))^3 + (\beta_F(\imath))^3 \leq 1, \quad \forall \imath \in \mathfrak{C}.$$

Here, $\alpha_F(\imath)$ and $\beta_F(\imath)$ represent the degree of membership and the degree of non-membership of the element x in the set F, respectively. For any FFS F and $\imath \in \mathfrak{C}$, the degree of indeterminacy of $\mathfrak{C}$ with respect to F is defined as

$$\pi_F(\imath) = \sqrt[3]{1 - [(\alpha_F(\imath))^3 + (\beta_F(\imath))^3]}.$$

For simplicity, we denote the FFS $F = \{\langle \imath, \alpha_F(\imath), \beta_F(\imath)\rangle : \imath \in \mathfrak{C}\}$ by $F = (\alpha_F, \beta_F)$.

Definition 2.2. [11] Let $\mathfrak{C}$ be a non-empty set. A Fermatean neutrosophic set (abbreviated as $\mathfrak{F}\mathfrak{N}s$) L is defined as

$$L = \{\langle \imath, \mu_L(\imath), \nu_L(\imath), \sigma_L(\imath)\rangle : \imath \in \mathfrak{C}\},$$

where

$$\mu_L(\imath) : \mathfrak{C} \to [0,1], \quad \nu_L(\imath) : \mathfrak{C} \to [0,1], \quad \sigma_L(\imath) : \mathfrak{C} \to [0,1],$$

denote the degree of membership, the degree of indeterminacy, and the degree of non-membership of each element $\imath \in \mathfrak{C}$ to the set L, respectively, satisfying

$$0 \leq (\mu_L(\imath))^3 + (\sigma_L(\imath))^3 \leq 1, \quad 0 \leq (\nu_L(\imath))^3 \leq 1.$$

Consequently,

$$0 \leq (\mu_L(\imath))^3 + (\nu_L(\imath))^3 + (\sigma_L(\imath))^3 \leq 2, \quad \forall \imath \in \mathfrak{C}.$$

Here, $\mu_L(\imath)$ and $\sigma_L(\imath)$ are dependent components, whereas $\nu_L(\imath)$ is an independent component.

Before proceeding to set operations, we provide the definitions of $1_{\mathfrak{F}\mathfrak{N}}$ and $0_{\mathfrak{F}\mathfrak{N}}$. Possible definitions of $1_{\mathfrak{F}\mathfrak{N}}$ and $0_{\mathfrak{F}\mathfrak{N}}$ neutrosophic sets were discussed in [6]. In this paper, we construct the theory by defining $0_{\mathfrak{F}\mathfrak{N}}$ and $1_{\mathfrak{F}\mathfrak{N}}$ Fermatean neutrosophic sets in a single, unified way:

$$0_{\mathfrak{F}\mathfrak{N}} = \{(\imath, 0, 0, 1) : \imath \in \mathfrak{C}\}, \quad 1_{\mathfrak{F}\mathfrak{N}} = \{(\imath, 1, 1, 0) : \imath \in \mathfrak{C}\}.$$

Next, we present the definitions of union, intersection, and complement, which are necessary for defining a topological space. While several definitions exist in classical neutrosophic spaces

[2], to avoid confusion, we provide here a single method suitable for sets with Fermatean structure. This method differs from that chosen in [6].

Definition 2.3. [11] Let $\mathfrak{C}$ be a non-empty set & the $\mathfrak{FN}\mathfrak{s}$'s L & M in the form $L = \{\langle \imath, \mu_L(\imath), \nu_L(\imath), \sigma_L(\imath)\rangle : \imath \in \mathfrak{C}\}$, $M = \{\langle \imath, \mu_M(\imath), \nu_M(\imath), \sigma_M(\imath)\rangle : \imath \in \mathfrak{C}\}$, then

(i) $0_{\mathfrak{FN}} = \langle \imath, 0, 0, 1\rangle$ and $1_{\mathfrak{FN}} = \langle \imath, 1, 1, 0\rangle$,

(ii) $L \subseteq M$ iff $\mu_L(\imath) \leq \mu_M(\imath)$, $\nu_L(\imath) \leq \nu_M(\imath)$ & $\sigma_L(\imath) \geq \sigma_M(\imath) : \imath \in \mathfrak{C}$,

(iii) $L = M$ iff $L \subseteq M$ and $M \subseteq L$,

(iv) $1_{\mathfrak{FN}} - L = \{\langle \imath, \sigma_L(\imath), 1_{\mathfrak{FN}} - \nu_L(\imath), \mu_L(\imath)\rangle : \imath \in \mathfrak{C}\} = L^c$ or $C(L)$,

(v) $L \cup M = \{\langle \imath, \max(\mu_L(\imath), \mu_M(\imath)), \max(\nu_L(\imath), \nu_M(\imath)), \min(\sigma_L(\imath), \sigma_M(\imath))\rangle : \imath \in \mathfrak{C}\}$,

(vi) $L \cap M = \{\langle \imath, \min(\mu_L(\imath), \mu_M(\imath)), \min(\nu_L(\imath), \nu_M(\imath)), \max(\sigma_L(\imath), \sigma_M(\imath))\rangle : \imath \in \mathfrak{C}\}$.

Definition 2.4. [5] A $\mathfrak{FN}$ topology ($\mathfrak{FN}t$) on a non-empty set $\mathfrak{C}$ is a family $\tau_{\mathfrak{FN}}$ of $\mathfrak{FN}$ subsets of $\mathfrak{C}$ satisfying:

(i) $0_{\mathfrak{FN}}, 1_{\mathfrak{FN}} \in \tau_{\mathfrak{FN}}$,

(ii) $L_1 \cap L_2 \in \tau_{\mathfrak{FN}}$ for any $L_1, L_2 \in \tau_{\mathfrak{FN}}$,

(iii) $\bigcup_{a\in A} L_a \in \tau_{\mathfrak{FN}}$ for any collection $\{L_a : a \in A\} \subseteq \tau_{\mathfrak{FN}}$.

Then, $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is called a $\mathfrak{FN}$ topological space (briefly, $\mathfrak{FN}ts$) on $\mathfrak{C}$. The elements of $\tau_{\mathfrak{FN}}$ are called $\mathfrak{FN}$ open sets (briefly, $\mathfrak{FN}os$) in $\mathfrak{C}$. A $\mathfrak{FN}$ set C is called a $\mathfrak{FN}$ closed set (briefly, $\mathfrak{FN}cs$) if and only if its complement C^c is a $\mathfrak{FN}os$.

Definition 2.5. [5] Let $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ be a $\mathfrak{FN}ts$ on $\mathfrak{C}$ and L be a $\mathfrak{FN}s$ on $\mathfrak{C}$. Then, the $\mathfrak{FN}$ interior of L (briefly, $\mathfrak{FN}int(L)$) and the $\mathfrak{FN}$ closure of L (briefly, $\mathfrak{FN}cl(L)$) are defined as

$$\mathfrak{FN}int(L) = \bigcup\{I : I \subseteq L \text{ and } I \text{ is a } \mathfrak{FN}os \text{ in } \mathfrak{C}\}, \tag{1}$$

$$\mathfrak{FN}cl(L) = \bigcap\{I : L \subseteq I \text{ and } I \text{ is a } \mathfrak{FN}cs \text{ in } \mathfrak{C}\}. \tag{2}$$

Theorem 2.6. [5] Let L be an $\mathfrak{FN}s$ on $\mathfrak{C}$. In this case, the following four properties hold:

(i) $\mathfrak{FN}cl(L)$ is a $\mathfrak{FN}cs$,

(ii) $\mathfrak{FN}cl(1_{\mathfrak{FN}}) = 1_{\mathfrak{FN}}$, $\mathfrak{FN}cl(0_{\mathfrak{FN}}) = 0_{\mathfrak{FN}}$,

(iii) $\mathfrak{FN}int(L)$ is an $\mathfrak{FN}os$.

(iv) $\mathfrak{FN}int(1_{\mathfrak{FN}}) = 1_{\mathfrak{FN}}$, $\mathfrak{FN}int(0_{\mathfrak{FN}}) = 0_{\mathfrak{FN}}$.

Lemma 2.7. [5] For any $\mathfrak{FN}$ set A in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$, we have $C(\mathfrak{FN}int(\mathfrak{G})) = \mathfrak{FN}cl(C(\mathfrak{G}))$ and $C(\mathfrak{FN}cl(\mathfrak{G})) = \mathfrak{FN}int(C(\mathfrak{G}))$. Here $C(\mathfrak{G})$ or $\overline{A}$ denotes complement of A.

3. **Fermatean neutrosophic e-open mapping**

Definition 3.1. Let $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ be an $\mathfrak{FN}ts$ and $\mathfrak{G}$ be an $\mathfrak{FN}s$. Then $\mathfrak{G}$ is said to be an $\mathfrak{FN}$ (i) regular open set ($\mathfrak{FN}ros$) if $\mathfrak{G} = \mathfrak{FN}int(\mathfrak{FN}cl(\mathfrak{G}))$. (ii) regular closed set ($\mathfrak{FN}rcs$) if $\mathfrak{G} = \mathfrak{FN}cl(\mathfrak{FN}int(\mathfrak{G}))$. By Lemma 2.7, we conclude that $\mathfrak{G}$ is an $\mathfrak{FN}ros$ iff $\bar{\mathfrak{G}}$ is an $\mathfrak{FN}rcs$.

Definition 3.2. Let $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ be an $\mathfrak{FN}ts$ and $\mathfrak{G} = \{< \imath, \mu_{\mathfrak{G}}(\imath), \nu_{\mathfrak{G}}(\imath), \sigma_{\mathfrak{G}}(\imath) > | \imath \in \mathfrak{C}\}$ be an $\mathfrak{FN}s$ in $\mathfrak{C}$. Then the δ-interior (closure) of $\mathfrak{G}$ are denoted by $\mathfrak{FN}\delta int(\mathfrak{G})$ and $\mathfrak{FN}\delta cl(\mathfrak{G})$ and are defined as follows. $\mathfrak{FN}\delta int(\mathfrak{G}) = \cup\{G|G$ is an $\mathfrak{FN}ros$ and $G \subseteq A\}$, $\mathfrak{FN}\delta cl(\mathfrak{G}) = \cap\{K|K$ is an $\mathfrak{FN}rcs$ and $\mathfrak{G} \subseteq K\}$.

A set $\mathfrak{G}$ is said to be $\mathfrak{FN}$

(i) δ-open set ($\mathfrak{FN}\delta os$) if $\mathfrak{G} = \mathfrak{FN}\delta int(\mathfrak{G})$,

(ii) δ-pre open set ($\mathfrak{FN}\delta\mathcal{P}os$) if $\mathfrak{G} \subseteq \mathfrak{FN}int(\mathfrak{FN}\delta cl(\mathfrak{G}))$.

(iii) δ-semi open set ($\mathfrak{FN}\delta\mathcal{S}os$) if $\mathfrak{G} \subseteq \mathfrak{FN}cl(\mathfrak{FN}\delta int(\mathfrak{G}))$.

(iv) $\delta\alpha$ open set or $\mathfrak{G}$-open set ($\mathfrak{FN}\delta\alpha os$ or $\mathfrak{FN}aos$) if $\mathfrak{G} \subseteq \mathfrak{FN}int(\mathfrak{FN}cl(\mathfrak{FN}\delta int(\mathfrak{G})))$.

(v) $\delta\beta$ open set or e^*-open set ($\mathfrak{FN}\delta\beta os$ or $\mathfrak{FN}e^*os$) if $\mathfrak{G} \subseteq \mathfrak{FN}cl(\mathfrak{FN}int(\mathfrak{FN}\delta cl(\mathfrak{G})))$.

(vi) e open set ($\mathfrak{FN}eos$) if $\mathfrak{G} \subseteq \mathfrak{FN}cl(\mathfrak{FN}\delta int(\mathfrak{G})) \cup \mathfrak{FN}int(\mathfrak{FN}\delta cl(\mathfrak{G}))$.

(vii) δ (resp. δ-pre, δ-semi, $\delta\alpha$, e and $\delta\beta$) dense if $\mathfrak{FN}\delta cl(\mathfrak{G})$ (resp. $\mathfrak{FN}\delta\mathcal{P}cl(\mathfrak{G})$, $\mathfrak{FN}\delta\mathcal{S}cl(\mathfrak{G})$, $\mathfrak{FN}\delta\alpha cl(\mathfrak{G})$, $\mathfrak{FN}ecl(\mathfrak{G})$ and $\mathfrak{FN}\delta\beta cl(\mathfrak{G})) = 1_{\mathfrak{FN}}$.

The complement of an $\mathfrak{FN}\delta os$ (resp. $\mathfrak{FN}\delta\mathcal{P}os$, $\mathfrak{FN}\delta\mathcal{S}os, \mathfrak{FN}\delta\alpha os$, $\mathfrak{FN}eos$ and $\mathfrak{FN}\delta\beta os$) is called an $\mathfrak{FN}\delta$ (resp. $\mathfrak{FN}\delta\mathcal{P}$, $\mathfrak{FN}\delta\mathcal{S}$, $\mathfrak{FN}\delta\alpha$, $\mathfrak{FN}e$ and $\mathfrak{FN}\delta\beta$) closed set (briefly, $\mathfrak{FN}\delta cs$ (resp. $\mathfrak{FN}\delta\mathcal{P}cs$, $\mathfrak{FN}\delta\mathcal{S}cs, \mathfrak{FN}\delta\alpha cs$, $\mathfrak{FN}ecs$ and $\mathfrak{FN}\delta\beta cs$)) in $\mathfrak{C}$.

The family of all $\mathfrak{FN}\delta os$ (resp. $\mathfrak{FN}\delta cs$, $\mathfrak{FN}\delta\mathcal{P}os$, $\mathfrak{FN}\delta\mathcal{P}cs$, $\mathfrak{FN}\delta\mathcal{S}os$, $\mathfrak{FN}\delta\mathcal{S}cs$, $\mathfrak{FN}\delta\alpha os$, $\mathfrak{FN}\delta\alpha cs$, $\mathfrak{FN}eos$, $\mathfrak{FN}ecs$, $\mathfrak{FN}\delta$ βos and $\mathfrak{FN}\delta\beta cs$) of $\mathfrak{C}$ is denoted by $\mathfrak{FN}\delta OS(\mathfrak{C})$ (resp. $\mathfrak{FN}\delta CS(\mathfrak{C})$, $\mathfrak{FN}\delta\mathcal{P}OS(\mathfrak{C})$, $\mathfrak{FN}\delta\mathcal{P}CS(\mathfrak{C}), \mathfrak{FN}\delta\mathcal{S}OS(\mathfrak{C}), \mathfrak{FN}\delta\mathcal{S}$ $CS(\mathfrak{C})$, $\mathfrak{FN}\delta\alpha OS(\mathfrak{C}), \mathfrak{FN}\delta\alpha CS(\mathfrak{C})$, $\mathfrak{FN}eOS(\mathfrak{C})$, $\mathfrak{FN}eCS(\mathfrak{C})$, $\mathfrak{FN}\delta\beta OS(\mathfrak{C})$ and $\mathfrak{FN}\delta\beta CS(\mathfrak{C})$).

Definition 3.3. Let $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ be an $\mathfrak{FN}ts$ and $\mathfrak{G} = \{< \imath, \mu_{\mathfrak{G}}(\imath), \nu_{\mathfrak{G}}(\imath), \sigma_{\mathfrak{G}}(\imath) > | \imath \in \mathfrak{C}\}$ be an $\mathfrak{FN}s$ in $\mathfrak{C}$. Then the $\mathfrak{FN}\delta$ ($\mathfrak{FN}\delta$-pre, $\mathfrak{FN}\delta$-semi, $\mathfrak{FN}\delta\alpha$, $\mathfrak{FN}e$ and $\mathfrak{FN}\delta\beta$)-interior and the $\mathfrak{FN}\delta$ ($\mathfrak{FN}\delta$-pre, $\mathfrak{FN}\delta$-semi, $\mathfrak{FN}\delta\alpha$, $\mathfrak{FN}e$ and $\mathfrak{FN}\delta\beta$)-closure of $\mathfrak{G}$ are denoted by $\mathfrak{FN}\delta int(\mathfrak{G})$ ($\mathfrak{FN}\delta\mathcal{P}int(\mathfrak{G})$, $\mathfrak{FN}\delta\mathcal{S}int(\mathfrak{G})$, $\mathfrak{FN}\delta\alpha int(\mathfrak{G})$, $\mathfrak{FN}eint(\mathfrak{G})$ and $\mathfrak{FN}\delta\beta int(\mathfrak{G})$) and the $\mathfrak{FN}\delta cl(\mathfrak{G})$ ($\mathfrak{FN}\delta\mathcal{P}cl(\mathfrak{G})$, $\mathfrak{FN}\delta\mathcal{S}cl(\mathfrak{G}), \mathfrak{FN}\delta\alpha cl(\mathfrak{G})$, $\mathfrak{FN}ecl(\mathfrak{G})$ and $\mathfrak{FN}\delta\beta cl(\mathfrak{G})$) and are defined as follows:

$\mathfrak{FN}\delta int(\mathfrak{G})$ ($\mathfrak{FN}\delta\mathcal{P}int(\mathfrak{G})$, $\mathfrak{FN}\delta\mathcal{S}int(\mathfrak{G}), \mathfrak{FN}\delta\alpha int(\mathfrak{G})$, $\mathfrak{FN}eint(\mathfrak{G})$ and $\mathfrak{FN}\delta\beta int(\mathfrak{G})$) $= \cup\{G|G$ in a $\mathfrak{FN}\delta os$ ($\mathfrak{FN}\delta\mathcal{P}os$, $\mathfrak{FN}\delta\mathcal{S}os, \mathfrak{FN}\delta\alpha os, \mathfrak{FN}eos$, and $\mathfrak{FN}\delta\beta os$) and $G \subseteq \mathfrak{G}\}$ and $\mathfrak{FN}\delta cl(\mathfrak{G})$ ($\mathfrak{FN}\delta\mathcal{P}cl(\mathfrak{G}), \mathfrak{FN}\delta\mathcal{S}cl(\mathfrak{G}), \mathfrak{FN}\delta\alpha cl(\mathfrak{G})$, $\mathfrak{FN}ecl(\mathfrak{G})$ and $\mathfrak{FN}\delta\beta cl(\mathfrak{G})$) $= \cap\{K|K$ is an $\mathfrak{FN}\delta cs$ ($\mathfrak{FN}\delta\mathcal{P}cs, \mathfrak{FN}\delta\mathcal{S}cs$, $\mathfrak{FN}\delta\alpha cs, \mathfrak{FN}ecs$ $\mathfrak{FN}\delta\beta cs$) and $\mathfrak{G} \subseteq K\}$.

Definition 3.4. A map $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is called $\mathfrak{FN}$

(i) continuous (briefly, $\mathfrak{FN}Cts$) if the inverse image of every $\mathfrak{FN}os$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$,

(ii) δ (resp. $\delta\mathcal{S}$, $\delta\mathcal{P}$, $\delta\alpha$, e and $\delta\beta$) -continuous (briefly, $\mathfrak{FN}\delta Cts$ (resp. $\delta\mathcal{S}Cts$, $\delta\mathcal{P}Cts$, $\delta\alpha Cts$, $eCts$ and $\delta\beta Cts$)) if the inverse image of every $\mathfrak{FN}os$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}\delta os$ (resp. $\delta\mathcal{S}os$, $\delta\mathcal{P}os$, $\delta\alpha os$, eos and $\delta\beta os$) in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$.

Definition 3.5. A map $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is called $\mathfrak{FN}$

(i) open (briefly, $\mathfrak{FN}O$) if the image of every $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is a $\mathfrak{FN}os$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$,

(ii) δ (resp. $\delta\mathcal{S}$, $\delta\mathcal{P}$, $\delta\alpha$, e and $\delta\beta$)-open (briefly, $\mathfrak{FN}\delta O$ (resp. $\delta\mathcal{S}O$, $\delta\mathcal{P}O$, $\delta\alpha O$, eO and $\delta\beta O$)) if the image of every $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is a $\mathfrak{FN}\delta os$ (resp. $\delta\mathcal{S}os$, $\delta\mathcal{P}os$, $\delta\alpha os$, eos and $\delta\beta os$) in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$.

Proposition 3.6. The statements are hold but the converse does not true. Every (i) $\mathfrak{FN}\delta O$ (resp. (ii) $\mathfrak{FN}\delta O$, (iii) $\mathfrak{FN}\delta O$, (iv) $\mathfrak{FN}\delta\mathcal{P}O$, (v) $\mathfrak{FN}\delta\mathcal{S}O$, (vi) $\mathfrak{FN}\delta\mathcal{P}O$, (vii) $\mathfrak{FN}\delta\mathcal{S}O$, (viii) $\mathfrak{FN}\delta\alpha O$ and (ix) $\mathfrak{FN}\delta\alpha O$) mapping is a $\mathfrak{FN}O$ (resp. $\mathfrak{FN}\delta\mathcal{S}O$, $\mathfrak{FN}\delta\mathcal{P}O$, $\mathfrak{FN}eO$, $\mathfrak{FN}eO$, $\mathfrak{FN}\delta\beta O$, $\mathfrak{FN}\delta\beta O$, $\mathfrak{FN}\delta\mathcal{S}O$ and $\mathfrak{FN}\delta\mathcal{P}O$) mapping.

Proof. We prove only (iv) and (v), the others are similar.

(iv) Let ζ be a $\mathfrak{FN}os$ in $\mathfrak{C}$. Since f is a $\mathfrak{FN}\delta\mathcal{P}O$ mapping, it follows that $f(\zeta)$ is a $\mathfrak{FN}\delta\mathcal{P}os$ in $\mathfrak{D}$. Moreover, since every $\mathfrak{FN}\delta\mathcal{P}os$ is a $\mathfrak{FN}eos$, we deduce that $f(\zeta)$ is a $\mathfrak{FN}eos$ in $\mathfrak{D}$. Hence, f is a $\mathfrak{FN}eO$ mapping.

(v) Let ζ be a $\mathfrak{FN}os$ in $\mathfrak{C}$. Since f is a $\mathfrak{FN}\delta\mathcal{S}O$ mapping, it follows that $f(\zeta)$ is a $\mathfrak{FN}\delta\mathcal{S}os$ in $\mathfrak{D}$. Moreover, since every $\mathfrak{FN}\delta\mathcal{S}os$ is a $\mathfrak{FN}eos$, we deduce that $f(\zeta)$ is a $\mathfrak{FN}eos$ in $\mathfrak{D}$. Hence, f is a $\mathfrak{FN}eO$ mapping. $\square$

Example 3.7. Let $\mathfrak{C} = \mathfrak{D} = \{\imath, \eta\}$ and the $\mathfrak{FN}s$'s $\mathfrak{G}_1$, $\mathfrak{G}_2$ and $\mathfrak{G}_3$ are defined as
$\mu_{\mathfrak{G}_1}(\imath) = 0.8,\ \nu_{\mathfrak{G}_1}(\imath) = 0.8,\ \sigma_{\mathfrak{G}_1}(\imath) = 0.1,$
$\mu_{\mathfrak{G}_1}(\eta) = 0.9,\ \nu_{\mathfrak{G}_1}(\eta) = 0.8,\ \sigma_{\mathfrak{G}_1}(\eta) = 0.2;$
$\mu_{\mathfrak{G}_2}(\imath) = 0.6,\ \nu_{\mathfrak{G}_2}(\imath) = 0.7,\ \sigma_{\mathfrak{G}_2}(\imath) = 0.2,$
$\mu_{\mathfrak{G}_2}(\eta) = 0.5,\ \nu_{\mathfrak{G}_2}(\eta) = 0.7,\ \sigma_{\mathfrak{G}_2}(\eta) = 0.6;$
$\mu_{\mathfrak{G}_3}(\imath) = 0.1,\ \nu_{\mathfrak{G}_3}(\imath) = 0.2,\ \sigma_{\mathfrak{G}_3}(\imath) = 0.8,$
$\mu_{\mathfrak{G}_3}(\eta) = 0.2,\ \nu_{\mathfrak{G}_3}(\eta) = 0.2,\ \sigma_{\mathfrak{G}_3}(\eta) = 0.9.$

Let $\tau_{\mathfrak{FN}} = \sigma_{\mathfrak{FN}} = \{0_{\mathfrak{FN}},\ 1_{\mathfrak{FN}},\ \mathfrak{G}_1,\ \mathfrak{G}_2,\ \mathfrak{G}_3\}$ be a $\mathfrak{FN}ts$ on $\mathfrak{C}$ and let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be an identity mapping, then f is $\mathfrak{FN}O$ (resp. $\mathfrak{FN}\delta\mathcal{P}O$, $\mathfrak{FN}\delta\mathcal{P}O$, $\mathfrak{FN}eO$ and $\mathfrak{FN}\delta\beta O$) but not $\mathfrak{FN}\delta O$ (resp. $\mathfrak{FN}\delta\alpha O$, $\mathfrak{FN}\delta O$, $\mathfrak{FN}\delta\mathcal{S}O$ and $\mathfrak{FN}\delta\mathcal{S}O$), the set $\mathfrak{G}_2$ is a $\mathfrak{FN}os$ in $\mathfrak{C}$ but $f(\mathfrak{G}_2) = \mathfrak{G}_2$ is not $\mathfrak{FN}\delta os$ (resp. $\mathfrak{FN}\delta\alpha os$, $\mathfrak{FN}\delta os$, $\mathfrak{FN}\delta\mathcal{S}os$ and $\mathfrak{FN}\delta\mathcal{S}os$) in $\mathfrak{D}$.

Example 3.8. Let $\mathfrak{C} = \mathfrak{D} = \{\imath, \eta\}$ and the $\mathfrak{FN}s$'s $\mathfrak{G}_1$, $\mathfrak{G}_2$ and $\mathfrak{G}_3$ are defined as
$\mu_{\mathfrak{G}_1}(\imath) = 0.2,\ \nu_{\mathfrak{G}_1}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_1}(\imath) = 0.8,$
$\mu_{\mathfrak{G}_1}(\eta) = 0.3,\ \nu_{\mathfrak{G}_1}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_1}(\eta) = 0.7;$
$\mu_{\mathfrak{G}_2}(\imath) = 0.1,\ \nu_{\mathfrak{G}_2}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_2}(\imath) = 0.9,$
$\mu_{\mathfrak{G}_2}(\eta) = 0.1,\ \nu_{\mathfrak{G}_2}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_2}(\eta) = 0.9;$
$\mu_{\mathfrak{G}_3}(\imath) = 0.2,\ \nu_{\mathfrak{G}_3}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_3}(\imath) = 0.8,$
$\mu_{\mathfrak{G}_3}(\eta) = 0.4,\ \nu_{\mathfrak{G}_3}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_3}(\eta) = 0.6.$

Let $\tau_{\mathfrak{FN}} = \sigma_{\mathfrak{FN}} = \{0_{\mathfrak{FN}},\ 1_{\mathfrak{FN}},\ \mathfrak{G}_1,\ \mathfrak{G}_2,\ \mathfrak{G}_3\}$ be a $\mathfrak{FN}ts$ on $\mathfrak{C}$ and let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be an identity mapping, then f is $\mathfrak{FN}\delta SO$ but not $\mathfrak{FN}\delta O$, the set $\mathfrak{G}_1$ is a $\mathfrak{FN}os$ in $\mathfrak{C}$ but $f(\mathfrak{G}_1) = \mathfrak{G}_1$ is not $\mathfrak{FN}\delta os$ in $\mathfrak{D}$.

Example 3.9. Let $\mathfrak{C} = \mathfrak{D} = \{\imath, \eta, c, d\}$ and the $\mathfrak{FN}s$'s $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$, $\mathfrak{G}_4$ and $\mathfrak{G}_5$ are defined as
$\mu_{\mathfrak{G}_1}(\imath) = 1,\ \nu_{\mathfrak{G}_1}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_1}(\imath) = 0,$
$\mu_{\mathfrak{G}_1}(\eta) = 1,\ \nu_{\mathfrak{G}_1}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_1}(\eta) = 0,$
$\mu_{\mathfrak{G}_1}(c) = 1,\ \nu_{\mathfrak{G}_1}(c) = 0.5,\ \sigma_{\mathfrak{G}_1}(c) = 0,$
$\mu_{\mathfrak{G}_1}(d) = 1,\ \nu_{\mathfrak{G}_1}(d) = 0.5,\ \sigma_{\mathfrak{G}_1}(d) = 0;$
$\mu_{\mathfrak{G}_2}(\imath) = 1,\ \nu_{\mathfrak{G}_2}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_2}(\imath) = 0,$
$\mu_{\mathfrak{G}_2}(\eta) = 0,\ \nu_{\mathfrak{G}_2}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_2}(\eta) = 1,$
$\mu_{\mathfrak{G}_2}(c) = 0.2,\ \nu_{\mathfrak{G}_2}(c) = 0.5,\ \sigma_{\mathfrak{G}_2}(c) = 0.7,$
$\mu_{\mathfrak{G}_2}(d) = 0,\ \nu_{\mathfrak{G}_2}(d) = 0.5,\ \sigma_{\mathfrak{G}_2}(d) = 1;$
$\mu_{\mathfrak{G}_3}(\imath) = 0,\ \nu_{\mathfrak{G}_3}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_3}(\imath) = 1,$
$\mu_{\mathfrak{G}_3}(\eta) = 1,\ \nu_{\mathfrak{G}_3}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_3}(\eta) = 0,$
$\mu_{\mathfrak{G}_3}(c) = 0,\ \nu_{\mathfrak{G}_3}(c) = 0.5,\ \sigma_{\mathfrak{G}_3}(c) = 0,$
$\mu_{\mathfrak{G}_3}(d) = 0,\ \nu_{\mathfrak{G}_3}(d) = 0.5,\ \sigma_{\mathfrak{G}_3}(d) = 1;$
$\mu_{\mathfrak{G}_4}(\imath) = 1,\ \nu_{\mathfrak{G}_4}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_4}(\imath) = 0,$
$\mu_{\mathfrak{G}_4}(\eta) = 1,\ \nu_{\mathfrak{G}_4}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_4}(\eta) = 0,$
$\mu_{\mathfrak{G}_4}(c) = 0.2,\ \nu_{\mathfrak{G}_4}(c) = 0.5,\ \sigma_{\mathfrak{G}_4}(c) = 0.7,$
$\mu_{\mathfrak{G}_4}(d) = 0,\ \nu_{\mathfrak{G}_4}(d) = 0.5,\ \sigma_{\mathfrak{G}_4}(d) = 0.1;$
$\mu_{\mathfrak{G}_5}(\imath) = 0,\ \nu_{\mathfrak{G}_5}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_5}(\imath) = 1,$
$\mu_{\mathfrak{G}_5}(\eta) = 0,\ \nu_{\mathfrak{G}_5}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_5}(\eta) = 1,$
$\mu_{\mathfrak{G}_5}(c) = 0,\ \nu_{\mathfrak{G}_5}(c) = 0.5,\ \sigma_{\mathfrak{G}_5}(c) = 1,$
$\mu_{\mathfrak{G}_5}(d) = 0,\ \nu_{\mathfrak{G}_5}(d) = 0.5,\ \sigma_{\mathfrak{G}_5}(d) = 1.$
Let $\tau_{\mathfrak{FN}} = \sigma_{\mathfrak{FN}} = \{0_{\mathfrak{FN}},\ 1_{\mathfrak{FN}},\ \mathfrak{G}_1,\ \mathfrak{G}_2,\ \mathfrak{G}_3,\ \mathfrak{G}_4,\ \mathfrak{G}_5\}$ be a $\mathfrak{FN}ts$ on $\mathfrak{C}$ and let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be an identity mapping, then f is $\mathfrak{FN}\delta SO$ (resp. $\mathfrak{FN}eO$ and $\mathfrak{FN}\delta\beta O$) but not $\mathfrak{FN}\delta\alpha O$ (resp. $\mathfrak{FN}\delta\mathcal{P}O$ and $\mathfrak{FN}\delta\mathcal{P}O$), the set $\mathfrak{G}_2$ is a $\mathfrak{FN}os$ in $\mathfrak{C}$ but $f(\mathfrak{G}_2) = \mathfrak{G}_2$ is not $\mathfrak{FN}\delta\alpha os$ (resp. $\mathfrak{FN}\delta\mathcal{P}os$ and $\mathfrak{FN}\delta\mathcal{P}os$) in $\mathfrak{D}$.

Diagram 3.10. From the above Proposition 3.6, and the following Example, the implications are hold.

Theorem 3.11. A mapping $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is $\mathfrak{FN}eO$ iff for every $\mathfrak{FN}s$ ζ of $(\mathfrak{C}, \tau_{\mathfrak{FN}})$, $f(int(\zeta)) \subseteq \mathfrak{FN}eint(f(\zeta))$.

$$\begin{tikzcd}
& \mathfrak{FN}O & \\
& \mathfrak{FN}\delta O \arrow[u] \arrow[ddl] \arrow[ddr] & \\
& \mathfrak{FN}\delta\alpha O \arrow[dl] \arrow[dr] & \\
\mathfrak{FN}\delta\mathcal{S}O \arrow[r] \arrow[dr] & \mathfrak{FN}eO & \mathfrak{FN}\delta\mathcal{P}O \arrow[l] \arrow[dl] \\
& \mathfrak{FN}\delta\beta O &
\end{tikzcd}$$

Note: $\mathfrak{G} \rightarrow B$ denotes $\mathfrak{G}$ implies B, but not conversely.

Proof. Necessity: Let f be a $\mathfrak{FN}eO$ mapping and let ζ be a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Since $\text{int}(\zeta) \subseteq \zeta$, it follows that

$$f(\text{int}(\zeta)) \subseteq f(\zeta).$$

As f is a $\mathfrak{FN}eO$ mapping, $f(\text{int}(\zeta))$ is a $\mathfrak{FN}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ and satisfies

$$f(\text{int}(\zeta)) \subseteq f(\zeta).$$

Therefore,

$$f(\text{int}(\zeta)) \subseteq \mathfrak{FN}e\,\text{int}(f(\zeta)).$$

Sufficiency: Assume ζ is a $\mathfrak{FN}os$ of $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then,

$$f(\zeta) = f(\text{int}(\zeta)) \subseteq \mathfrak{FN}e\,\text{int}(f(\zeta)).$$

But

$$\mathfrak{FN}e\,\text{int}(f(\zeta)) \subseteq f(\zeta).$$

Hence,

$$f(\zeta) = \mathfrak{FN}e\,\text{int}(f(\zeta)),$$

which implies that $f(\zeta)$ is a $\mathfrak{FN}eos$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Therefore, f is a $\mathfrak{FN}eO$ mapping. $\square$

Theorem 3.12. If $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \rightarrow (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}eO$ mapping, then

$$\text{int}\left(f^{-1}(\zeta)\right) \subseteq f^{-1}\big(\mathfrak{FN}e\,\text{int}(\zeta)\big)$$

for every $\mathfrak{FN}s$ ζ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$.

Proof. Let ζ be a $\mathfrak{F}\mathfrak{N}s$ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Then $\mathrm{int}(f^{-1}(\zeta))$ is a $\mathfrak{F}\mathfrak{N}os$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Since f is a $\mathfrak{F}\mathfrak{N}eO$ mapping, $f\big(\mathrm{int}(f^{-1}(\zeta))\big)$ is a $\mathfrak{F}\mathfrak{N}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$, and hence

$$f\big(\mathrm{int}(f^{-1}(\zeta))\big) \subseteq \mathfrak{F}\mathfrak{N}e\,\mathrm{int}\left(f(f^{-1}(\zeta))\right) \subseteq \mathfrak{F}\mathfrak{N}e\,\mathrm{int}(\zeta).$$

Thus,

$$\mathrm{int}(f^{-1}(\zeta)) \subseteq f^{-1}\big(\mathfrak{F}\mathfrak{N}e\,\mathrm{int}(\zeta)\big).$$

□

Theorem 3.13. A mapping $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ is a $\mathfrak{F}\mathfrak{N}eO$ mapping if and only if, for each $\mathfrak{F}\mathfrak{N}s$ μ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ and for each $\mathfrak{F}\mathfrak{N}cs$ ζ of $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$ containing $f^{-1}(\mu)$, there exists a $\mathfrak{F}\mathfrak{N}ecs$ γ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ such that

$$\mu \subseteq \gamma \quad \text{and} \quad f^{-1}(\gamma) \subseteq \zeta.$$

Proof. Necessity: Assume f is a $\mathfrak{F}\mathfrak{N}eO$ mapping. Let μ be a $\mathfrak{F}\mathfrak{N}cs$ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ and let ζ be a $\mathfrak{F}\mathfrak{N}cs$ of $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$ such that

$$f^{-1}(\mu) \subseteq \zeta.$$

Define

$$\gamma = \left(f^{-1}(\zeta^c)\right)^c.$$

Then γ is a $\mathfrak{F}\mathfrak{N}ecs$ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ such that

$$f^{-1}(\gamma) \subseteq \zeta.$$

Sufficiency: Assume δ is a $\mathfrak{F}\mathfrak{N}os$ of $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Then

$$f^{-1}\big((f(\delta))^c\big) \subseteq \delta^c,$$

and δ^c is a $\mathfrak{F}\mathfrak{N}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. By hypothesis, there exists a $\mathfrak{F}\mathfrak{N}ecs$ γ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ such that

$$(f(\delta))^c \subseteq \gamma \quad \text{and} \quad f^{-1}(\gamma) \subseteq \delta^c.$$

Therefore,

$$\delta \subseteq (f^{-1}(\gamma))^c.$$

Hence,

$$\gamma^c \subseteq f(\delta) \subseteq f\big((f^{-1}(\gamma))^c\big) \subseteq \gamma^c,$$

which implies

$$f(\delta) = \gamma^c.$$

Since γ^c is a $\mathfrak{F}\mathfrak{N}eos$ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$, it follows that $f(\delta)$ is a $\mathfrak{F}\mathfrak{N}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Thus, f is a $\mathfrak{F}\mathfrak{N}eO$ mapping. □

Theorem 3.14. A mapping $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ is a $\mathfrak{F}\mathfrak{N}eO$ mapping iff

$$f^{-1}\big(\mathfrak{F}\mathfrak{N}e\,\mathrm{cl}(\zeta)\big) \subseteq \mathrm{cl}\left(f^{-1}(\zeta)\right)$$

for every $\mathfrak{F}\mathfrak{N}s$ ζ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$.

Proof. Necessity: Assume f is a $\mathfrak{F}\mathfrak{N}eO$ mapping. For any $\mathfrak{F}\mathfrak{N}s$ ζ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$, we have

$$f^{-1}(\zeta) \subseteq \mathrm{cl}\left(f^{-1}(\zeta)\right).$$

Therefore, by Theorem 3.13, there exists a $\mathfrak{F}\mathfrak{N}ecs$ μ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ such that

$$\zeta \subseteq \mu \quad \text{and} \quad f^{-1}(\mu) \subseteq \mathrm{cl}\left(f^{-1}(\zeta)\right).$$

Hence, we obtain

$$f^{-1}\big(\mathfrak{F}\mathfrak{N}e\,\mathrm{cl}(\zeta)\big) \subseteq f^{-1}(\mu) \subseteq \mathrm{cl}\left(f^{-1}(\zeta)\right).$$

Sufficiency: Assume ζ is a $\mathfrak{F}\mathfrak{N}s$ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ and μ is a $\mathfrak{F}\mathfrak{N}cs$ of $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$ containing $f^{-1}(\zeta)$. Put $\alpha = \mathrm{cl}(\zeta)$. Then $\zeta \subseteq \alpha$, and α is a $\mathfrak{F}\mathfrak{N}ec$. Moreover,

$$f^{-1}(\alpha) \subseteq \mathrm{cl}\left(f^{-1}(\zeta)\right) \subseteq \mu.$$

Then, by Theorem 3.13, f is a $\mathfrak{F}\mathfrak{N}eO$ mapping. □

Theorem 3.15. If $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ and $g : (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ be two neutrosophic mappings and $g \circ f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ is $\mathfrak{F}\mathfrak{N}eO$. If $g : (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ is $\mathfrak{F}\mathfrak{N}eIrr$ then $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ is $\mathfrak{F}\mathfrak{N}eO$ mapping.

Proof. Let β be a $\mathfrak{F}\mathfrak{N}os$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Then $g \circ f(\beta)$ is a $\mathfrak{F}\mathfrak{N}eos$ of $(\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ because $g \circ f$ is a $\mathfrak{F}\mathfrak{N}eO$ mapping. Since g is $\mathfrak{F}\mathfrak{N}eIrr$ and $g \circ f(\beta)$ is a $\mathfrak{F}\mathfrak{N}eos$ of $(\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$, we have

$$g^{-1}\big(g \circ f(\beta)\big) = f(\beta),$$

which is a $\mathfrak{F}\mathfrak{N}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Hence, f is a $\mathfrak{F}\mathfrak{N}eO$ mapping. □

Theorem 3.16. If $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ is a $\mathfrak{F}\mathfrak{N}O$ mapping and $g : (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ is a $\mathfrak{F}\mathfrak{N}eO$ mapping, then

$$g \circ f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$$

is a $\mathfrak{F}\mathfrak{N}eO$ mapping.

Proof. Let β be a $\mathfrak{F}\mathfrak{N}os$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Then $f(\beta)$ is a $\mathfrak{F}\mathfrak{N}os$ of $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ because f is a $\mathfrak{F}\mathfrak{N}O$ mapping. Since g is a $\mathfrak{F}\mathfrak{N}eO$ mapping, we have

$$g(f(\beta)) = (g \circ f)(\beta),$$

which is a $\mathfrak{F}\mathfrak{N}eos$ of $(\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$. Hence, $g \circ f$ is a $\mathfrak{F}\mathfrak{N}eO$ mapping. □

4. Fermatean neutrosophic e-closed mapping

Definition 4.1. A map $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is called $\mathfrak{FN}$

(i) closed (briefly, $\mathfrak{FN}C$) if the image of every $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is a $\mathfrak{FN}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$,

(ii) δ (resp. $\delta\mathcal{S}$, $\delta\mathcal{P}$, $\delta\alpha$, e and $\delta\beta$)-closed (briefly, $\mathfrak{FN}\delta C$ (resp. $\delta\mathcal{S}C$, $\delta\mathcal{P}C$, $\delta\alpha C$, eC and $\delta\beta C$)) if the image of every $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is a $\mathfrak{FN}\delta cs$ (resp. $\delta\mathcal{S}cs$, $\delta\mathcal{P}cs$, $\delta\alpha cs$, ecs and $\delta\beta cs$) in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$.

Proposition 4.2. The statements are hold but the converse does not true. Every (i) $\mathfrak{FN}\delta C$ (resp. (ii) $\mathfrak{FN}\delta C$, (iii) $\mathfrak{FN}\delta C$, (iv) $\mathfrak{FN}\delta\mathcal{P}C$, (v) $\mathfrak{FN}\delta\mathcal{S}C$, (vi) $\mathfrak{FN}\delta\mathcal{P}C$, (vii) $\mathfrak{FN}\delta\mathcal{S}C$, (viii) $\mathfrak{FN}\delta\alpha C$ and (ix) $\mathfrak{FN}\delta\alpha C$) mapping is a $\mathfrak{FN}C$ (resp. $\mathfrak{FN}\delta\mathcal{S}C$, $\mathfrak{FN}\delta\mathcal{P}C$, $\mathfrak{FN}eC$, $\mathfrak{FN}eC$, $\mathfrak{FN}\delta\beta C$, $\mathfrak{FN}\delta\beta C$, $\mathfrak{FN}\delta\mathcal{S}C$ and $\mathfrak{FN}\delta\mathcal{P}C$) mapping.

Proof. We prove only (iv) and (v), since the others follow similarly.

(iv) Let ζ be a $\mathfrak{FN}cs$ in $\mathfrak{C}$. Since f is $\mathfrak{FN}\delta\mathcal{P}C$ mapping, $f(\zeta)$ is a $\mathfrak{FN}\delta\mathcal{P}cs$ in $\mathfrak{D}$. Since every $\mathfrak{FN}\delta\mathcal{P}cs$ is a $\mathfrak{FN}ecs$, $f(\zeta)$ is a $\mathfrak{FN}ecs$ in $\mathfrak{D}$. Hence f is a $\mathfrak{FN}eC$ mapping.

(v) Let ζ be a $\mathfrak{FN}cs$ in $\mathfrak{C}$. Since f is $\mathfrak{FN}\delta\mathcal{S}C$ mapping, $f(\zeta)$ is a $\mathfrak{FN}\delta\mathcal{S}cs$ in $\mathfrak{D}$. Since every $\mathfrak{FN}\delta\mathcal{S}cs$ is a $\mathfrak{FN}ecs$, $f(\zeta)$ is a $\mathfrak{FN}ecs$ in $\mathfrak{D}$. Hence f is a $\mathfrak{FN}eC$ mapping. □

Example 4.3. In Example 3.7, f is $\mathfrak{FN}C$ (resp. $\mathfrak{FN}\delta\mathcal{P}C$, $\mathfrak{FN}\delta\mathcal{P}C$, $\mathfrak{FN}eC$ and $\mathfrak{FN}\delta\beta C$) but not $\mathfrak{FN}\delta C$ (resp. $\mathfrak{FN}\delta\alpha C$, $\mathfrak{FN}\delta C$, $\mathfrak{FN}\delta\mathcal{S}C$ and $\mathfrak{FN}\delta\mathcal{S}C$), the set $\mathfrak{G}_2^c$ is a $\mathfrak{FN}cs$ in $\mathfrak{C}$ but $f(\mathfrak{G}_2^c) = \mathfrak{G}_2^c$ is not $\mathfrak{FN}\delta cs$ (resp. $\mathfrak{FN}\delta\alpha cs$, $\mathfrak{FN}\delta cs$, $\mathfrak{FN}\delta\mathcal{S}cs$ and $\mathfrak{FN}\delta\mathcal{S}cs$) in $\mathfrak{D}$.

Example 4.4. In Example 3.8, f is $\mathfrak{FN}\delta\mathcal{S}C$ but not $\mathfrak{FN}\delta C$, the set $\mathfrak{G}_1^c$ is a $\mathfrak{FN}cs$ in $\mathfrak{C}$ but $f(\mathfrak{G}_1^c) = \mathfrak{G}_1^c$ is not $\mathfrak{FN}\delta cs$ in $\mathfrak{D}$.

Example 4.5. In Example 3.9, f is $\mathfrak{FN}\delta\mathcal{S}C$ (resp. $\mathfrak{FN}eC$ and $\mathfrak{FN}\delta\beta C$) but not $\mathfrak{FN}\delta\alpha C$ (resp. $\mathfrak{FN}\delta\mathcal{P}C$ and $\mathfrak{FN}\delta\mathcal{P}C$), the set $\mathfrak{G}_2^c$ is a $\mathfrak{FN}cs$ in $\mathfrak{C}$ but $f(\mathfrak{G}_2^c) = \mathfrak{G}_2^c$ is not $\mathfrak{FN}\delta\alpha cs$ (resp. $\mathfrak{FN}\delta\mathcal{P}cs$ and $\mathfrak{FN}\delta\mathcal{P}cs$) in $\mathfrak{D}$.

Theorem 4.6. A mapping $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}eC$ mapping iff, for each $\mathfrak{FN}s$ μ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ and for each $\mathfrak{FN}os$ ζ of $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ containing $f^{-1}(\mu)$, there exists a $\mathfrak{FN}eos$ γ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ such that

$$\mu \subseteq \gamma \quad \text{and} \quad f^{-1}(\gamma) \subseteq \zeta.$$

Proof. Necessity: Assume f is a $\mathfrak{FN}eC$ mapping. Let μ be a $\mathfrak{FN}cs$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ and let ζ be a $\mathfrak{FN}os$ of $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ such that

$$f^{-1}(\mu) \subseteq \zeta.$$

Then

$$\gamma = \mathfrak{D} - f^{-1}(\zeta^c)$$

is a $\mathfrak{FN}eos$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ such that

$$f^{-1}(\gamma) \subseteq \zeta.$$

Sufficiency: Assume δ is a $\mathfrak{FN}cs$ of $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then $(f(\delta))^c$ is a $\mathfrak{FN}s$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ and δ^c is a $\mathfrak{FN}os$ of $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ such that

$$f^{-1}\big((f(\delta))^c\big) \subseteq \delta^c.$$

By hypothesis, there exists a $\mathfrak{FN}eos$ γ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ such that

$$(f(\delta))^c \subseteq \gamma \quad \text{and} \quad f^{-1}(\gamma) \subseteq \delta^c.$$

Therefore,

$$\delta \subseteq (f^{-1}(\gamma))^c.$$

Hence,

$$\gamma^c \subseteq f(\delta) \subseteq f\big((f^{-1}(\gamma))^c\big) \subseteq \gamma^c,$$

which implies

$$f(\delta) = \gamma^c.$$

Since γ^c is a $\mathfrak{FN}ecs$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$, it follows that $f(\delta)$ is a $\mathfrak{FN}ec$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Thus, f is a $\mathfrak{FN}eC$ mapping. □

Theorem 4.7. If $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is $\mathfrak{FN}C$ and $g : (\mathfrak{D}, \sigma_{\mathfrak{FN}}) \to (\mathfrak{Z}, \rho_{\mathfrak{FN}})$ is $\mathfrak{FN}eC$. Then $g \circ f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{Z}, \rho_{\mathfrak{FN}})$ is $\mathfrak{FN}eC$.

Proof. Let β be a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then $f(\beta)$ is a $\mathfrak{FN}cs$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ because f is a $\mathfrak{FN}C$ mapping. Now,

$$(g \circ f)(\beta) = g(f(\beta))$$

is a $\mathfrak{FN}ecs$ in $(\mathfrak{Z}, \rho_{\mathfrak{FN}})$ because g is a $\mathfrak{FN}eC$ mapping. Thus, $g \circ f$ is a $\mathfrak{FN}eC$ mapping. □

Theorem 4.8. If $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is $\mathfrak{FN}eC$ map, then $\mathfrak{FN}ecl(f(\gamma)) \subsetneq f(cl(\gamma))$.

Theorem 4.9. Let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ and $g : (\mathfrak{D}, \sigma_{\mathfrak{FN}}) \to (\mathfrak{Z}, \rho_{\mathfrak{FN}})$ are $\mathfrak{FN}eC$ mappings. If every $\mathfrak{FN}ecs$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is $\mathfrak{FN}c$ then, $g \circ f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{Z}, \rho_{\mathfrak{FN}})$ is $\mathfrak{FN}eC$.

Proof. Let β be a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then $f(\beta)$ is $\mathfrak{FN}ecs$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ because f is $\mathfrak{FN}eC$ mapping. By hypothesis $f(\beta)$ is $\mathfrak{FN}cs$ of $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$.Now $g(f(\beta)) = (g \circ f)(\beta)$ is $\mathfrak{FN}ecs$ in $(\mathfrak{Z}, \rho_{\mathfrak{FN}})$ because g is $\mathfrak{FN}eC$ mapping. Thus $g \circ f$ is $\mathfrak{FN}eC$ mapping. □

Theorem 4.10. Let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be a bijective mapping. Then the following statements are equivalent:

(i) f is a $\mathfrak{FN}eO$ mapping.

(ii) f is a $\mathfrak{FN}eC$ mapping.

(iii) f^{-1} is $\mathfrak{FN}eCts$ mapping.

Proof. **(i) $\Rightarrow$ (ii):** Assume that f is a $\mathfrak{FN}eO$ mapping. By definition, if β is a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$, then $f(\beta)$ is a $\mathfrak{FN}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Now, if β is a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$, then $\mathfrak{C} - \beta$ is a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. By assumption, $f(\mathfrak{C} - \beta)$ is a $\mathfrak{FN}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Hence,

$$\mathfrak{D} - f(\mathfrak{C} - \beta)$$

is a $\mathfrak{FN}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Therefore, f is a $\mathfrak{FN}eC$ mapping.

(ii) $\Rightarrow$ (iii): Let β be a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. By (ii), $f(\beta)$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Since $f(\beta) = (f^{-1})^{-1}(\beta)$, it follows that f^{-1} is a $\mathfrak{FN}ecs$ mapping in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Hence, f^{-1} is $\mathfrak{FN}eCts$.

(iii) $\Rightarrow$ (i): Let β be a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. By (iii), we have

$$(f^{-1})^{-1}(\beta) = f(\beta),$$

which shows that $f(\beta)$ is a $\mathfrak{FN}eos$. Hence, f is a $\mathfrak{FN}eO$ mapping. □

5. Fermatean neutrosophic e-homeomorphism

Definition 5.1. A bijection $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is called a $\mathfrak{FN}e$-homeomorphism (briefly $\mathfrak{FN}eHom$) if f and f^{-1} are $\mathfrak{FN}eCts$.

Theorem 5.2. Each $\mathfrak{FN}Hom$ is a $\mathfrak{FN}eHom$. But not conversely.

Proof. Let f be a $\mathfrak{FN}Hom$. Then f and f^{-1} are $\mathfrak{FN}Cts$. Since every $\mathfrak{FN}Cts$ function is $\mathfrak{FN}eCts$, it follows that f and f^{-1} are $\mathfrak{FN}eCts$. Therefore, f is a $\mathfrak{FN}eHom$. □

Example 5.3. Let $\mathfrak{C} = \{\imath, \eta\}$, $\mathfrak{D} = \{p, q\}$ and the $\mathfrak{FN}s$'s $\mathfrak{G}_1$, $\mathfrak{G}_2$, B_1 and B_2 are defined as
$\mu_{\mathfrak{G}_1}(\imath) = 0.1,\ \nu_{\mathfrak{G}_1}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_1}(\imath) = 0.9,$
$\mu_{\mathfrak{G}_1}(\eta) = 0.3,\ \nu_{\mathfrak{G}_1}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_1}(\eta) = 0.7;$
$\mu_{\mathfrak{G}_2}(\imath) = 0.2,\ \nu_{\mathfrak{G}_2}(\imath) = 0.5,\ \sigma_{\mathfrak{G}_2}(\imath) = 0.8,$
$\mu_{\mathfrak{G}_2}(\eta) = 0.4,\ \nu_{\mathfrak{G}_2}(\eta) = 0.5,\ \sigma_{\mathfrak{G}_2}(\eta) = 0.6;$
$\mu_{B_1}(p) = 0.3,\ \nu_{B_1}(p) = 0.5,\ \sigma_{B_1}(p) = 0.7,$
$\mu_{B_1}(q) = 0.4,\ \nu_{B_1}(q) = 0.5,\ \sigma_{B_1}(q) = 0.6;$
$\mu_{B_2}(p) = 0.2,\ \nu_{B_2}(p) = 0.5,\ \sigma_{B_2}(p) = 0.8,$
$\mu_{B_2}(q) = 0.3,\ \nu_{B_2}(q) = 0.5,\ \sigma_{B_2}(q) = 0.7;$
Let $\tau_{\mathfrak{FN}} = \{0_{\mathfrak{FN}},\ 1_{\mathfrak{FN}},\ \mathfrak{G}_1\}$ and $\sigma_{\mathfrak{FN}} = \{0_{\mathfrak{FN}},\ 1_{\mathfrak{FN}},\ B_1,\ B_2\}$ be a $\mathfrak{FN}ts$ and let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be defined as $f(\imath) = p$, $f(\eta) = q$, then f is $\mathfrak{FN}eHom$ but not $\mathfrak{FN}Hom$, the set B_1 is a $\mathfrak{FN}os$ in $\mathfrak{D}$ but $f^{-1}(B_1) = B_1$ is not $\mathfrak{FN}os$ in $\mathfrak{C}$.

Theorem 5.4. Let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be a bijective mapping. If f is $\mathfrak{FN}eCts$, then the following statements are equivalent:

(i) f is a $\mathfrak{FN}eC$ mapping.

(ii) f is a $\mathfrak{FN}eO$ mapping.

(iii) f^{-1} is a $\mathfrak{FN}eHom$.

Proof. (i) ⇒ (ii): Assume that f is a bijective mapping and a $\mathfrak{FN}eC$ mapping. Then f^{-1} is a $\mathfrak{FN}eCts$ mapping. Since each $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is a $\mathfrak{FN}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$, it follows that f is a $\mathfrak{FN}eO$ mapping.

(ii) ⇒ (iii): Let f be bijective and a $\mathfrak{FN}O$ mapping. Then f^{-1} is a $\mathfrak{FN}eCts$ mapping. Hence, both f and f^{-1} are $\mathfrak{FN}eCts$. Therefore, f is a $\mathfrak{FN}eHom$.

(iii) ⇒ (i): Let f be a $\mathfrak{FN}eHom$. Then f and f^{-1} are $\mathfrak{FN}eCts$. Since each $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$, it follows that f is a $\mathfrak{FN}eC$ mapping. □

Definition 5.5. A $\mathfrak{FN}ts$ $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is said to be a $\mathfrak{FN}$ $eT_{\frac{1}{2}}$ (briefly, $\mathfrak{FN}eT_{\frac{1}{2}}$)-space if every $\mathfrak{FN}ecs$ is $\mathfrak{FN}c$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$.

Theorem 5.6. Let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be a $\mathfrak{FN}eHom$, then f is a $\mathfrak{FN}Hom$ if $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ and $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ are $\mathfrak{FN}eT_{\frac{1}{2}}$-space.

Proof. Assume that β is a $\mathfrak{FN}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Then $f^{-1}(\beta)$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Since $(\mathfrak{C}, \tau_{\mathfrak{FN}})$ is an $\mathfrak{FN}eT_{\frac{1}{2}}$-space, $f^{-1}(\beta)$ is a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Therefore, f is $\mathfrak{FN}Cts$.

By hypothesis, f^{-1} is $\mathfrak{FN}eCts$. Let δ be a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then $(f^{-1})^{-1}(\delta) = f(\delta)$ is a $\mathfrak{FN}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ by presumption. Since $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}eT_{\frac{1}{2}}$-space, $f(\delta)$ is a $\mathfrak{FN}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Hence, f^{-1} is $\mathfrak{FN}Cts$. Therefore, f is a $\mathfrak{FN}Hom$. □

Theorem 5.7. Let $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ be a $\mathfrak{FN}ts$ mapping. Then the following statements are equivalent if $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}eT_{\frac{1}{2}}$-space:

(i) f is a $\mathfrak{FN}eC$ mapping.

(ii) If β is a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$, then $f(\beta)$ is a $\mathfrak{FN}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$.

(iii) $f(\mathrm{int}(\beta)) \subseteq \mathrm{cl}(\mathrm{int}(f(\beta)))$ for every $\mathfrak{FN}s$ β in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$.

Proof. (i) ⇒ (ii): Obvious.

(ii) ⇒ (iii): Let β be a $\mathfrak{FN}s$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then $\mathrm{int}(\beta)$ is a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Thus, $f(\mathrm{int}(\beta))$ is a $\mathfrak{FN}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Since $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$ is a $\mathfrak{FN}eT_{\frac{1}{2}}$-space, $f(\mathrm{int}(\beta))$ is a $\mathfrak{FN}os$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Therefore,

$$f(\mathrm{int}(\beta)) = \mathrm{int}(f(\mathrm{int}(\beta))) \subseteq \mathrm{cl}(\mathrm{int}(f(\beta))).$$

(iii) ⇒ (i): Let β be a $\mathfrak{FN}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then β^c is a $\mathfrak{FN}os$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. From (iii),

$$f(\mathrm{int}(\beta^c)) \subseteq \mathrm{cl}(\mathrm{int}(f(\beta^c))).$$

Hence,

$$f(\beta^c) \subseteq \mathrm{cl}(\mathrm{int}(f(\beta^c))).$$

Therefore, $f(\beta^c)$ is a $\mathfrak{F}\mathfrak{N}eos$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$, which implies that $f(\beta)$ is a $\mathfrak{F}\mathfrak{N}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Hence, f is a $\mathfrak{F}\mathfrak{N}C$ mapping. □

Theorem 5.8. Let $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ and $g : (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ be $\mathfrak{F}\mathfrak{N}eC$, where $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$ and $(\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ are two $\mathfrak{F}\mathfrak{N}ts$'s and $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ a $\mathfrak{F}\mathfrak{N}eT_{\frac{1}{2}}$-space, then the composition $g \circ f$ is $\mathfrak{F}\mathfrak{N}eC$.

Proof. Let β be a $\mathfrak{F}\mathfrak{N}cs$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Since f is $\mathfrak{F}\mathfrak{N}eC$ and $f(\beta)$ is a $\mathfrak{F}\mathfrak{N}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$, by assumption, $f(\beta)$ is a $\mathfrak{F}\mathfrak{N}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Since g is $\mathfrak{F}\mathfrak{N}eC$, $g(f(\beta))$ is $\mathfrak{F}\mathfrak{N}eC$ in $(\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ and

$$g(f(\beta)) = (g \circ f)(\beta).$$

Therefore, $g \circ f$ is $\mathfrak{F}\mathfrak{N}eC$. □

Theorem 5.9. Let $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ and $g : (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{Z}, \rho_{\mathfrak{F}\mathfrak{N}})$ be two $\mathfrak{F}\mathfrak{N}ts$ mappings. Then the following statements hold:

(i) If $g \circ f$ is $\mathfrak{F}\mathfrak{N}eO$ and f is $\mathfrak{F}\mathfrak{N}Cts$, then g is $\mathfrak{F}\mathfrak{N}eO$.

(ii) If $g \circ f$ is $\mathfrak{F}\mathfrak{N}O$ and g is $\mathfrak{F}\mathfrak{N}eCts$, then f is $\mathfrak{F}\mathfrak{N}eO$.

6. **Fermatean neutrosophic e-C homeomorphism**

Definition 6.1. A bijection $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ is called a $\mathfrak{F}\mathfrak{N}e$-C homeomorphism (briefly, $\mathfrak{F}\mathfrak{N}eCHom$) if f and f^{-1} are $\mathfrak{F}\mathfrak{N}eIrr$ mappings.

Theorem 6.2. Each $\mathfrak{F}\mathfrak{N}eCHom$ is a $\mathfrak{F}\mathfrak{N}eHom$. But not conversely.

Proof. Let us assume that β is a $\mathfrak{F}\mathfrak{N}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Then β is a $\mathfrak{F}\mathfrak{N}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. By assumption, $f^{-1}(\beta)$ is a $\mathfrak{F}\mathfrak{N}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$. Hence, f is a $\mathfrak{F}\mathfrak{N}eCts$ mapping. Thus, both f and f^{-1} are $\mathfrak{F}\mathfrak{N}eCts$ mappings. Therefore, f is a $\mathfrak{F}\mathfrak{N}eHom$. □

Example 6.3. In Example 5.3, the mapping f is $\mathfrak{F}\mathfrak{N}eHom$ but not $\mathfrak{F}\mathfrak{N}eCHom$, the set A_2^c is a $\mathfrak{F}\mathfrak{N}os$ in $\mathfrak{C}$ but $f(A_2^c) = A_2^c$ is not $\mathfrak{F}\mathfrak{N}eos$ in $\mathfrak{D}$.

Theorem 6.4. If $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ is a $\mathfrak{F}\mathfrak{N}eCHom$, then $\mathfrak{F}\mathfrak{N}ecl(f^{-1}(\xi)) \subseteq f^{-1}(\mathfrak{F}\mathfrak{N}cl(\xi))$ for each $\mathfrak{F}\mathfrak{N}ts$ ξ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$.

Proof. Let ξ be a $\mathfrak{F}\mathfrak{N}ts$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Then, $\mathfrak{F}\mathfrak{N}cl(\xi)$ is a $\mathfrak{F}\mathfrak{N}cs$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$, and every $\mathfrak{F}\mathfrak{N}cs$ is a $\mathfrak{F}\mathfrak{N}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. Assume f is $\mathfrak{F}\mathfrak{N}eIrr$, $f^{-1}(\mathfrak{F}\mathfrak{N}cl(\zeta))$ is a $\mathfrak{F}\mathfrak{N}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}})$, then $\mathfrak{F}\mathfrak{N}cl(f^{-1}(\mathfrak{F}\mathfrak{N}cl(\xi))) = f^{-1}(\mathfrak{F}\mathfrak{N}cl(\xi))$. Here, $\mathfrak{F}\mathfrak{N}ecl(f^{-1}(\xi)) \subseteq \mathfrak{F}\mathfrak{N}ecl(f^{-1}(\mathfrak{F}\mathfrak{N}cl(\xi))) = f^{-1}(\mathfrak{F}\mathfrak{N}cl(\xi))$. Therefore, $\mathfrak{F}\mathfrak{N}ecl(f^{-1}(\xi)) \subseteq f^{-1}(\mathfrak{F}\mathfrak{N}cl(\xi))$ for every $\mathfrak{F}\mathfrak{N}s$ ξ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$. □

Theorem 6.5. Let $f : (\mathfrak{C}, \tau_{\mathfrak{F}\mathfrak{N}}) \to (\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$ be a $\mathfrak{F}\mathfrak{N}eCHom$, then $\mathfrak{F}\mathfrak{N}ecl(f^{-1}(\xi)) = f^{-1}(\mathfrak{F}\mathfrak{N}ecl(\xi))$ for each $\mathfrak{F}\mathfrak{N}s$ ξ in $(\mathfrak{D}, \sigma_{\mathfrak{F}\mathfrak{N}})$.

Proof. Since f is a $\mathfrak{FN}eCHom$, then f is a $\mathfrak{FN}eIrr$ mapping. Let ξ be a $\mathfrak{FN}s$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. Clearly, $\mathfrak{FN}ecl(\xi)$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then $\mathfrak{FN}ecl(\xi)$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Since $f^{-1}(\xi) \subseteq f^{-1}(\mathfrak{FN}ecl(\xi))$, then $\mathfrak{FN}ecl(f^{-1}(\xi)) \subseteq \mathfrak{FN}ecl(f^{-1}(\mathfrak{FN}ecl(\xi))) = f^{-1}(\mathfrak{FN}ecl(\xi))$. Therefore, $\mathfrak{FN}ecl(f^{-1}(\xi)) \subseteq f^{-1}(\mathfrak{FN}ecl(\xi))$. Let f be a $\mathfrak{FN}eCHom$. f^{-1} is a $\mathfrak{FN}eIrr$ mapping. Let us consider $\mathfrak{FN}s$ $f^{-1}(\xi)$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$, which implies $\mathfrak{FN}ecl(f^{-1}(\xi))$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Hence, $\mathfrak{FN}ecl(f^{-1}(\xi))$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. This implies that $(f^{-1})^{-1}(\mathfrak{FN}ecl(f^{-1}(\xi))) = f(\mathfrak{FN}ecl(f^{-1}(\xi)))$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. This proves $\xi = (f^{-1})^{-1}(f^{-1}(\xi)) \subseteq (f^{-1})^{-1}(\mathfrak{FN}ecl(f^{-1}(\xi))) = f(\mathfrak{FN}ecl(f^{-1}(\xi)))$. Therefore, $\mathfrak{FN}ecl(\xi) \subseteq \mathfrak{FN}ecl(f(\mathfrak{FN}ecl(f^{-1}(\xi)))) = f(\mathfrak{FN}ecl(f^{-1}(\xi)))$, since f^{-1} is a $\mathfrak{FN}eIrr$ mapping. Hence, $f^{-1}(\mathfrak{FN}ecl(\xi)) \subseteq f^{-1}(f(\mathfrak{FN}ecl(f^{-1}(\xi)))) = \mathfrak{FN}ecl(f^{-1}(\xi))$. That is, $f^{-1}(\mathfrak{FN}ecl(\xi)) \subseteq \mathfrak{FN}ecl(f^{-1}(\xi))$. Hence, $\mathfrak{FN}ecl(f^{-1}(\xi)) = f^{-1}(\mathfrak{FN}ecl(\xi))$. □

Theorem 6.6. If $f : (\mathfrak{C}, \tau_{\mathfrak{FN}}) \to (\mathfrak{D}, \sigma_{\mathfrak{FN}})$ and $g : (\mathfrak{D}, \sigma_{\mathfrak{FN}}) \to (\mathfrak{Z}, \rho_{\mathfrak{FN}})$ are $\mathfrak{FN}eCHom$'s, then $g \circ f$ is a $\mathfrak{FN}eCHom$.

Proof. Let f and g be two $\mathfrak{FN}eCHom$ mappings. Assume that ξ is a $\mathfrak{FN}ecs$ in $(\mathfrak{Z}, \rho_{\mathfrak{FN}})$. Then $g^{-1}(\xi)$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. By hypothesis, $f^{-1}(g^{-1}(\xi))$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Hence, $g \circ f$ is a $\mathfrak{FN}eIrr$ mapping.

Now, let δ be a $\mathfrak{FN}ecs$ in $(\mathfrak{C}, \tau_{\mathfrak{FN}})$. Then, by presumption, $f(\delta)$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{D}, \sigma_{\mathfrak{FN}})$. By hypothesis, $g(f(\delta))$ is a $\mathfrak{FN}ecs$ in $(\mathfrak{Z}, \rho_{\mathfrak{FN}})$. This shows that $g \circ f$ is a $\mathfrak{FN}eIrr$ mapping.

Hence, $g \circ f$ is a $\mathfrak{FN}eCHom$. □

7. Application

Entropy as a measure of fuzziness was first proposed by Zadeh [18]. Subsequently, many mathematicians have introduced various entropy measures. In this section, we focus on defining an entropy measure for $\mathfrak{FN}fs$ that captures the degrees of membership, indeterminacy, and non-membership. As an illustration, we apply the proposed entropy measure in the context of decision making.

Algortihm:

Step 1: Input each of the FNS's $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_k$.

Step 2: Compute the entropy of each FNS using the expression

$$\varepsilon_{\mathfrak{FN}fs}(\mathfrak{G}) = 1 - \frac{1}{n}\sum_{i=1}^{n}(\mu_{\mathfrak{G}} - \sigma_{\mathfrak{G}})^2 \cdot (1 - \nu_{\mathfrak{G}}), \quad \text{for every } x_i \in \mathfrak{G}.$$

Step 3: Identify $\mathfrak{G}_r$ with the minimum value of $\varepsilon_{\mathfrak{FN}fs}(\mathfrak{G}_r)$.

Step 4: The optimal choice is to select the $\mathfrak{G}_r$ from Step 3.

Step 5: If more than one ideal solution is found, the user may select any one of them.

Since FNS's are extensions of existing sets such as IFS's and PFS's, they provide an effective tool for representing information in decision-making.

Consider a set of k options $V_1, V_2, \ldots, V_k$ evaluated by n experts $P_1, P_2, \ldots, P_n$. Each expert P_j assesses the alternatives using the parameters $K = \{k_1, k_2, \ldots, k_m\}$ and assigns ratings to FNS's. The challenge is to select the best option among them. The proposed methodology provides a systematic way to solve this problem using entropy measures.

Definition 7.1. Let $\mathfrak{G} = \{< \imath, \mu_{\mathfrak{G}}(\imath), \nu_{\mathfrak{G}}(\imath), \sigma_{\mathfrak{G}}(\imath) | \imath \in \mathfrak{C}\}$ be a $\mathfrak{F}\mathfrak{N}fs$ in U. The new entropy measure for $\mathfrak{G}$ denoted by $\varepsilon_{\mathfrak{F}\mathfrak{N}fs}(A)$, is a function, $\varepsilon_{\mathfrak{F}\mathfrak{N}fs} : \tau_{\mathfrak{F}\mathfrak{N}fs}(U) \to [0, 1]$ and is defined as $\varepsilon_{\mathfrak{F}\mathfrak{N}fs}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}(\mu_{\mathfrak{G}} - \sigma_{\mathfrak{G}})^2 * (1 - \nu_{\mathfrak{G}});$ for every$x_i \in A$, where $\tau_{\mathfrak{F}\mathfrak{N}fs}(U)$ denote the family of all $\mathfrak{F}\mathfrak{N}fs$'s on U.

Example 7.2. The Public Service commission(PSC) wants to select the director of education for district to select among the three education department $(D1, D2, D3)$. Assume the following three criteria $(C1, C2, C3)$ Educational management skills, Policy development & implementation and Good communications. The PSC to appoint the best director from the three departments, $D1$ - State council of educational research, $D2$ - State board of Education, $D3$ - The Text book board.

Table 1. Weights assigned to the criteria.

	$C1$	$C2$	$C3$
$D1$	$< D1, C1; 0.52, 0.20, 0.15 >$	$< D1, C2; 0.48, 0.30, 0.19 >$	$< D1, C3; 0.26, 0.25, 0.10 >$
$D2$	$< D2, C1; 0.59, 0.25, 0.32 >$	$< D2, C2; 0.45, 0.50, 0.25 >$	$< D2, C3; 0.21, 0.50, 0.47 >$
$D3$	$< D3, C1; 0.45, 0.30, 0.58 >$	$< D3, C2; 0.90, 0.20, 0.25 >$	$< D3, C3; 0.25, 0.30, 0.50 >$

Table 3. Entropy measure based on the criteria

	$C1$	$C2$	$C3$
$D1$	0.89	0.94	0.98
$D2$	0.95	0.99	0.97
$D3$	0.99	0.66	0.96

Table 4. Selection result for the director

	Result
$D1$	2.81
$D2$	2.90
$D3$	**2.61**

After evaluating the criteria, D3 is selecting for the position Dirctor of education for the district using entropy measure of minimum fuzziness.

8. Conclusions

This study introduced $\mathfrak{FN}$ e-open sets, e-continuous mappings, e-homomorphisms, e-C homomorphisms, and $\mathfrak{FN}eT_{\frac{1}{2}}$-spaces, establishing their key properties and enriching neutrosophic topology. The framework has direct applications in entropy-based decision-making and related real-life problems. Future work may explore $\mathfrak{FN}e$-compactness, $\mathfrak{FN}e$-connectedness, contra e-continuous functions, and applications in multi-criteria decision-making, entropy measures, and AI-driven uncertainty modeling.

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