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A Note On Different Types of Product of Fermatean Quadripartitioned Neutrosophic Graphs

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Abstract. The focus of this paper is to propose a new concept of Fermatean Quadripartitioned Neutrosophic Graphs(FQNGs) and to present some different operations on FQNGs such as rejection, symmetric difference, maximal product and residue product with appropriate examples and also described some of their important theorems. Further we have discussed the concept of total degree of a fermatean quadripartitioned neutrosophic graph with some interesting examples. An efficient approach to solve a decision making problem using FQNG is presented.

Keywords: Fermatean Quadripartitioned Neutrosophic set, Fermatean Quadripartitioned Neutrosophic Graph, Maximal product, Symmetric difference, Residue product.

1. **Introduction**

In 1965, Lofti A. Zadeh [14] developed Fuzzy logic, which was successfully used by him to handle uncertainty in decision making. The concept is based on the observation that people make decisions based on imprecise and non - numerical information. Fuzzy sets are mathematical means of representing vagueness and imprecise information (hence the term Fuzzy). In fuzzy set concept, a membership function is defined to assign each element of the reference system, a real value in the interval [0,1]. The membership value of an element is zero indicates that the element does not belong to the class. The membership value of an element belongs to that class and other values between zero to one indicate the degree of membership to a class. The idea of intuitionistic fuzzy set was first defined by Atanassov [1] in 1986 as a generalization of Zadeh's fuzzy set. The prominent feature of intuitionistic fuzzy set is that a membership degree and a non-membership degree are assigned to each element in the set.

Florentin Smarandache [8] familiarized with the neutrosophic sets as a generalization of fuzzy and intuitionistics fuzzy sets. Neutrosophy is a branch of philosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideas. Smarandache introduced the term ”neutrosophic” because ”neutrosophic” etymologically comes from ”neutrosophy” which means knowledge of neutral thought and this neutral represents the main distinction between fuzzy, intuitionistic fuzzy and neutrosophic logic /set.

Neutrosophic sets are the more summed up class by which one can deal with uncertain informations in a more successful way. Neutrosophic sets are characterized by three functions truth - membership, indeterminacy - membership and falsity - membership and they are represented independently. The concept of Interval Valued Neutrospphic Set was first made into existence by Smarandache, Wang [11]. Many other Single Valued Neutrosophic sets are found by Smarandache [10]. M. Ramya outlined a brand new mathematical approach of Fermatean Quadripartitioned Neutrosophic Sets (FQNS) in 2022 [7].

Graphs serve as a powerful tool for representing real-life problems. In a graph model, objects and their connections are represented as nodes and edges. Real-life problems encompass various types of information, necessitating multiple graph types for effective modeling, including fuzzy graphs, intuitionistic fuzzy graphs, and neutrosophic graph theory [1, 2, 3, 6]. The most general form of graph as today, the n - SuperHyperGraph with super vertices was introduced by Smarandache[9]. Recently many scientist have researched on graph in neutrosophic environment for instance, Yang et al.[12], Arkam [2,3], Ye[13], Naz et al.[6].

The primary goal of this research is to introduce effective techniques for modeling and addressing the decision-making issue through the use of FQNG. In this study, we define regular FQNG, complete FQNG, and strong FQNG. We present various operations on FQNG, including rejection, symmetric difference, maximal product, and residue product, accompanied by suitable examples. Following this, we explain the concept of degree and total degree of a FQNG in relation to these operations, illustrated with examples. Additionally, we propose two novel operators (the fermatean quadriparitioned neutrosophic weighted arithmetic average operator and the fermatean quadriparitioned neutrosophic weighted geometric average operator) to tackle the fermatean quadriparitioned neutrosophic decision-making challenge. We have applied this framework to a real-world scenario involving the selection of the best holiday destination, utilizing FQNG and the two operators to solve the issue. The layout of this article is in such kind : Introduction, Preliminary, The concept of FQNGs along with its operations and application and finally the Conclusion.

2. PRELIMINARIES

In this section some basic definitions and results that are needed in the sequel are recalled. The following are also used in the rest of the paper:

Definition 2.1. [14] Let U be any non-empty set. A fuzzy set denoted by $\hat{\mathbb{L}}$ in U is characterized by a membership function $\hat{\lambda}_{\mathbb{T}} : \mathsf{U} \to \hat{\mathbb{I}}$, where $\hat{\mathbb{I}}$ is the closed unit interval [0,1], and one can write the fuzzy set $\hat{\mathbb{L}}$ by the set of points,
$\hat{\mathbb{L}} = \{(\mathsf{x}, \hat{\lambda}_{\mathbb{T}}(\mathsf{x}) : \mathsf{x} \in \mathsf{U}\}$

Definition 2.2. [4] Let U be a non empty fixed set. An intuitionistic fuzzy set (IFS in short) $\hat{\mathbb{L}}$ on the universe U is an object having the form,
$\hat{\mathbb{L}} = \{\langle(\mathsf{x}, \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}), \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x})) : \mathsf{x} \in \mathsf{U}\rangle\}$.
where the functions $\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}) : \mathsf{U} \to [0,1]$ and $\hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x}) : \mathsf{U} \to [0,1]$ denote the degree of membership and the degree of non - membership of each element $\mathsf{x} \in \mathsf{U}$ to the set $\hat{\mathbb{L}}$, respectively and $0 \leq \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}) + \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x}) \leq 1$ for each $\mathsf{x} \in \mathsf{U}$.
Denote by IFS(U), the set of all intuitionistic fuzzy sets in U.

Definition 2.3. [8] A Neutrosophic Set $\hat{\mathbb{L}}$ on the universe U is defined as $\hat{\mathbb{L}} = \{\langle(\mathsf{x}, \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}), \hat{\lambda}_{\mathbb{I}_{\hat{\mathbb{L}}}}(\mathsf{x})\hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x})) : \mathsf{x} \in \mathsf{U}\rangle\}$ where $\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}), \hat{\lambda}_{\mathbb{I}_{\hat{\mathbb{L}}}}(\mathsf{x}), \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x}) : \mathsf{U} \to Int([0,1])$ satisfying $\forall\, \mathsf{x} \in \mathsf{U}$, $0 \leq \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}) + \hat{\lambda}_{\mathbb{I}_{\hat{\mathbb{L}}}}(\mathsf{x}) + \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x}) \leq 3$.

Here $\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x}), \hat{\lambda}_{\mathbb{I}_{\hat{\mathbb{L}}}}(\mathsf{x}) and \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x})$ represents the truthness, indeterminacy and falsity membership values respectively of $\mathsf{x} \in \mathsf{U}$.

Definition 2.4. [7] Let X be a universe. A Fermatean Quadripartitioned Neutrosophic Set(FQN) $\hat{\mathbb{L}}$ on X is an object of the form
$\hat{\mathbb{L}} = \{\langle(\mathsf{x}, \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}, \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{L}}}}, \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{L}}}}, \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}) : \mathsf{x} \in \mathsf{X}\rangle\}$.
$(\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}})^3 + (\hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{L}}}})^3 + (\hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{L}}}})^3 + (\hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}})^3 \leq 2$
Here $\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the truth membership,
$\hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the contradiction membership,
$\hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the ignorance membership,
$\hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{L}}}}(\mathsf{x})$ is the falsity membership.

3. FERMATEAN QUADRIPARTITIONED NEUTROSOPHIC GRAPH(FQNG)

Here, we provide a novel idea of Fermatean Quadripartitioned Neutrosophic graph and examine some fundamental characteristics.

Definition 3.1. Let $\hat{\mathsf{G}} = (\mathbb{V}_{\mathsf{G}}, \mathbb{E}_{\mathsf{G}})$ be a graph containing no self-loop and parallel edges, where $\mathbb{V}_{\mathsf{G}}$ be the set of vertices and $\mathbb{E}_{\mathsf{G}}$ be the set of edges. Then a fermatean quadripartitioned neutrosophic graph of G is denoted by $\hat{\mathsf{G}} = (\mathbb{V}_{\mathsf{G}}, \hat{\lambda}, \hat{\mu})$, where $\hat{\lambda} = (\hat{\lambda}_{\mathbb{T}}, \hat{\lambda}_{\mathbb{C}}, \hat{\lambda}_{\mathbb{U}}, \hat{\lambda}_{\mathbb{F}})$ is a FQNS on $\mathbb{V}_{\mathsf{G}}$ and $\hat{\mu} = (\hat{\mu}_{\mathbb{T}}, \hat{\mu}_{\mathbb{C}}, \hat{\mu}_{\mathbb{U}}, \hat{\mu}_{\mathbb{F}})$ is a fermatean quadripartitioned neutrosophic symmetric relation on $\mathbb{E}_{\mathsf{G}} \subseteq \mathbb{V}_{\mathsf{G}} \times \mathbb{V}_{\mathsf{G}}$ ($\hat{\lambda}_{\mathbb{T}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\lambda}_{\mathbb{C}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\lambda}_{\mathbb{U}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\lambda}_{\mathbb{F}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\mu}_{\mathbb{T}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\mu}_{\mathbb{C}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\mu}_{\mathbb{U}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$, $\hat{\mu}_{\mathbb{F}} : \mathbb{V}_{\mathsf{G}} \to [0,1]$), and is defined as follows:

i) $\hat{\mu}_{\mathbb{T}}((\text{x,y})) \leq \hat{\lambda}_{\mathbb{T}}(x) \wedge \hat{\lambda}_{\mathbb{T}}(y)$, $\forall$ (x,y) $\in \mathbb{V}_{\mathsf{G}} \times \mathbb{V}_{\mathsf{G}}$;

ii) $\hat{\mu}_{\mathbb{C}}((\text{x,y})) \leq \hat{\lambda}_{\mathbb{C}}(x) \wedge \hat{\lambda}_{\mathbb{C}}(y)$, $\forall$ (x,y) $\in \mathbb{V}_{\mathsf{G}} \times \mathbb{V}_{\mathsf{G}}$;

iii) $\hat{\mu}_{\mathbb{U}}((\text{x,y})) \geq \hat{\lambda}_{\mathbb{U}}(x) \vee \hat{\lambda}_{\mathbb{U}}(y)$, $\forall$ (x,y) $\in \mathbb{V}_{\mathsf{G}} \times \mathbb{V}_{\mathsf{G}}$;

iv) $\hat{\mu}_{\mathbb{F}}((\text{x,y})) \geq \hat{\lambda}_{\mathbb{F}}(x) \vee \hat{\lambda}_{\mathbb{F}}(y)$, $\forall$ (x,y) $\in \mathbb{V}_{\mathsf{G}} \times \mathbb{V}_{\mathsf{G}}$;

If there is no edge between the vertices β and γ, then $\hat{\mu}_{\mathbb{T}}((\beta,\gamma)) = 0$, $\hat{\mu}_{\mathbb{C}}((\beta,\gamma)) = 0$, $\hat{\mu}_{\mathbb{U}}((\beta,\gamma)) = 0$ and $\hat{\mu}_{\mathbb{F}}((\beta,\gamma)) = 0$. The FQNG $\hat{\mathsf{G}}$ is called **STRONG** FQNG, if $\forall$ (x,y) $\in \mathbb{E}_{\mathsf{G}}$, $\hat{\mu}_{\mathbb{T}}((\text{x,y})) = \hat{\lambda}_{\mathbb{T}}(x) \wedge \hat{\lambda}_{\mathbb{T}}(y)$, $\hat{\mu}_{\mathbb{C}}((\text{x,y})) = \hat{\lambda}_{\mathbb{C}}(x) \wedge \hat{\lambda}_{\mathbb{C}}(y)$, $\hat{\mu}_{\mathbb{U}}((\text{x,y})) = \hat{\lambda}_{\mathbb{U}}(x) \vee \hat{\lambda}_{\mathbb{U}}(y)$, $\hat{\mu}_{\mathbb{F}}((\text{x,y})) = \hat{\lambda}_{\mathbb{F}}(x) \vee \hat{\lambda}_{\mathbb{F}}(y)$. $\hat{\mathsf{G}}$ is called **COMPLETE** FQNG, if $\forall$ (x,y) $\in \mathbb{V}_{\mathsf{G}}$, $\hat{\mu}_{\mathbb{T}}((\text{x,y})) = \hat{\lambda}_{\mathbb{T}}(x) \wedge \hat{\lambda}_{\mathbb{T}}(y)$, $\hat{\mu}_{\mathbb{C}}((\text{x,y})) = \hat{\lambda}_{\mathbb{C}}(x) \wedge \hat{\lambda}_{\mathbb{C}}(y)$, $\hat{\mu}_{\mathbb{U}}((\text{x,y})) = \hat{\lambda}_{\mathbb{U}}(x) \vee \hat{\lambda}_{\mathbb{U}}(y)$, $\hat{\mu}_{\mathbb{F}}((\text{x,y})) = \hat{\lambda}_{\mathbb{F}}(x) \vee \hat{\lambda}_{\mathbb{F}}(y)$. $\hat{\mathsf{G}}$ is called **REGULAR** FQNG if $\sum_{x\neq y} \hat{\mu}_{\mathbb{T}}((\text{x,y})) =$ constant, $\sum_{x\neq y} \hat{\mu}_{\mathbb{C}}((\text{x,y})) =$ constant, $\sum_{x\neq y} \hat{\mu}_{\mathbb{U}}((\text{x,y})) =$ constant and $\sum_{x\neq y} \hat{\mu}_{\mathbb{F}}((\text{x,y})) =$ constant.

Definition 3.2. (Degree and Total Degree)
Let $\hat{\mathsf{G}} = (\mathbb{V}_{\mathsf{G}}, \hat{\lambda}, \hat{\mu})$ be the FQNG defined in the above definition. The degree of a vertex $\beta \in \mathbb{V}_{\mathsf{G}}$ is denoted by $d_{\hat{\mathsf{G}}}(\beta) = ((d_{\mathbb{T}})_{\hat{\mathsf{G}}}\ (\beta)$, $(d_{\mathbb{C}})_{\hat{\mathsf{G}}}\ (\beta)$, $(d_{\mathbb{U}})_{\hat{\mathsf{G}}}\ (\beta)$, $(d_{\mathbb{F}})_{\hat{\mathsf{G}}}\ (\beta)$. Here $(d_{\mathbb{T}})_{\hat{\mathsf{G}}}\ (\beta) = \sum_{\beta\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{T}}(\beta\gamma)$, $(d_{\mathbb{C}})_{\hat{\mathsf{G}}}\ (\beta) = \sum_{\beta\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{C}}(\beta\gamma)$, $(d_{\mathbb{U}})_{\hat{\mathsf{G}}}\ (\beta) = \sum_{\beta\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{U}}(\beta\gamma)$ and $(d_{\mathbb{F}})_{\hat{\mathsf{G}}}\ (\beta) = \sum_{\beta\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{F}}(\beta\gamma)$. The total degree of a vertex $v \in \mathbb{V}_{\mathsf{G}}$ is denoted by $td_{\hat{\mathsf{G}}}(v) = ((td_{\mathbb{T}})_{\hat{\mathsf{G}}}(v)$, $(td_{\mathbb{C}})_{\hat{\mathsf{G}}}(v)$, $(td_{\mathbb{U}})_{\hat{\mathsf{G}}}(v)$, $(td_{\mathbb{F}})_{\hat{\mathsf{G}}}(v))$. Here $(td_{\mathbb{T}})_{\hat{\mathsf{G}}}(v) = \sum_{v\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{T}}(vu) + \hat{\lambda}_{\mathbb{T}}(u)$, $(td_{\mathbb{C}})_{\hat{\mathsf{G}}}(v) = \sum_{v\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{C}}(vu) + \hat{\lambda}_{\mathbb{C}}(u)$, $(td_{\mathbb{U}})_{\hat{\mathsf{G}}}(v) = \sum_{v\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{U}}(vu) + \hat{\lambda}_{\mathbb{U}}(u)$ and $(td_{\mathbb{F}})_{\hat{\mathsf{G}}}(v) = \sum_{v\gamma\in\mathbb{E}_{\mathsf{G}}} \hat{\mu}_{\mathbb{F}}(vu) + \hat{\lambda}_{\mathbb{F}}(u)$.

Definition 3.3. (Rejection of two FQNG)
Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathbb{R}_1 = (\mathbb{V}_{\mathbb{R}_1}, \mathbb{E}_{\mathbb{R}_1})$ and $\mathbb{R}_2 = (\mathbb{V}_{\mathbb{R}_2}, \mathbb{E}_{\mathbb{R}_2})$, respectively, where $\hat{\lambda}_1 = (\hat{\lambda}_{\mathbb{T}_1}, \hat{\lambda}_{\mathbb{C}_1}, \hat{\lambda}_{\mathbb{U}_1}, \hat{\lambda}_{\mathbb{F}_1})$, $\hat{\mu}_1 = (\hat{\mu}_{\mathbb{T}_1}, \hat{\mu}_{\mathbb{C}_1}, \hat{\mu}_{\mathbb{U}_1}, \hat{\mu}_{\mathbb{F}_1})$, $\hat{\lambda}_2 = (\hat{\lambda}_{\mathbb{T}_2}, \hat{\lambda}_{\mathbb{C}_2}, \hat{\lambda}_{\mathbb{U}_2}, \hat{\lambda}_{\mathbb{F}_2})$, $\hat{\mu}_2 = (\hat{\mu}_{\mathbb{T}_2}, \hat{\mu}_{\mathbb{C}_2}, \hat{\mu}_{\mathbb{U}_2}, \hat{\mu}_{\mathbb{F}_2})$. Then the rejection of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is denoted by $\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2} = (\hat{\lambda}_1 \mid \hat{\lambda}_2, \hat{\mu}_1 \mid \hat{\mu}_2)$ and defined as:

i) $\forall$ (x,y) $\in \mathbb{V}_{\mathbb{R}_1} \times \mathbb{V}_{\mathbb{R}_2}$, $\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2}(x,y) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y)$, $\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2}(x,y) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y)$, $\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2}(x,y) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y)$, $\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2}(x,y) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y)$

ii) $\forall$ x $\in \mathbb{V}_{\mathbb{R}_1}$ and yz $\notin \mathbb{E}_{\mathbb{R}_2}$,

a) $\hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((x,y)(x,z)) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y) \wedge \hat{\lambda}_{\mathbb{T}_2}(z)$;

b) $\hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((x,y)(x,z)) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y) \wedge \hat{\lambda}_{\mathbb{C}_2}(z)$;

c) $\hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((x,y)(x,z)) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y) \vee \hat{\lambda}_{\mathbb{U}_2}(z)$;

d) $\hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((x,y)(x,z)) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y) \vee \hat{\lambda}_{\mathbb{F}_2}(z)$.

iii) $\forall$ x $\in \mathbb{V}_{\mathbb{R}_2}$ and yz $\notin \mathbb{E}_{\mathbb{R}_1}$,

a) $\hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((y,x)(z,x)) = \hat{\lambda}_{\mathbb{T}_1}(y) \wedge \hat{\lambda}_{\mathbb{T}_2}(z) \wedge \hat{\lambda}_{\mathbb{T}_2}(x)$;

b) $\hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((y,x)(z,x)) = \hat{\lambda}_{\mathbb{C}_1}(y) \wedge \hat{\lambda}_{\mathbb{C}_2}(z) \wedge \hat{\lambda}_{\mathbb{C}_2}(x)$;

c) $\hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((y,x)(z,x)) = \hat{\lambda}_{\mathbb{U}_1}(y) \vee \hat{\lambda}_{\mathbb{U}_2}(z) \vee \hat{\lambda}_{\mathbb{U}_2}(x)$;

d) $\hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((y,x)(z,x)) = \hat{\lambda}_{\mathbb{F}_1}(y) \vee \hat{\lambda}_{\mathbb{F}_2}(z) \vee \hat{\lambda}_{\mathbb{F}_2}(x)$.

iv) $\forall$ xy $\notin \mathbb{E}_{\mathbb{R}_1}$ and zw $\notin \mathbb{E}_{\mathbb{R}_2}$,

a) $\hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((x,z)(y,w)) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y) \wedge \hat{\lambda}_{\mathbb{T}_2}(z) \wedge \hat{\lambda}_{\mathbb{T}_2}(w)$;

b) $\hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((x,z)(y,w)) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y) \wedge \hat{\lambda}_{\mathbb{C}_2}(z) \wedge \hat{\lambda}_{\mathbb{C}_2}(w)$;

c) $\hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((x,z)(y,w)) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y) \vee \hat{\lambda}_{\mathbb{U}_2}(z) \vee \hat{\lambda}_{\mathbb{U}_2}(w)$;

d) $\hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((x,z)(y,w)) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y) \vee \hat{\lambda}_{\mathbb{F}_2}(z) \vee \hat{\lambda}_{\mathbb{F}_2}(w)$.

Example 3.4. Consider two FQNGs $\hat{\mathbb{R}_1}$ (shown in Fig. 1a) and $\hat{\mathbb{R}_2}$ (shown in Fig. 1b) of the graphs $\mathbb{R}_1$ and $\mathbb{R}_2$, respectively, the rejection $\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}$ of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is shown in Fig. 1.

Theorem 3.5. *Let $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ be two FQNGs of the graph's G_1 and G_2, respectively. Then, the rejection $\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}$ of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is a FQNG.*

Proof. Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively.

i) Let $\alpha \in \mathbb{V}_1$ and $(\gamma, \delta) \notin \mathbb{E}_2$. Then :

$(\hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((\alpha,\gamma),(\alpha,\delta)))$
$= \hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta)$
$= (\hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma))(\hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta))$
$= ((\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2})(\alpha,\gamma)) \wedge ((\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2})(\alpha,\delta))$
$(\hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((\alpha,\gamma),(\alpha,\delta)))$
$= \hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta)$
$= (\hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma))(\hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta))$
$= ((\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2})(\alpha,\gamma)) \wedge ((\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2})(\alpha,\delta))$
$(\hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((\alpha,\gamma),(\alpha,\delta)))$
$= \hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta)$

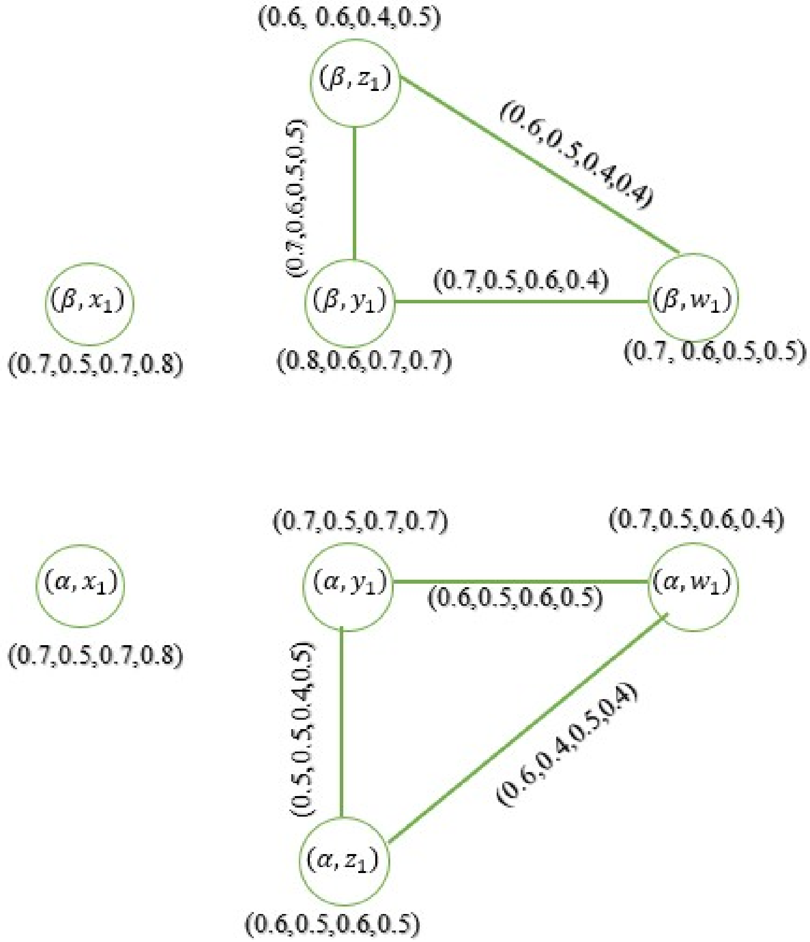
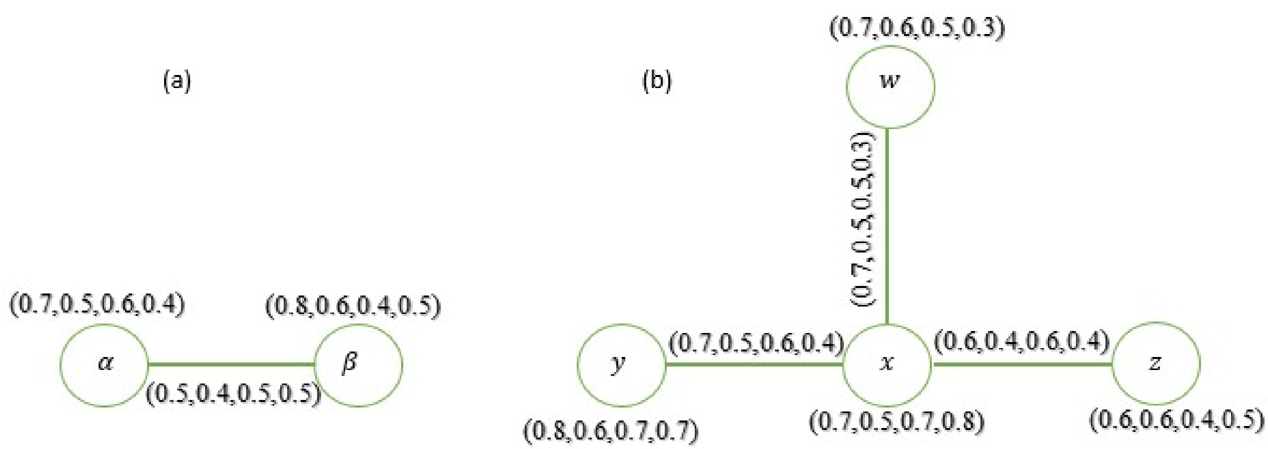


Fig. 1 $\widehat{\mathbb{R}}_1 \mid \widehat{\mathbb{R}}_2$



$= (\hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma))(\hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta))$

$= ((\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2})(\alpha, \gamma)) \vee ((\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2})(\alpha, \delta))$

$(\hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((\alpha, \gamma), (\alpha, \delta)))$

$= \hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta)$

$= (\hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma))(\hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta))$
$= ((\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2})(\alpha, \gamma)) \vee ((\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2})(\alpha, \delta))$

ii) Let $(\alpha_1, \beta_1) \notin \mathbb{E}_1$ and $\gamma_1 \in \mathbb{V}_2$. Then :

$(\hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((\alpha, \gamma), (\beta, \gamma)))$
$= \hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma)$
$= (\hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma))(\hat{\lambda}_{\mathbb{T}_1}(\beta) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma))$
$= ((\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2})(\alpha, \gamma)) \wedge ((\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2})(\beta, \gamma))$
$(\hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((\alpha, \gamma), (\beta, \gamma)))$
$= \hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_1}(\beta) \wedge \hat{\lambda}_{\mathbb{C}_1}(\gamma)$
$= (\hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma))(\hat{\lambda}_{\mathbb{C}_1}(\beta) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma))$
$= ((\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2})(\alpha, \gamma)) \wedge ((\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2})(\beta, \gamma))$
$(\hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((\alpha, \gamma), (\beta, \gamma)))$
$= \hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\beta) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma)$
$= (\hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma))(\hat{\lambda}_{\mathbb{U}_1}(\beta) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma))$
$= ((\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2})(\alpha, \gamma)) \vee ((\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2})(\beta, \gamma))$
$(\hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((\alpha, \gamma), (\beta, \gamma)))$
$= \hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_1}(\beta) \vee \hat{\lambda}_{\mathbb{F}_1}(\gamma)$
$= (\hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma))(\hat{\lambda}_{\mathbb{F}_1}(\beta) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma))$
$= ((\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2})(\alpha, \gamma)) \vee ((\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2})(\beta, \gamma))$

iii) Let $(\alpha_1, \beta_1) \notin \mathbb{E}_1$ and $(\gamma_1, \delta) \in \mathbb{E}_2$. Then :

$(\hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((\alpha, \gamma), (\beta, \delta)))$
$= \hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta)$
$= (\hat{\lambda}_{\mathbb{T}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma))(\hat{\lambda}_{\mathbb{T}_1}(\beta) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta))$
$= ((\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2})(\alpha, \gamma)) \wedge ((\hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2})(\beta, \gamma))$
$(\hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((\alpha, \gamma), (\beta, \delta)))$
$= \hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta)$
$= (\hat{\lambda}_{\mathbb{C}_1}(\alpha) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma))(\hat{\lambda}_{\mathbb{C}_1}(\beta) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta))$
$= ((\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2})(\alpha, \gamma)) \wedge ((\hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2})(\beta, \delta))$
$(\hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((\alpha, \gamma), (\beta, \delta)))$
$= \hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\beta) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma) \wedge \hat{\lambda}_{\mathbb{U}_2}(\delta)$
$= (\hat{\lambda}_{\mathbb{U}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma))(\hat{\lambda}_{\mathbb{U}_1}(\beta) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta))$
$= ((\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2})(\alpha, \gamma)) \vee ((\hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2})(\beta, \delta))$
$(\hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_1}((\alpha, \gamma), (\beta, \delta)))$
$= \hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_2}(\beta) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma) \wedge \hat{\lambda}_{\mathbb{F}_2}(\delta)$
$= (\hat{\lambda}_{\mathbb{F}_1}(\alpha) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma))(\hat{\lambda}_{\mathbb{F}_1}(\beta) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta))$

$= ((\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2})(\alpha,\gamma)) \vee ((\hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2})(\beta,\delta))$

This completes the proof. □

Definition 3.6. (Degree of the rejection of two FQNGs)

Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$(d_T)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$$
$$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$$
$$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{T}_2}(\beta_1,\delta_1) + \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) + \sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}((\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1),$$

$$(d_C)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$$
$$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$$
$$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{C}_1}(\beta_1,\delta_1) + \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) + \sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1),$$

$$(d_U)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$$
$$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$$
$$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{U}_2}(\beta_1,\delta_1) + \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) + \sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1)\ ,$$

$$(d_F)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$$
$$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$$
$$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{F}_2}(\beta_1,\delta_1) + \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) + \sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}((\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1).$$

Definition 3.7. (Total Degree of the Rejection of two FQNGs)

Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$(td_T)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$$
$$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} \mid \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2}(\alpha_1,\beta_1)$$
$$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1,\delta_1) + \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) + \sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1) + \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1)\ ,$$

$$(td_C)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$$
$$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} \mid \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2}(\alpha_1,\beta_1)$$

$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{C}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1) + \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1)$,
$(td_U)_{\hat{\mathbb{P}_1}|\hat{\mathbb{P}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} \mid \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{U}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{U}_2}((\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1) + \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\beta_1)$,
$(td_F)_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} \mid \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{F}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{F}_2}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1) + \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\beta_1)$.

Example 3.8. We consider two FQNG $\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}$ as shown in Fig. 1. Then, the degree of the vertex (α, z_1) is $d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, z_1) = ((d_T)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1), ((d_C)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1), ((d_U)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1), ((d_F)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1)) = (1.1, 0.9, 0.9, 0.9)$, and the total degree of the vertex (α, z_1) is $td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, z_1) = ((td_T)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1), ((td_C)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1), ((td_U)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1),$
$((td_F)\hat{\mathbb{R}_1} \mid \hat{\mathbb{R}_2}(\alpha, z_1)) = (1.7, 1.4, 1.5, 1.4)$. Similarly the degree and the total degree of the remaining vertices is given by:

$d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, x_1) = (0.0, 0.0, 0.0, 0.0), td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, x_1) = (0.7, 0.5, 0.7, 0.8), d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, x_1) = (0.0, 0.0, 0.0, 0.0), td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, x_1) = (0.7, 0.5, 0.7, 0.8), d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, y_1) = (1.1, 1.0, 1.0, 1.0), td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, y_1) = (1.8, 1.5, 1.7, 1.7), d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, w_1) = (1.2, 0.9, 1.1, 0.9)$,
$td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\alpha, w_1) = (1.9, 1.4, 1.7, 1.3)$, $d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, y_1) = (1.4, 1.1, 1.1, 0.9), td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, y_1) = (2.2, 1.7, 1.8, 1.6), d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, w_1) = (1.3, 1.0, 1.0, 0.8), td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, w_1) = (2.0, 1.6, 1.5, 1.3)$,
$d_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, z_1) = (1.3, 1.1, 0.9, 0.9), td_{\hat{\mathbb{R}_1}|\hat{\mathbb{R}_2}}(\beta, z_1) = (1.9, 1.7, 1.3, 1.4)$

Definition 3.9. (Symmetric Difference)
Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Then the symmetric difference of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is denoted as $\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2} = (\hat{\lambda}_1 \bigoplus \hat{\lambda}_2), (\hat{\mu}_1 \bigoplus \hat{\mu}_2)$ and is defined as follows:

i) $\forall$ (x,y) $\in \mathbb{V}_1 \times \mathbb{V}_2$, $\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2}(x, y) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y)$,
$\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2}(x, y) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y)$,
$\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2}(x, y) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y)$,
$\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2}(x, y) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y)$.

ii) $\forall$ x $\in \mathbb{V}_2$ and (y,z) $\in \mathbb{E}_1$,

a) $\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((x, y), (x, z)) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y, z)$;

b) $\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y,z)$;

c) $\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y,z)$;

d) $\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y,z)$.

iii)$\forall$ sx $\in \mathbb{V}_2$ and (y,z) $\in \mathbb{E}_1$,

a) $\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{T}_1}(y,z) \wedge \hat{\lambda}_{\mathbb{T}_2}(x)$;

b) $\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{C}_1}(y,z) \wedge \hat{\lambda}_{\mathbb{C}_2}(x)$;

c) $\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{U}_1}(y,z) \vee \hat{\lambda}_{\mathbb{U}_2}(x)$;

d) $\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{F}_1}(y,z) \vee \hat{\lambda}_{\mathbb{F}_2}(x)$.

iv)$\forall$ (x,y) $\notin \mathbb{E}_1$ and (z,w) $\in \mathbb{E}_2$,

a) $\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((x,z),(y,w)) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y) \wedge \hat{\lambda}_{\mathbb{T}_2}(z,w)$;

b) $\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((x,z),(y,w)) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y) \wedge \hat{\lambda}_{\mathbb{C}_2}(z,w)$;

c) $\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((x,z),(y,w)) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y) \vee \hat{\lambda}_{\mathbb{U}_2}(z,w)$;

d) $\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((x,z),(y,w)) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y) \vee \hat{\lambda}_{\mathbb{F}_2}(z,w)$.

v)$\forall$ (x,y) $\in \mathbb{E}_1$ and (z,w) $\notin \mathbb{E}_2$,

a) $\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((x,z),(y,w)) = \hat{\mu}_{\mathbb{T}_1}(x,y) \wedge \hat{\lambda}_{\mathbb{T}_2}(z) \wedge \hat{\mu}_{\mathbb{T}_2}(w)$;

b) $\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((x,z),(y,w)) = \hat{\mu}_{\mathbb{C}_1}(x,y) \wedge \hat{\lambda}_{\mathbb{C}_2}(z) \wedge \hat{\mu}_{\mathbb{C}_2}(w)$;

c) $\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((x,z),(y,w)) = \hat{\mu}_{\mathbb{U}_1}(x,y) \vee \hat{\lambda}_{\mathbb{U}_2}(z) \vee \hat{\mu}_{\mathbb{U}_2}(w)$;

d) $\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((x,z),(y,w)) = \hat{\mu}_{\mathbb{F}_1}(x,y) \vee \hat{\lambda}_{\mathbb{F}_2}(z) \vee \hat{\mu}_{\mathbb{F}_2}(w)$.

Example 3.10. Consider two FQNGs $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ of the graphs G_1 and G_2 respectively as shown in Figure 2a and Figure 2b. The symmetric difference $\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}$ is shown in Figure 2.

Fig.(2a) FQNG $\hat{\mathbb{R}}_1$

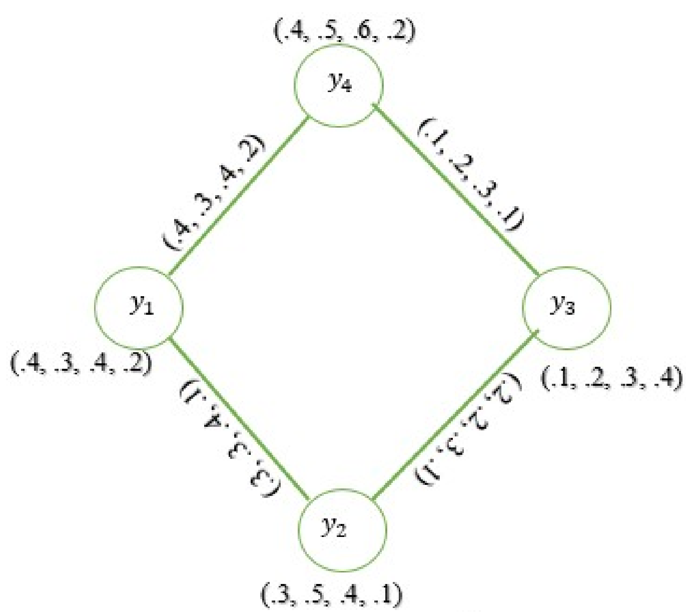


Fig.(2b) FQNG $\hat{\mathbb{R}}_2$

Theorem 3.11. *Let $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ be two FQNGs of the graph's G_1 and G_2 respectively. Then, the symmetric difference $\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}$ of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is a FQNG.*

Proof. Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively.

i) Let $\alpha_1 \in \mathbb{V}_1$ and $(\gamma_1, \delta_1) \in \mathbb{E}_2$. Then :

$$\begin{aligned}
&\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1, \gamma_1), (\alpha_1, \delta_1)) \\
&\quad = \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1, \delta_1) \\
&\quad = (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\alpha_1, \gamma_1) \wedge (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\alpha_1, \delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1, \gamma_1), (\alpha_1, \delta_1)) \\
&\quad = \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1, \delta_1) \\
&\quad = (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\alpha_1, \gamma_1) \wedge (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\alpha_1, \delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((\alpha_1, \gamma_1), (\alpha_1, \delta_1)) \\
&\quad = \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1, \delta_1) \\
&\quad = (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\alpha_1, \gamma_1) \vee (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\alpha_1, \delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\gamma_1),(\alpha_1,\delta_1))\\
&\quad= \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1,\delta_1)\\
&\quad= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))\\
&\quad= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))\\
&\quad= (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\alpha_1,\gamma_1) \vee (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\alpha_1,\delta_1)
\end{aligned}$$

ii) Let $(\alpha_1,\beta_1) \in \mathbb{E}_1$ and $\gamma_1 \in \mathbb{V}_2$. Then :

$$\begin{aligned}
&\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\gamma_1),(\beta_1,\gamma_1))\\
&\quad= \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1)\\
&\quad= (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\alpha_1,\gamma_1) \wedge (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\beta_1,\gamma_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\gamma_1),(\beta_1,\gamma_1))\\
&\quad= \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)\\
&\quad= (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\alpha_1,\gamma_1) \wedge (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\beta_1,\gamma_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\gamma_1),(\beta_1,\gamma_1))\\
&\quad= \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)\\
&\quad= (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\alpha_1,\gamma_1) \vee (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\beta_1,\gamma_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\gamma_1),(\beta_1,\gamma_1))\\
&\quad= \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)\\
&\quad= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1))\\
&\quad= (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\alpha_1,\gamma_1) \vee (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\beta_1,\gamma_1)
\end{aligned}$$

iii) Let $(\alpha_1,\beta_1) \notin \mathbb{E}_1$ and $(\gamma_1,\delta_1) \in \mathbb{E}_2$. Then :

$$\begin{aligned}
&\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&\quad= \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\mu}_{\mathbb{T}_2}(\gamma_1,\delta_1)\\
&\quad= (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1))\\
&\quad= (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1))\\
&\quad= (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\alpha_1,\gamma_1) \wedge (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\beta_1,\delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&= \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\mu}_{\mathbb{C}_2}(\gamma_1,\delta_1)\\
&= (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\alpha_1,\gamma_1) \wedge (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\beta_1,\delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&= \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\mu}_{\mathbb{U}_2}(\gamma_1,\delta_1)\\
&= (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\alpha_1,\gamma_1) \vee (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\beta_1,\delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&= \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\mu}_{\mathbb{F}_2}(\gamma_1,\delta_1)\\
&= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\alpha_1,\gamma_1) \vee (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\beta_1,\delta_1)
\end{aligned}$$

iv) Let $(\alpha_1,\beta_1) \in \mathbb{E}_1$ and $(\gamma_1,\delta_1) \notin \mathbb{E}_2$. Then :

$$\begin{aligned}
&\hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&= \hat{\lambda}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1)\\
&= (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\alpha_1,\gamma_1) \wedge (\hat{\lambda}_{\mathbb{T}_1} \bigoplus \hat{\lambda}_{\mathbb{T}_2})(\beta_1,\delta_1)
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&= \hat{\lambda}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1)\\
&= (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\alpha_1,\gamma_1) \wedge (\hat{\lambda}_{\mathbb{C}_1} \bigoplus \hat{\lambda}_{\mathbb{C}_2})(\beta_1,\delta_1)
\end{aligned}$$

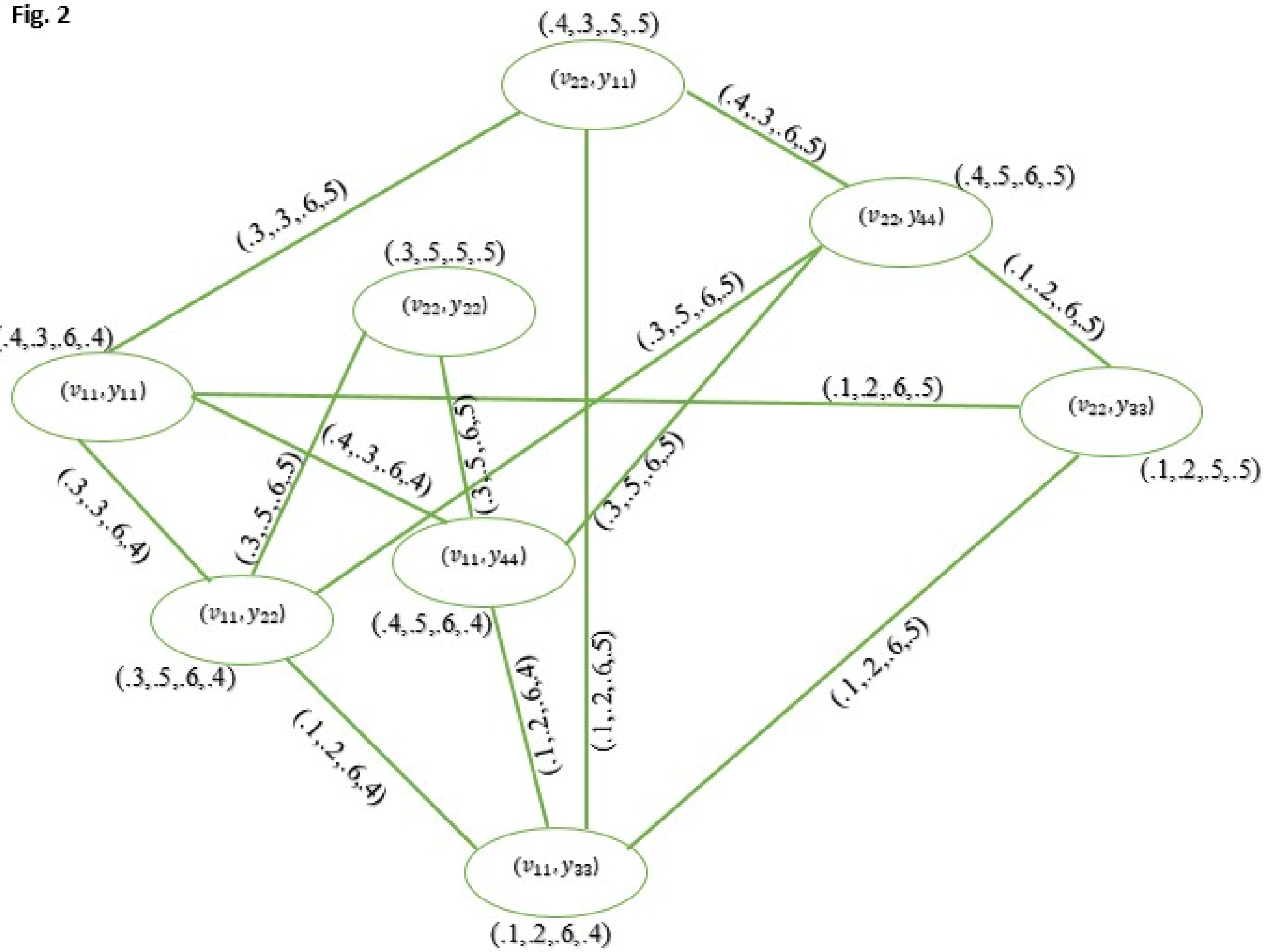
$$\begin{aligned}
&\hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\gamma_1),(\beta_1,\delta_1))\\
&= \hat{\lambda}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1)\\
&= (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1))\\
&= (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\alpha_1,\gamma_1) \vee (\hat{\lambda}_{\mathbb{U}_1} \bigoplus \hat{\lambda}_{\mathbb{U}_2})(\beta_1,\delta_1)
\end{aligned}$$

$$\hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((\alpha_1, \gamma_1), (\beta_1, \delta_1))$$
$$= \hat{\lambda}_{\mathbb{F}_1}(\alpha_1, \gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1)$$
$$= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))$$
$$= (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))$$
$$= (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\alpha_1, \gamma_1) \vee (\hat{\lambda}_{\mathbb{F}_1} \bigoplus \hat{\lambda}_{\mathbb{F}_2})(\beta_1, \delta_1)$$

This completes the proof.

□

Fig. 2



Definition 3.12. (Degree of the symmetric difference of two FQNGs).

Let $\hat{\mathbb{R}}_1 = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}}_2 = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$(d_T)_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(\alpha_1, \beta_1)$$
$$= \textstyle\sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in \mathbb{E}_1 \times \mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1))$$
$$= \textstyle\sum_{\alpha_1=\gamma_1\in \mathbb{V}_1,(\beta_1,\delta_1)\in \mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{T}_2}(\beta_1, \delta_1) +$$
$$\textstyle\sum_{(\alpha_1,\gamma_1)\in \mathbb{E}_1,\beta_1=\delta_1\in \mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1, \gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) +$$

$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\gamma_1) \wedge \hat{\mu}_{\mathbb{T}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1)$.

$(d_C)_{\hat{\mathbb{R}_1}\bigoplus\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{C}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\gamma_1) \wedge \hat{\mu}_{\mathbb{C}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1)$.

$(d_U)_{\hat{\mathbb{R}_1}\bigoplus\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{U}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\gamma_1) \vee \hat{\mu}_{\mathbb{U}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1)$.

$(d_F)_{\hat{\mathbb{P}_1}\bigoplus\hat{\mathbb{P}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{F}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\gamma_1) \vee \hat{\mu}_{\mathbb{F}_2}(\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1)$.

Definition 3.13. (Total Degree of the Symmetric Difference of two FQNGs)
Let $\hat{\mathbb{R}}_1 = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}}_2 = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$(td_T)_{\hat{\mathbb{R}_1}\bigoplus\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} \bigoplus \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{T}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}((\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_1}(\gamma_1) \wedge \hat{\mu}_{\mathbb{T}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1) + \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1)$,

$(td_C)_{\hat{\mathbb{R}_1}\bigoplus\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} \bigoplus \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{C}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}((\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1) + \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1)$,

$(td_U)_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} \bigoplus \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{U}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}((\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{U}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \mathbb{U}_{\hat{\zeta_1}}(\alpha_1,\gamma_1) \wedge \mathbb{U}_{\hat{\zeta_2}}(\beta_1) \wedge \hat{\lambda}_{\mathbb{U}_2}(\delta_1) + \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{U}_2}(\beta_1)$,
$(td_F)_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} \bigoplus \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{F}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}((\gamma_1) +$
$\sum_{(\alpha_1,\gamma_1)\notin\mathbb{E}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_1}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{F}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,(\beta_1,\delta_1)\notin\mathbb{E}_2} \mathbb{F}_{\hat{\zeta_1}}(\alpha_1,\gamma_1) \wedge \mathbb{F}_{\hat{\zeta_2}}(\beta_1) \wedge \hat{\lambda}_{\mathbb{F}_2}(\delta_1) + \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{F}_2}(\beta_1)$

Example 3.14. We consider two FQNG $\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}$ as shown in Fig. 2. Then, the degree of the vertex (v_{11}, y_{44}) is $d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{44}) = ((d_T)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44}), ((d_C)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44}), ((d_U)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44}), ((d_F)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44})) = (1.1, 1.5, 2.4, 1.8)$, and the total degree of the vertex (v_{11}, y_{44}) is $td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{44}) = ((td_T)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44}),$
$((td_C)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44}), ((td_U)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44}), ((td_F)\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}(v_{11}, y_{44})) = (1.5, 2.0, 3.0, 2.2)$. Similarly the degree and the total degree of the remaining vertices is given by:

$d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{11}) = (1.1, 1.1, 2.4, 1.8), td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{11}) = (1.5, 1.4, 3.0, 2.2), d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{22}) = (1.0, 1.5, 2.4, 1.8)$,

$td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{22}) = (1.3, 2.0, 3.0, 2.2), d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{33}) = (0.4, 0.8, 2.4, 1.8), td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{11}, y_{33}) = (0.5, 1.0, 3.0, 2.2)$,

$d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{22}, y_{11}) = (0.8, 0.8, 1.8, 1.5), td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{22}, y_{11}) = (1.2, 1.1, 2.3, 2.0)$, $d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{22}, y_{22}) = (0.6, 1.0, 1.2, 1.0)$,

$td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{22}, y_{22}) = (0.9, 1.5, 1.7, 1.5), d_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{22}, y_{33}) = (0.3, 0.6, 1.8, 1.5), td_{\hat{\mathbb{R}_1} \bigoplus \hat{\mathbb{R}_2}}(v_{22}, y_{33}) = (0.4, 0.8, 2.3, 2.0)$

Definition 3.15. (Residue Product of two FQNGs)

Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Then the residue product of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is denoted as $\hat{\mathbb{R}_1} \bullet \hat{\mathbb{R}_2} = (\hat{\lambda}_1 \bullet \hat{\lambda}_2, \hat{\mu}_1 \bullet \hat{\mu}_2)$ and is defined as follows:

i) $\forall$ (x,y) $\in \mathbb{V}_1 \times \mathbb{V}_2$, $\hat{\lambda}_{\mathbb{T}_1} \bullet \hat{\lambda}_{\mathbb{T}_2}(x,y) = \hat{\lambda}_{\mathbb{T}_1}(x) \wedge \hat{\lambda}_{\mathbb{T}_2}(y)$,
$\hat{\lambda}_{\mathbb{C}_1} \bullet \hat{\lambda}_{\mathbb{C}_2}(x,y) = \hat{\lambda}_{\mathbb{C}_1}(x) \wedge \hat{\lambda}_{\mathbb{C}_2}(y)$,

$\hat{\lambda}_{\mathbb{U}_1} \bullet \hat{\lambda}_{\mathbb{U}_2}(x, y) = \hat{\lambda}_{\mathbb{U}_1}(x) \vee \hat{\lambda}_{\mathbb{U}_2}(y)$,

$\hat{\lambda}_{\mathbb{F}_1} \bullet \hat{\lambda}_{\mathbb{F}_2}(x, y) = \hat{\lambda}_{\mathbb{F}_1}(x) \vee \hat{\lambda}_{\mathbb{F}_2}(y)$

ii)$\forall$ (x,y) $\in \mathbb{E}_1$ and $z \neq w \in \mathbb{V}_2$,

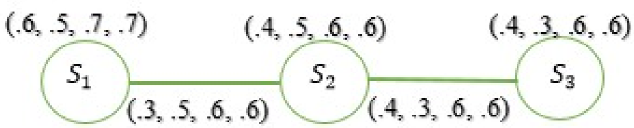
a) $\hat{\mu}_{\mathbb{T}_1} \bullet \hat{\mu}_{\mathbb{T}_2}((x, z), (y, w)) = \hat{\mu}_{\mathbb{T}_1}(x, y)$;

b) $\hat{\mu}_{\mathbb{C}_1} \bullet \hat{\mu}_{\mathbb{C}_2}((x, z), (y, w)) = \hat{\mu}_{\mathbb{C}_1}(x, y)$;

c) $\hat{\mu}_{\mathbb{U}_1} \bullet \hat{\mu}_{\mathbb{U}_2}((x, z), (y, w)) = \hat{\mu}_{\mathbb{U}_1}(x, y)$

d) $\hat{\mu}_{\mathbb{F}_1} \bullet \hat{\mu}_{\mathbb{F}_2}((x, z), (y, w)) = \hat{\mu}_{\mathbb{F}_1}(x, y)$.

Example 3.16. Consider two FQNGs $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ of the graphs G_1 and G_2 respectively as shown in Figure 3a and Figure 3b. The residue product $\hat{\mathbb{R}_1} \bullet \hat{\mathbb{R}_2}$ of FQNGs $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is shown in Fig. 3.



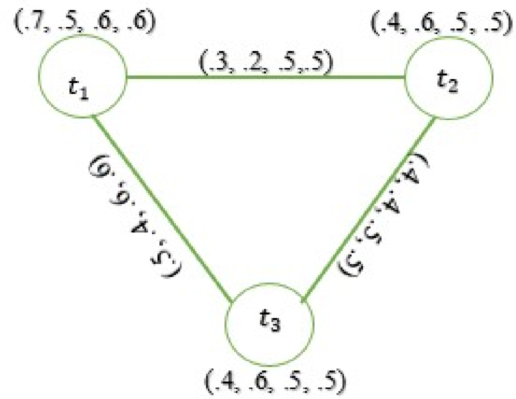


Fig. 3(a) FQNG $\hat{\mathbb{R}}_1$

Fig. 3(b) FQNG $\hat{\mathbb{R}}_2$

Theorem 3.17. *Let $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ be two FQNGs of the graph's G_1 and G_2 respectively. Then, the residue product of $\hat{\mathbb{R}_1} \bullet \hat{\mathbb{R}_2}$ of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is a FQNG.*

Definition 3.18. (Degree of the Residue Product of two FQNGs)

Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$(d_T)_{\hat{\mathbb{R}_1} \bullet \hat{\mathbb{R}_2}}(\alpha_1, \beta_1)$

$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} \bullet \hat{\mu}_{\mathbb{T}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1))$

$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1, \gamma_1)$

$(d_C)_{\hat{\mathbb{R}_1} \bullet \hat{\mathbb{R}_2}}(\alpha_1, \beta_1)$

$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} \bullet \hat{\mu}_{\mathbb{C}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1))$

$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1, \gamma_1)$

$(d_U)_{\hat{\mathbb{R}_1} \bullet \hat{\mathbb{R}_2}}(\alpha_1, \beta_1)$

$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} \bullet \hat{\mu}_{\mathbb{U}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1))$

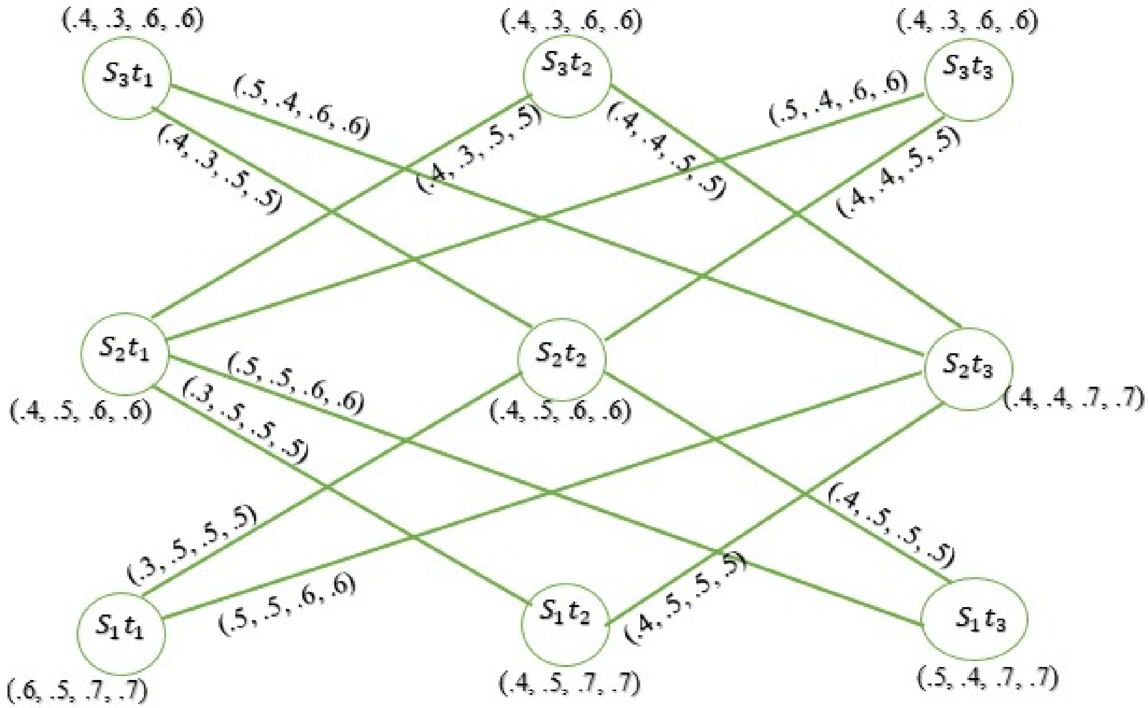


Fig. 3 FQNG $\widehat{\mathbb{R}}_1 \cdot \widehat{\mathbb{R}}_2$

$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1, \beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1)$

$(d_F)_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$

$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} \bullet \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1))$

$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1, \beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1)$

Definition 3.19. (Total Degree of the Residue Product of two FQNGs)
Let $\hat{\mathbb{R}}_1 = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}}_2 = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$(td_T)_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$

$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} \bullet \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) +$
$\hat{\lambda}_{\mathbb{T}_1} \bullet \hat{\lambda}_{\mathbb{T}_2}(\alpha_1,\beta_1)$

$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1, \beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\gamma_1) + \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_1}((\beta_1),$

$(td_C)_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$

$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} \bullet \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) +$
$\hat{\lambda}_{\mathbb{C}_1} \bullet \hat{\lambda}_{\mathbb{C}_2}(\alpha_1,\beta_1)$

$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1, \beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\gamma_1) + \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_1}((\beta_1),$

$(td_U)_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1}\bullet\hat{\mu}_{\mathbb{U}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) +$
$\hat{\lambda}_{\mathbb{U}_1}\bullet\hat{\lambda}_{\mathbb{U}_2}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1) + \hat{\lambda}_{\mathbb{U}_1}(\alpha_1)\vee\hat{\lambda}_{\mathbb{U}_1}((\beta_1),$

$(td_F)_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1}\bullet\hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) +$
$\hat{\lambda}_{\mathbb{F}_1}\bullet\hat{\lambda}_{\mathbb{F}_1}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1\neq\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) + \hat{\lambda}_{\mathbb{F}_1}(\alpha_1)\vee\hat{\lambda}_{\mathbb{F}_1}((\beta_1).$

Example 3.20. We consider two FQNG $\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}$ as shown in Fig. 3. Then, the degree of the vertex (S_1,t_1) is $d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_1,t_1) = ((d_T)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1),((d_C)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1),((d_U)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1),((d_F)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1)) = (0.8,1.0,1.1,1.1)$, and the total degree of the vertex (S_1,t_1) is $td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_1,t_1) = ((td_T)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1),((td_C)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1),((td_U)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1),$
$((td_F)\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}(S_1,t_1)) = (1.4,1.5,1.8,1.8)$. Similarly the degree and the total degree of the remaining vertices is given by:
$d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_1,t_2) = (0.7,1.0,1.0,1.0), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_1,t_2) = (1.1,1.5,1.7,1.7), d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_1,t_3) = (0.9,1.0,1.1,1.1), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_1,t_3) = (1.4,1.4,1.8,1.8), d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_2,t_1) = (1.7,1.7,2.2,2.2), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_2,t_1) = (2.1,2.2,2.8,2.8), d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_2,t_2) = (1.5,1.7,2.0,2.0)$,
$td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_2,t_2) = (1.9,2.2,2.6,2.6),\ d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_2,t_3) = (1.8,1.9,2.2,2.2), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_2,t_3) = (2.2,2.3,2.9,2.9), d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_3,t_1) = (0.9,0.7,1.1,1.1), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_3,t_1) = (1.3,1.0,1.7,1.7), d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_3,t_2) = (0.8,0.7,1.0,1.0), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_3,t_2) = (1.2,1.0,1.6,1.6)$,
$d_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_3,t_3) = (0.9,0.8,1.1,1.1), td_{\hat{\mathbb{R}_1}\bullet\hat{\mathbb{R}_2}}(S_3,t_3) = (1.3,1.1,1.7,1.7)$

Definition 3.21. (Maximal Product)
Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1,\hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2,\hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1,\mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2,\mathbb{E}_2)$, respectively. Then the maximal product of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is denoted as $\hat{\mathbb{R}_1} * \hat{\mathbb{R}_2} = (\hat{\lambda}_1 * \hat{\lambda}_2),(\hat{\mu}_1 * \hat{\mu}_2)$ and is defined as follows:

i) ∀ (x,y) ∈ $\mathbb{V}_1\times\mathbb{V}_2$, $\hat{\lambda}_{\mathbb{T}_1} * \hat{\lambda}_{\mathbb{T}_1}(x,y) = \hat{\lambda}_{\mathbb{T}_1}(x)\wedge\hat{\lambda}_{\mathbb{T}_2}(y)$,
$\hat{\lambda}_{\mathbb{C}_1} * \hat{\lambda}_{\mathbb{C}_2}(x,y) = \hat{\lambda}_{\mathbb{C}_1}(x)\wedge\hat{\lambda}_{\mathbb{C}_2}(y)$,
$\hat{\lambda}_{\mathbb{U}_1} * \hat{\lambda}_{\mathbb{U}_2}(x,y) = \hat{\lambda}_{\mathbb{U}_1}(x)\vee\hat{\lambda}_{\mathbb{U}_2}(y)$,
indent $\hat{\lambda}_{\mathbb{F}_1} * \hat{\lambda}_{\mathbb{F}_2}(x,y) = \hat{\lambda}_{\mathbb{F}_1}(x)\vee\hat{\lambda}_{\mathbb{F}_2}(y)$.

ii) ∀ x ∈ $\mathbb{V}_2$ and (y,z) ∈ $\mathbb{E}_1$,
a) $\hat{\mu}_{\mathbb{T}_1} * \hat{\mu}_{\mathbb{T}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{T}_1}(x)\wedge\hat{\lambda}_{\mathbb{T}_2}(y,z)$;
b) $\hat{\mu}_{\mathbb{C}_1} * \hat{\mu}_{\mathbb{C}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{C}_1}(x)\wedge\hat{\lambda}_{\mathbb{C}_2}(y,z)$;
c) $\hat{\mu}_{\mathbb{U}_1} * \hat{\mu}_{\mathbb{U}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{U}_1}(x)\vee\hat{\lambda}_{\mathbb{U}_2}(y,z)$;
d) $\hat{\mu}_{\mathbb{F}_1} * \hat{\mu}_{\mathbb{F}_2}((x,y),(x,z)) = \hat{\lambda}_{\mathbb{F}_1}(x)\vee\hat{\lambda}_{\mathbb{F}_2}(y,z)$.

iii)$\forall$ x $\in \mathbb{V}_2$ and (y,z) $\in \mathbb{E}_1$,

a) $\hat{\mu}_{\mathbb{T}_1} * \hat{\mu}_{\mathbb{T}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{T}_1}(y,z) \wedge \hat{\lambda}_{\mathbb{T}_2}(x)$;

b) $\hat{\mu}_{\mathbb{C}_1} * \hat{\mu}_{\mathbb{C}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{C}_1}(y,z) \wedge \hat{\lambda}_{\mathbb{C}_2}(x)$;

c) $\hat{\mu}_{\mathbb{U}_1} * \hat{\mu}_{\mathbb{U}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{U}_1}(y,z) \vee \hat{\lambda}_{\mathbb{U}_2}(x)$;

d) $\hat{\mu}_{\mathbb{F}_1} * \hat{\mu}_{\mathbb{F}_2}((y,x),(z,x)) = \hat{\lambda}_{\mathbb{F}_1}(y,z) \vee \hat{\lambda}_{\mathbb{F}_2}(x)$.

Example 3.22. Consider two FQNGs $\hat{\mathbb{M}}_1$ and $\hat{\mathbb{M}}_2$ of the graphs G_1 and G_2 respectively as shown in Figure 4a and Figure 4b. The rejection $\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2$ of $\hat{\mathbb{M}}_1$ and $\hat{\mathbb{M}}_2$ is shown in Fig. 4.

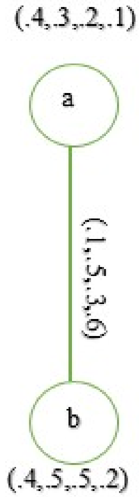


Fig. 4(a) FQNG $\widehat{\mathbb{M}}_1$

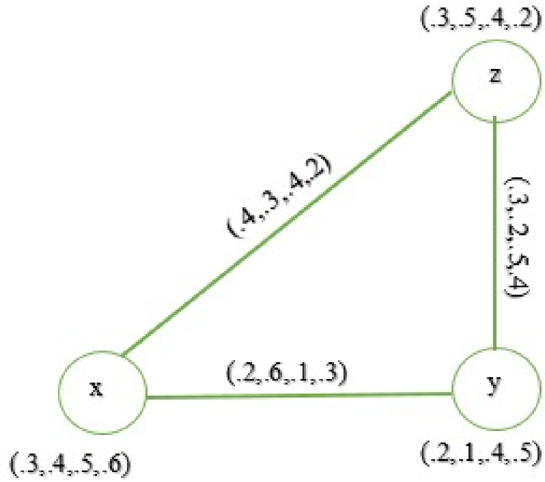


Fig. 4(b) FQNG $\widehat{\mathbb{M}}_2$

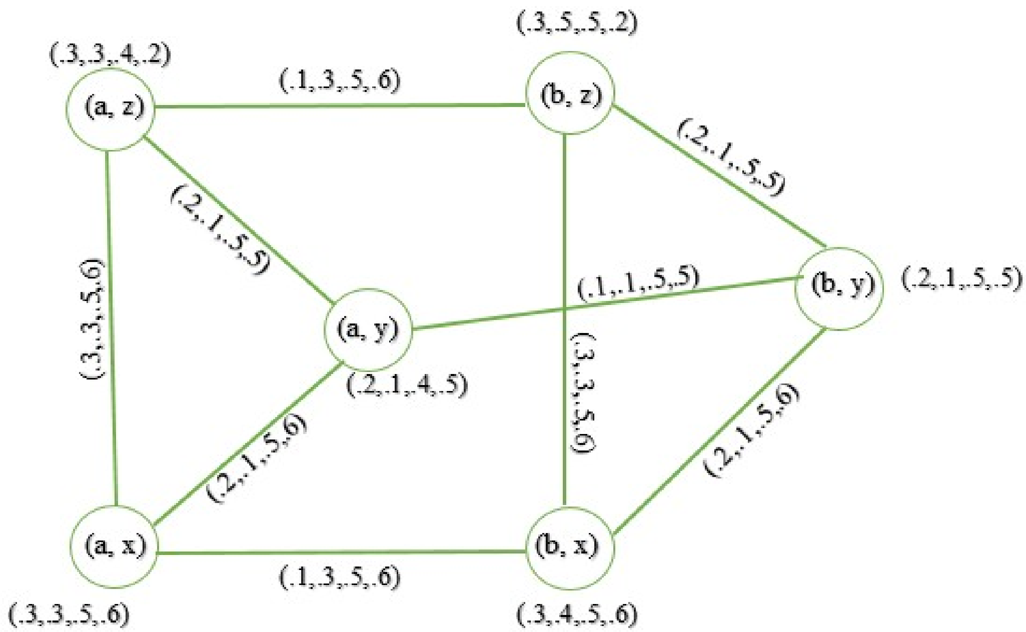


Fig. 4 FQNG $\widehat{\mathbb{M}}_1 * \widehat{\mathbb{M}}_2$

Theorem 3.23. *Let $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ be two FQNGs of the graph's G_1 and G_2 respectively. Then, the maximal product $\hat{\mathbb{R}_1} * \hat{\mathbb{R}_2}$ of $\hat{\mathbb{R}_1}$ and $\hat{\mathbb{R}_2}$ is a FQNG.*

Proof. Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively.

i) Let $\alpha_1 \in \mathbb{V}_1$ and $(\gamma_1, \delta_1) \in \mathbb{E}_2$. Then :

$$\begin{aligned}
&\hat{\mu}_{\mathbb{T}_1} * \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\gamma_1),(\alpha_1,\delta_1))\\
&\quad = \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1,\delta_1)\\
&\quad = (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{T}_1} * \hat{\lambda}_{\mathbb{T}_2})(\alpha_1,\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{T}_1} * \hat{\lambda}_{\mathbb{T}_2})(\alpha_1,\delta_1))
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{C}_1} * \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\gamma_1),(\alpha_1,\delta_1))\\
&\quad = \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1,\delta_1)\\
&\quad = (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{C}_1} * \hat{\lambda}_{\mathbb{C}_2})(\alpha_1,\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{C}_1} * \hat{\lambda}_{\mathbb{C}_2})(\alpha_1,\delta_1))
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{U}_1} * \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\gamma_1),(\alpha_1,\delta_1))\\
&\quad = \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1,\delta_1)\\
&\quad = (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{U}_1} * \hat{\lambda}_{\mathbb{U}_2})(\alpha_1,\gamma_1)) \vee (\hat{\lambda}_{\mathbb{U}_1} * \hat{\lambda}_{\mathbb{U}_2})(\alpha_1,\delta_1))
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{F}_1} * \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\gamma_1),(\alpha_1,\delta_1))\\
&\quad = \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1,\delta_1)\\
&\quad = (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\delta_1))\\
&\quad = (\hat{\lambda}_{\mathbb{F}_1} * \hat{\lambda}_{\mathbb{F}_2})(\alpha_1,\gamma_1)) \vee (\hat{\lambda}_{\mathbb{F}_1} * \hat{\lambda}_{\mathbb{F}_2})(\alpha_1,\delta_1))
\end{aligned}$$

ii) Let $(\alpha_1, \beta_1) \in \mathbb{E}_1$ and $\gamma_1 \in \mathbb{V}_2$. Then :

$$\begin{aligned}
&\hat{\mu}_{\mathbb{T}_1} * \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\gamma_1),(\beta_1,\gamma_1))\\
&\quad = \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1)\\
&\quad = (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1))\\
&\quad = (\hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{T}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\gamma_1))\\
&\quad = (\hat{\lambda}_{\mathbb{T}_1} * \hat{\lambda}_{\mathbb{T}_2})(\alpha_1,\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{T}_1} * \hat{\lambda}_{\mathbb{T}_2})(\beta_1,\gamma_1))
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{C}_1} * \hat{\mu}_{\mathbb{C}_2}((\alpha_1, \gamma_1), (\beta_1, \gamma_1)) \\
&\quad = \hat{\mu}_{\mathbb{C}_1}(\alpha_1, \beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1) \\
&\quad = (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)) \wedge (\hat{\lambda}_{\mathbb{C}_1}(\beta_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\gamma_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{C}_1} * \hat{\lambda}_{\mathbb{C}_2})(\alpha_1, \gamma_1)) \wedge (\hat{\lambda}_{\mathbb{C}_1} * \hat{\lambda}_{\mathbb{C}_2})(\beta_1, \gamma_1))
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{U}_1} * \hat{\mu}_{\mathbb{U}_2}((\alpha_1, \gamma_1), (\beta_1, \gamma_1)) \\
&\quad = \hat{\mu}_{\mathbb{U}_1}(\alpha_1, \beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1) \\
&\quad = (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{U}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{U}_2}(\gamma_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{U}_1} * \hat{\lambda}_{\mathbb{U}_2})(\alpha_1, \gamma_1)) \vee (\hat{\lambda}_{\mathbb{U}_1} * \hat{\lambda}_{\mathbb{U}_2})(\beta_1, \gamma_1))
\end{aligned}$$

$$\begin{aligned}
&\hat{\mu}_{\mathbb{F}_1} * \hat{\mu}_{\mathbb{F}_2}((\alpha_1, \gamma_1), (\beta_1, \gamma_1)) \\
&\quad = \hat{\mu}_{\mathbb{F}_1}(\alpha_1, \beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1) \\
&\quad = (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee (\hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)) \vee (\hat{\lambda}_{\mathbb{F}_1}(\beta_1) \vee \hat{\lambda}_{\mathbb{F}_2}(\gamma_1)) \\
&\quad = (\hat{\lambda}_{\mathbb{F}_1} * \hat{\lambda}_{\mathbb{F}_2})(\alpha_1, \gamma_1)) \vee (\hat{\lambda}_{\mathbb{F}_1} * \hat{\lambda}_{\mathbb{F}_2})(\beta_1, \gamma_1)).
\end{aligned}$$

This completes the proof. □

Definition 3.24. (Degree of the Maximal product of two FQNGs).
Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$\begin{aligned}
&(d_T)_{\hat{\mathbb{R}_1} * \hat{\mathbb{R}_2}}(\alpha_1, \beta_1) \\
&\quad = \textstyle\sum_{(\alpha_1, \beta_1), (\gamma_1, \delta_1)) \in \mathbb{E}_1 \times \mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} * \hat{\mu}_{\mathbb{T}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1)) \\
&\quad = \textstyle\sum_{\alpha_1 = \gamma_1 \in \mathbb{V}_1, (\beta_1, \delta_1) \in \mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{T}_2}((\beta_1, \delta_1) + \\
&\quad \textstyle\sum_{(\alpha_1, \gamma_1) \in \mathbb{E}_1, \beta_1 = \delta_1 \in \mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1, \gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}((\gamma_1) \\
&(d_C)_{\hat{\mathbb{R}_1} * \hat{\mathbb{R}_2}}(\alpha_1, \beta_1) \\
&\quad = \textstyle\sum_{(\alpha_1, \beta_1), (\gamma_1, \delta_1)) \in \mathbb{E}_1 \times \mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} * \hat{\mu}_{\mathbb{C}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1)) \\
&\quad = \textstyle\sum_{\alpha_1 = \gamma_1 \in \mathbb{V}_1, (\beta_1, \delta_1) \in \mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{C}_2}((\beta_1, \delta_1) + \\
&\quad \textstyle\sum_{(\alpha_1, \gamma_1) \in \mathbb{E}_1, \beta_1 = \delta_1 \in \mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1, \gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}((\gamma_1) \\
&(d_U)_{\hat{\mathbb{R}_1} * \hat{\mathbb{R}_2}}(\alpha_1, \beta_1) \\
&\quad = \textstyle\sum_{(\alpha_1, \beta_1), (\gamma_1, \delta_1)) \in \mathbb{E}_1 \times \mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} * \hat{\mu}_{\mathbb{U}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1)) \\
&\quad = \textstyle\sum_{\alpha_1 = \gamma_1 \in \mathbb{V}_1, (\beta_1, \delta_1) \in \mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{U}_2}((\beta_1, \delta_1) + \\
&\quad \textstyle\sum_{(\alpha_1, \gamma_1) \in \mathbb{E}_1, \beta_1 = \delta_1 \in \mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1, \gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}((\gamma_1) \\
&(d_F)_{\hat{\mathbb{R}_1} * \hat{\mathbb{R}_2}}(\alpha_1, \beta_1) \\
&\quad = \textstyle\sum_{(\alpha_1, \beta_1), (\gamma_1, \delta_1)) \in \mathbb{E}_1 \times \mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} * \hat{\mu}_{\mathbb{F}_2}((\alpha_1, \beta_1), (\gamma_1, \delta_1))
\end{aligned}$$

$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{F}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}((\gamma_1)$

Definition 3.25. (Total Degree of the Maximal Product of two FQNGs)
Let $\hat{\mathbb{R}_1} = (\hat{\lambda}_1, \hat{\mu}_1)$ and $\hat{\mathbb{R}_2} = (\hat{\lambda}_2, \hat{\mu}_2)$ be two FQNGs of the graphs $\mathsf{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathsf{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. For any vertex $(\alpha_1, \beta_1) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$(td_T)_{\hat{\mathbb{R}_1}*\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{T}_1} * \hat{\mu}_{\mathbb{T}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{T}_1} \mid \hat{\lambda}_{\mathbb{T}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{T}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{T}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{T}_2}((\gamma_1) + \hat{\lambda}_{\mathbb{T}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{T}_2}(\beta_1)$,
$(td_C)_{\hat{\mathbb{R}_1}*\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{C}_1} * \hat{\mu}_{\mathbb{C}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{C}_1} \mid \hat{\lambda}_{\mathbb{C}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\mu}_{\mathbb{C}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{C}_1}(\alpha_1,\gamma_1) \wedge \hat{\lambda}_{\mathbb{C}_2}((\gamma_1) + \hat{\lambda}_{\mathbb{C}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{C}_2}(\beta_1)$,
$(td_U)_{\hat{\mathbb{R}_1}*\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{U}_1} * \hat{\mu}_{\mathbb{U}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{U}_1} \mid \hat{\lambda}_{\mathbb{U}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{U}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{U}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{U}_2}((\gamma_1) + \hat{\lambda}_{\mathbb{U}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{U}_2}(\beta_1)$,
$(td_F)_{\hat{\mathbb{R}_1}*\hat{\mathbb{R}_2}}(\alpha_1,\beta_1)$
$= \sum_{(\alpha_1,\beta_1),(\gamma_1,\delta_1))\in\mathbb{E}_1\times\mathbb{E}_2} \hat{\mu}_{\mathbb{F}_1} * \hat{\mu}_{\mathbb{F}_2}((\alpha_1,\beta_1),(\gamma_1,\delta_1)) + \hat{\lambda}_{\mathbb{F}_1} \mid \hat{\lambda}_{\mathbb{F}_2}(\alpha_1,\beta_1)$
$= \sum_{\alpha_1=\gamma_1\in\mathbb{V}_1,(\beta_1,\delta_1)\in\mathbb{E}_2} \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \vee \hat{\mu}_{\mathbb{F}_2}((\beta_1,\delta_1) +$
$\sum_{(\alpha_1,\gamma_1)\in\mathbb{E}_1,\beta_1=\delta_1\in\mathbb{V}_2} \hat{\mu}_{\mathbb{F}_1}(\alpha_1,\gamma_1) \vee \hat{\lambda}_{\mathbb{F}_2}((\gamma_1) + \hat{\lambda}_{\mathbb{F}_1}(\alpha_1) \wedge \hat{\lambda}_{\mathbb{F}_2}(\beta_1)$.

Example 3.26. We consider two FQNG $\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2$ as shown in Fig. 4. Then, the degree of the vertex (a,x) is $d_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(a,x) = ((d_T)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x), ((d_C)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x), ((d_U)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x), ((d_F)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x)) = (0.6, 0.7, 1.5, 1.8)$, and the total degree of the vertex (a,x) is $td_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(a,x) = ((td_T)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x), ((td_C)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x), ((td_U)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x),$
$((td_F)\hat{\mathbb{M}}_1 * \hat{\mathbb{M}}_2(a,x)) = (0.9, 1.0, 2.02.4)$. Similarly the degree and the total degree of the remaining vertices is given by:

$d_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(b,x) = (0.6, 0.7, 1.5, 1.8), td_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(b,x) = (0.9, 1.1, 2.0, 2.4), d_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(a,y) = (0.5, 0.3, 1.5, 1.6), td_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(a,y) = (0.7, 0.4, 1.9, 2.1), d_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(b,y) = (0.5, 0.3, 1.5, 1.6), td_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(b,y) = (0.7, 0.4, 2.0, 2.1), d_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(b,z) = (0.6, 0.7, 1.5, 1.7),$
$td_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(b,z) = (0.9, 1.2, 2.0, 1.9), d_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(a,z) = (0.6, 0.7, 1.5, 1.7), td_{\hat{\mathbb{M}}_1*\hat{\mathbb{M}}_2}(a,z) = (0.9, 1.0, 1.9, 1.9)$

4. Arithmetic Operator and Geometric Average Operator in Fermatean Quadripartitioned Neutrosophic Environment

Definition 4.1. Let $\hat{\mathbb{R}} = (\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{R}}}}, \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{R}}}}, \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{R}}}}, \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{R}}}})$ and $\hat{\mathbb{S}} = (\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{S}}}}, \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{S}}}}, \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{S}}}}, \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{S}}}})$ be two single-valued fermatean quadripartitioned neutrosophic sets. Then, the operational relations between $\hat{\mathbb{R}}$ and $\hat{\mathbb{S}}$ are defined as follows:

i) $\hat{\mathbb{R}} + \hat{\mathbb{S}} = (\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{R}}}}(x) + \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{S}}}}(x) - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{S}}}}(x), \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{R}}}}(x) + \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{S}}}}(x) - \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{S}}}}(x),$
$\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{R}}}}(x) + \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{S}}}}(x) - \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{S}}}}(x), \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{R}}}}(x) + \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{S}}}}(x) - \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{S}}}}(x),$

ii) $\hat{\mathbb{R}} \cdot \hat{\mathbb{S}} = (\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{S}}}}(x), \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{S}}}}(x), \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{S}}}}(x), \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{R}}}}(x)\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{S}}}}(x),$

iii) $\mathrm{a}\hat{\mathbb{R}} = (1 - (\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{R}}}}(x))^a, 1 - (\hat{\lambda}^3_{\mathbb{C}\hat{\mathbb{R}}}(x))^a, \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{R}}}}(x))^a, \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{R}}}}(x))^a),$

iv) $\hat{\mathbb{R}}^a = (\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{R}}}}(x))^a, (\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{R}}}}(x))^a, 1 - (\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{R}}}}(x))^a, 1 - (\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{R}}}}(x))^a), a > 0.$

Definition 4.2. Let $\hat{\mathbb{P}}_r(r = 1, 2, 3, ..., n)$ be n numbers single- valued fermatean quadripartitioned neutrosophic sets. Then, the fermatean quadripartitioned neutrosophic weighted arithmetic average operator of $\hat{\mathbb{R}}_r$ with weight $w = (w_1, w_2, w_3, ..., w_n)^t$ is denoted by $FQNWA^w(\hat{\mathbb{R}_1}, \hat{\mathbb{R}_2}, \hat{\mathbb{R}_3}, ..., \hat{\mathbb{R}_n})$ and is defined as follows:

$FQNWA^w(\hat{\mathbb{R}_1}, \hat{\mathbb{R}_2}, \hat{\mathbb{R}_3}, ..., \hat{\mathbb{R}_n}) = \sum_{r=1}^n w_r \hat{\mathbb{R}}_r$

$= (1 - \prod_{r=1}^n (1 - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{R}_r}}}(x))^{w_r}, 1 - \prod_{r=1}^n (1 - \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{R}_r}}}(x))^{w_r}, \prod_{r=1}^n (\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{R}_r}}}(x))^{w_r}, \prod_{r=1}^n (\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{R}_r}}}(x))^{w_r})$

where $\sum_{r=1}^n w_r = 1$.

Definition 4.3. Let $\hat{\mathbb{Q}}_r(r = 1, 2, 3, ..., n)$ be n numbers single- valued fermatean quadripartitioned neutrosophic sets. Then, the fermatean quadripartitioned neutrosophic weighted geometric average operator of $\hat{\mathbb{S}}_r$ with weight $w = (w_1, w_2, w_3, ..., w_n)^t$ is denoted by $FQNWG^w(\hat{\mathbb{S}_1}, \hat{\mathbb{S}_2}, \hat{\mathbb{S}_3}, ..., \hat{\mathbb{S}_n})$ and is defined as follows:

$FQNWG^w(\hat{\mathbb{S}_1}, \hat{\mathbb{S}_2}, \hat{\mathbb{S}_3}, ..., \hat{\mathbb{S}_n}) = \prod_{r=1}^n \hat{\mathbb{S}_r}^{w_r}$

$= (\prod_{r=1}^n (\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{S}_r}}}(x))^{w_r}, \prod_{r=1}^n (\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{S}_r}}}(x))^{w_r}, (1 - \prod_{r=1}^n (1 - \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{S}_r}}}(x))^{w_r}, (1 - \prod_{r=1}^n (1 - \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{S}_r}}}(x))^{w_r}),$

where $\prod_{r=1}^n w_r = 1$.

Score and Accuracy Value

Let $\hat{\mathbb{R}} = (\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{R}}}}, \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{R}}}}, \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{R}}}}, \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{R}}}})$ be a single-valued fermatean quadripartitioned neutrosophic number. Then, its score and accuracy value is denoted by $\mathrm{S}(\hat{\mathbb{R}})$ and $\mathrm{H}(\hat{\mathbb{R}})$, respectively, and is defined as follows:

$\mathrm{S}(\hat{\mathbb{R}}) = \frac{2 + \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{R}}}} + \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{R}}}} - \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{R}}}} - \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{R}}}}}{4}$ and $\mathrm{H}(\hat{\mathbb{R}}) = (\hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{R}}}} - \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{R}}}})$.

Comparison Between Quadripartitioned Neutrosophic Numbers

Let $\hat{\mathbb{R}}$ and $\hat{\mathbb{T}}$ be two fermatean quadripartitioned neutrosophic numbers on the universe of discourse $\mathbb{X}$. Then the relation between $\hat{\mathbb{R}}$ and $\hat{\mathbb{T}}$ is defined as follows:

i) If $\mathrm{S}(\hat{\mathbb{R}}) > \mathrm{S}(\hat{\mathbb{T}}), \hat{\mathbb{R}} > \hat{\mathbb{T}}$.

ii) $\mathrm{S}(\hat{\mathbb{R}}) < \mathrm{S}(\hat{\mathbb{T}}), \hat{\mathbb{R}} < \hat{\mathbb{T}}$.

iii) $\mathrm{S}(\hat{\mathbb{R}}) = \mathrm{S}(\hat{\mathbb{T}}), \hat{\mathbb{R}} \approx \hat{\mathbb{T}}$, then we consider the following cases:

a) If $\mathrm{H}(\hat{\mathbb{R}}) > \mathrm{H}(\hat{\mathbb{T}}), \hat{\mathbb{R}} > \hat{\mathbb{T}}$.

b) $\mathrm{H}(\hat{\mathbb{R}}) < \mathrm{H}(\hat{\mathbb{T}}), \hat{\mathbb{R}} < \hat{\mathbb{T}}$.

c) $\mathrm{H}(\hat{\mathbb{R}}) = \mathrm{H}(\hat{\mathbb{T}}), \hat{\mathbb{R}} \approx \hat{\mathbb{T}}$.

Let $\hat{\mathbb{A}} = \{\hat{\mathbb{A}}_1, \hat{\mathbb{A}}_2, ..., \hat{\mathbb{A}}_m\}$ be the set of alternatives and $\mathrm{C} = \{\mathrm{C}_1, \mathrm{C}_2, ..., \mathrm{C}_n$ be the set of criteria. We consider that w_j be the weight corresponding to the criteria $\mathrm{C}_j (j = 1, 2, ..., n)$, such that $\sum_j = 1^n w_j = 1$. In the decision-making problem, the evaluation information of the alternative $\hat{\mathbb{A}}_i (i = 1, 2, ..., m)$ on the criteria is represented by the form of a single-valued neutrosophic number.,

$\hat{\mathbb{A}}_i = \{(\mathrm{C}_j, \hat{\lambda}_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j), \hat{\lambda}_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j), \hat{\lambda}_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j), \hat{\lambda}_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j)) : 0 \leq \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j) + \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j) + \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j) + \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j) \leq 2, \forall j = 1, 2, ..., n\}$

$= (\hat{\lambda}_{\mathbb{T}_{i,j}}, \hat{\lambda}_{\mathbb{C}_{i,j}}, \hat{\lambda}_{\mathbb{U}_{i,j}}, \hat{\lambda}_{\mathbb{F}_{i,j}})$.

Let $\hat{\eth}_{i,j} = (\hat{\lambda}_{\mathbb{T}_{i,j}}, \hat{\lambda}_{\mathbb{C}_{i,j}}, \hat{\lambda}_{\mathbb{U}_{i,j}}, \hat{\lambda}_{\mathbb{F}_{i,j}})$, i = 1,2,...,m , j = 1,2,...,n.

Step 1 : We construct single-valued fermatean quadripartitioned neutrosophic decision matrix:

$$\hat{\mathbb{D}}_{m,n} = (\hat{\eth}_{i,j})_{m,n}$$

$$= \begin{matrix} (\hat{\lambda}_{\mathbb{T}_{1,1}}, \hat{\lambda}_{\mathbb{C}_{1,1}}, \hat{\lambda}_{\mathbb{U}_{1,1}}, \hat{\lambda}_{\mathbb{F}_{1,1}}) & (\hat{\lambda}_{\mathbb{T}_{1,2}}, \hat{\lambda}_{\mathbb{C}_{1,2}}, \hat{\lambda}_{\mathbb{U}_{1,2}}, \hat{\lambda}_{\mathbb{F}_{1,2}}) \cdots & (\hat{\lambda}_{\mathbb{T}_{1,n}}, \hat{\lambda}_{\mathbb{C}_{1,n}}, \hat{\lambda}_{\mathbb{U}_{1,n}}, \hat{\lambda}_{\mathbb{F}_{1,n}}) \\ (\hat{\lambda}_{\mathbb{T}_{2,1}}, \hat{\lambda}_{\mathbb{C}_{2,1}}, \hat{\lambda}_{\mathbb{U}_{2,1}}, \hat{\lambda}_{\mathbb{F}_{2,1}}) & (\hat{\lambda}_{\mathbb{T}_{2,2}}, \hat{\lambda}_{\mathbb{C}_{2,2}}, \hat{\lambda}_{\mathbb{U}_{2,2}}, \hat{\lambda}_{\mathbb{F}_{2,2}}) \cdots & (\hat{\lambda}_{\mathbb{T}_{2,n}}, \hat{\lambda}_{\mathbb{C}_{2,n}}, \hat{\lambda}_{\mathbb{U}_{2,n}}, \hat{\lambda}_{\mathbb{F}_{2,n}}) \\ \vdots & \vdots & \vdots \\ (\hat{\lambda}_{\mathbb{T}_{m,1}}, \hat{\lambda}_{\mathbb{C}_{m,1}}, \hat{\lambda}_{\mathbb{U}_{m,1}}, \hat{\lambda}_{\mathbb{F}_{m,1}}) & (\hat{\lambda}_{\mathbb{T}_{m,2}}, \hat{\lambda}_{\mathbb{C}_{m,2}}, \hat{\lambda}_{\mathbb{U}_{m,2}}, \hat{\lambda}_{\mathbb{F}_{m,2}}) \cdots & (\hat{\lambda}_{\mathbb{T}_{m,n}}, \hat{\lambda}_{\mathbb{C}_{m,n}}, \hat{\lambda}_{\mathbb{U}_{m,n}}, \hat{\lambda}_{\mathbb{F}_{m,n}}) \end{matrix}$$

Step 2 : We aggregate the elements of each row of the decision matrix using fermatean quadripartitioned neutrosophic arithmetic average operator:

$$\hat{\mathbb{R}}_i = FQNWA^w(\hat{\eth}_{i,1}, \hat{\eth}_{i,2}, \hat{\eth}_{i,3}, ..., \hat{\eth}_{i,n})$$

$$= (1 - \textstyle\prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{T}_{i,j}}(x))^{w_j}, 1 - \prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{C}_{i,j}}(x))^{w_j}, \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{U}_{i,j}}(x))^{w_j}, \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{F}_{i,j}}(x))^{w_j})$$

$$= (1 - \textstyle\prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, 1 - \prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j})$$

Step 3: Now, we evaluate score and accuracy value of $\hat{\mathbb{R}}_i$:

$$\mathrm{S}(\hat{\mathbb{R}}_i) = \frac{2 + \{1 - \prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\} + \{1 - \prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\} - \{\prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\} - \{\prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\}}{4}$$

$$= \frac{4 - \prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j} + \prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j} - \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j} - \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}}{4}$$

and

$$\mathrm{H}(\hat{\mathbb{R}}_i) = 1 - \textstyle\prod_{j=1}^n (1 - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j} - \prod_{j=1}^n (\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}$$

Step 4 : According to score and accuracy value, we ranking $\hat{\mathbb{R}}_i$, i.e., $\hat{\mathbb{A}}_i$, i = 1,2,...,n.

Now, using fermatean quadripartitioned neutrosophic arithmetic average operator on the fermatean quadripartitioned neutrosophic decision matrix, we have solved the decision - making problem.

Step 2 : We aggregate the elements of each row of the decision matrix using fermatean quadripartitioned neutrosophic geometric average operator.

$\hat{\mathbb{Q}}_i = FQNWG^w(\hat{\eth}_{i,1}, \hat{\eth}_{i,2}, \hat{\eth}_{i,3}, ..., \hat{\eth}_{i,n})$

$= (\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{T}_{i,j}}(x))^{w_j}, \prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{C}_{i,j}}(x))^{w_j}), 1-\prod_{j=1}^{n}(1-\hat{\lambda}^3_{\mathbb{U}_{i,j}}(x))^{w_j}, 1-\prod_{j=1}^{n}(1-\hat{\lambda}^3_{\mathbb{F}_{i,j}}(x))^{w_j})$

$= (\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, \prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, 1 - \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, 1 - \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j},)$

Step 3:

$$\mathrm{S}(\hat{\mathbb{Q}}_i) = \frac{2+\{\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\}+\{\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\}-\{1-\prod_{j=1}^{n}(1-\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\}-\{1-\prod_{j=1}^{n}(1-\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}\}}{3}$$

$$= \frac{\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}+\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}+\prod_{j=1}^{n}(1-\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}+\prod_{j=1}^{n}(1-\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}}{3}$$

and

$$\mathrm{H}(\hat{\mathbb{Q}}_i) = \prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j} - 1 + \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}$$

Step 4 : According to score and accuracy value, we ranking $\hat{\mathbb{R}}_i$, i.e., $\hat{\mathbb{A}}_i$, i = 1,2,...,n.

Fermatean Quadripartitioned Neutrosophic Decision - Making Problem for the Selection of Best

We consider a fermatean quadripartitioned neutrosophic decision - making problem for choosing the best holiday destination from four destination. Let the four holiday destination be $D_1, D_2, D_3 and D_4$. The decision - maker must decide according to the following four criteria:

1. C_1 is the Travel costs.
2. C_2 is the Weather preferences.
3. C_3 is the Available activities.
4. C_4 is the Cultural Experiences.

The weight vector of the criteria is $\omega = (0.2, 0.3, 0.4, 0.1)$. When an expert gives his/her opinion about choosing holiday destination D_1 under criteria C_1, he/she must consider that the possibility in which the report is correct is 0.4, and the report is accurate is 0.2, the report is not sure is 0.3 and he/she ignoring the report is 0.5. Therefore, in Fermatean Quadripartitioned Neutrosophic sense it can be written as $\hat{\eth}_{1,1} = (0.4, 0.3, 0.5, 0.4)$. If the expert considers the four alternatives A_1, A_2, A_3, A_4 concerning four different criteria C_1, C_2, C_3, C_4 we obtain the Fermatean Quadripartitioned Neutrosophic decision matrix as follows:

$\hat{\mathbb{D}}_{m,n} = (\hat{\eth}_{i,j})_{m,n}$

$$= \begin{pmatrix} (0.4, 0.3, 0.5, 0.4) & (0.4, 0.6, 0.3, 0.2) & (0.1, 0.3, 0.5, 0.6) & (0.4, 0.3, 0.4, 0.6) \\ (0.5, 0.5, 0.4, 0.3) & (0.2, 0.5, 0.4, 0.5) & (0.4, 0.4, 0.5, 0.4) & (0.3, 0.4, 0.3, 0.4) \\ (0.5, 0.6, 0.4, 0.2) & (0.3, 0.7, 0.3, 0.5) & (0.4, 0.4, 0.5, 0.5) & (0.5, 0.6, 0.4, 0.1) \\ (0.8, 0.5, 0.5, 0.1) & (0.5, 0.6, 0.4, 0.3) & (0.2, 0.4, 0.6, 0.7) & (0.3, 0.4, 0.5, 0.4) \end{pmatrix}$$

By aggregating the elements of each row of the decision matrix by using fermatean quadripartitioned neutrosophic arithmetic average operator, we get:

$\hat{\mathbb{R}}_i = FQNWA^w(\hat{\eth}_{i,1}, \hat{\eth}_{i,2}, \hat{\eth}_{i,3}, ..., \hat{\eth}_{i,n})$

$=$

$(1 - \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, 1 - \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, \prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, \prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j})$

Hence :

$\hat{\mathbb{R}}_1 = (0.0391,\ 0.093,\ 0.0149,\ 0.345)$

$\hat{\mathbb{R}}_2 = (0.0459,\ 0.1129,\ 0.0114,\ 0.3266)$

$\hat{\mathbb{R}}_3 = (0.0895,\ 0.2531,\ 0.0114,\ 0.2040)$

$\hat{\mathbb{R}}_4 = (0.1625,\ 0.1417,\ 0.0342,\ 0.2384)$.

Now, we find the score value of $\hat{\mathbb{R}}_i$. S($\hat{\mathbb{R}}_1$) = 0.4899, S($\hat{\mathbb{R}}_2$) = 0.4917, S($\hat{\mathbb{R}}_3$) = 0.5021, S($\hat{\mathbb{R}}_4$) = 0.4984. Based on the value of S($\hat{\mathbb{R}}_i$) (i = 1,2,3,4), we rank the holiday destinations$\hat{D}_i$ (i = 1,2,3,4) as follows $\hat{D}_3 \succ \hat{D}_4 \succ \hat{D}_2 \succ \hat{D}_1$. Hence the best holiday destination is $\hat{D}_3$.

Now by aggregating the elements of each row of the decision matrix using fermatean quadripartitioned neutrosophic geometric average operator:

$\hat{\mathbb{Q}}_i = FQNWG^w(\hat{\eth}_{i,1}, \hat{\eth}_{i,2}, \hat{\eth}_{i,3}, ..., \hat{\eth}_{i,n})$

$= (\prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{T}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, \prod_{j=1}^{n}(\hat{\lambda}^3_{\mathbb{C}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}), 1 - \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{U}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j}, 1 - \prod_{j=1}^{n}(1 - \hat{\lambda}^3_{\mathbb{F}_{\hat{\mathbb{A}}_i}}(\mathrm{C}_j))^{w_j})$

$\hat{\mathbb{Q}}_1 = (0.0483,\ 0.0245,\ 0.1343,\ 0.0545)$

$\hat{\mathbb{Q}}_2 = (0.0704,\ 0.0552,\ 0.1107,\ 0.0289)$

$\hat{\mathbb{Q}}_3 = (0.1220,\ 0.1268,\ 0.1107,\ 0.0272)$

$\hat{\mathbb{Q}}_4 = (0.1067,\ 0.0650,\ 0.2060,\ 0.0502)$.

Now, we find the score value of $\hat{\mathbb{R}}_i$. S($\hat{\mathbb{Q}}_1$) = 0.4994, S($\hat{\mathbb{Q}}_2$) = 0.0.4998, S($\hat{\mathbb{Q}}_3$) = 0.0.5006, S($\hat{\mathbb{Q}}_4$) = 0.4816. Based on the value of S($\hat{\mathbb{Q}}_i$) (i = 1,2,3,4), we rank the holiday destinations$\hat{D}_i$ (i = 1,2,3,4) as follows $\hat{D}_3 \succ \hat{D}_2 \succ \hat{D}_1 \succ \hat{D}_4$. Hence the best holiday destination is $\hat{D}_3$.

5. CONCLUSION

Graph theory is applicable in various real-world situations, including operations research, computer networking, economics, systems analysis, urban traffic management, and transportation. However, in practical scenarios, uncertainty can arise in nearly every problem related to graph theory. FQNG is a well-established and valuable approach in graph theory that models uncertain decision-making issues. This paper introduces several new operations associated with FQNG. First, we outline the definitions of regular FQNG, complete FQNG, and strong FQNG. Various operations on FQNG, such as rejection, symmetric difference, maximal product, and residue product, are provided along with suitable examples, and significant theorems related to them are discussed. We have also covered theorems concerning total degree and degree within these operations, supported by relevant examples. We have also developed two new fermatean quadripartitioned neutrosophic operators (the fermatean quadripartitioned neutrosophic weighted arithmetic average operator and the fermatean quadripartitioned neutrosophic weighted geometric average operator) to address the fermatean quadripartitioned neutrosophic decision-making challenge. We applied these operators to a real-world scenario involving the selection of the best holiday destination using FQNG.

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