



Neutrosophic UP (BCC)-Subalgebras of Several Types: A Threshold-Based Analysis in UP (BCC)-Algebras

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Abstract. This paper presents a comprehensive exploration of neutrosophic UP-subalgebras within the framework of BCC-algebras, introducing a novel framework for their classification and characterization. By integrating neutrosophic set theory—which generalizes fuzzy and intuitionistic fuzzy sets—we systematically investigate three types of neutrosophic subalgebras: (\in, \in) -neutrosophic, $(q, \in \vee q)$ -neutrosophic, and their hybrid variants. A key contribution lies in establishing rigorous conditions under which neutrosophic \in -subsets, q -subsets, and $\in \vee q$ -subsets qualify as BCC-subalgebras, leveraging threshold-based analysis (e.g., λ, μ, ν parameters) to delineate membership and non-membership criteria. We further derive algebraic characterizations for (Φ, Ψ) -neutrosophic BCC-subalgebras, where $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and demonstrate their theoretical consistency through illustrative examples and counterexamples. Notably, the study reveals that neutrosophic q -subsets and $\in \vee q$ -subsets exhibit subalgebraic properties under specific threshold constraints ($\lambda, \mu > 0.5$; $\nu < 0.5$), extending classical results in fuzzy algebraic structures. Our findings contribute to the theoretical foundations of neutrosophic algebraic systems while offering a versatile tool for modeling uncertainty in logical algebras, decision-making, and beyond.

Keywords: BCC-algebra; (\in, \in) -neutrosophic BCC-subalgebra; (\in, q) -neutrosophic BCC-subalgebra; (q, \in) -neutrosophic BCC-subalgebra; (q, \in) -neutrosophic BCC-subalgebra; $(q, \in \vee q)$ -neutrosophic BCC-subalgebra.

1. Introduction

The concept of fuzzy sets (FSs) was first introduced by Zadeh [22], and it has since been widely applied in various real-world contexts. Numerous researchers have further developed this theory, leading to a variety of generalizations. Among them, the integration of FSs with other uncertainty models, such as soft sets and rough sets, has been actively explored [1, 3, 4].

Atanassov [2] extended the idea of FSs by proposing intuitionistic fuzzy sets (IFSs), which have shown greater flexibility and broader applicability. These sets have been applied in diverse fields, including medical diagnosis, optimization, and multicriteria decision-making [6–8]. Later, Smarandache [18] introduced neutrosophic sets in 1999 as a more comprehensive framework, which encompasses classical sets, FSs, IFSs, and interval-valued variants. Neutrosophic set theory has since found applications in many areas, with additional resources available at <http://fs.unm.edu/neutrosophy.htm>.

Iampan [11] introduced UP-algebras as a novel algebraic structure, which motivated extensive studies on their fuzzy and generalized forms. Somjanta et al. [19] and Guntasow et al. [9] investigated fuzzy sets within this setting, while Dokkhamdang et al. [5] proposed threshold-based fuzzy UP-subalgebras. Pongsumpao et al. [16] extended the study to fuzzy UP-subalgebras and fuzzy UP-ideals, and Tanamoon et al. [21] advanced the framework further to Q -fuzzy UP-subalgebras and Q -fuzzy UP-ideals. Intuitionistic fuzzy extensions were also explored: Kesorn et al. [14] examined intuitionistic fuzzy structures in UP-algebras, and Jana et al. [12] introduced quasi-coincidence for intuitionistic fuzzy points and analyzed $(\in, \in \vee q)$ -intuitionistic fuzzy BCI-subalgebras. Neutrosophic approaches have equally enriched the field: Songsaeng and Iampan [20] defined neutrosophic UP-subalgebras, filters, near filters, and strongly UP-ideals, while Rangasuk et al. [17] studied the more specialized class of neutrosophic \mathcal{N} -UP-structures. Finally, Jun et al. [13] demonstrated the equivalence of UP-algebras [11] and BCC-algebras [15], and following Komori's original formulation, the present study employs the term BCC-algebra for consistency.

This study investigates neutrosophic BCC-subalgebras in the setting of BCC-algebras and provides a structured classification of their types. Drawing on neutrosophic set theory, which extends both fuzzy and intuitionistic fuzzy frameworks, we focus on three central categories: (\in, \in) -neutrosophic, $(q, \in \vee q)$ -neutrosophic, and hybrid forms. The main contribution is to specify the precise conditions under which neutrosophic \in -subsets, q -subsets, and $\in \vee q$ -subsets constitute valid BCC-subalgebras, using threshold parameters (λ, μ, ν) to regulate truth, indeterminacy, and falsity membership. In addition, algebraic properties of generalized (Φ, Ψ) -neutrosophic BCC-subalgebras $(\Phi, \Psi \in \{\in, q, \in \vee q\})$ are derived, supported by illustrative examples and counterexamples. It is shown that neutrosophic q -subsets and $\in \vee q$ -subsets possess subalgebraic characteristics under particular thresholds $(\lambda, \mu > 0.5; \nu < 0.5)$, thereby extending earlier results in fuzzy algebra. Overall, this work enriches the theoretical foundations of neutrosophic algebraic systems and highlights their potential applications in uncertainty modeling, logical structures, and decision-making.

2. Preliminaries

Building upon Komori’s seminal work [15], the notion of BCC-algebras can be reformulated without explicitly relying on condition (6), yielding a more concise axiomatic framework as follows:

Definition 2.1. [11] An algebra $\mathfrak{U} = (\mathfrak{U}, \star, 0)$ of type $(2, 0)$ is referred to as a BCC-algebra (cf. [10]) provided that the following axioms are satisfied:

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) ((\hat{k} \star \hat{z}) \star ((\hat{j} \star \hat{k}) \star (\hat{j} \star \hat{z})) = 0), \tag{1}$$

$$(\forall \hat{j} \in \mathfrak{U}) (0 \star \hat{j} = \hat{j}), \tag{2}$$

$$(\forall \hat{j} \in \mathfrak{U}) (\hat{j} \star 0 = 0), \tag{3}$$

$$(\forall \hat{j}, \hat{k} \in \mathfrak{U}) (\hat{j} \star \hat{k} = 0 = \hat{k} \star \hat{j} \Rightarrow \hat{j} = \hat{k}). \tag{4}$$

Throughout this paper, we shall simply write \mathfrak{U} instead of $(\mathfrak{U}, \star, 0)$, unless otherwise specified. A binary relation \leq on \mathfrak{U} is defined by

$$(\forall \hat{j}, \hat{k} \in \mathfrak{U}) (\hat{j} \leq \hat{k} \Leftrightarrow \hat{j} \star \hat{k} = 0). \tag{5}$$

In the context of \mathfrak{U} , one can verify the validity of the following properties (cf. [11]).

$$(\forall \hat{j} \in \mathfrak{U}) (\hat{j} \leq \hat{j}) \tag{6}$$

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) (\hat{j} \leq \hat{k} \Rightarrow \hat{k} \star \hat{j} \leq \hat{k} \star \hat{z}) \tag{7}$$

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) (\hat{j} \leq \hat{k} \Rightarrow \hat{z} \star \hat{j} \leq \hat{z} \star \hat{k}) \tag{8}$$

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) (\hat{j} \leq \hat{k} \star \hat{z}) \tag{9}$$

$$(\forall \hat{j}, \hat{k} \in \mathfrak{U}) (\hat{k} \star \hat{j} \leq \hat{j} \Leftrightarrow \hat{j} = \hat{k} \star \hat{j}) \tag{10}$$

$$(\forall \hat{j}, \hat{k} \in \mathfrak{U}) (\hat{j} \leq \hat{k} \star \hat{j}) \tag{11}$$

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) ((\hat{j} \star \hat{k}) \star \hat{z} \leq \hat{k} \star \hat{z}) \tag{12}$$

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) (\hat{j} \leq \hat{k} \Rightarrow \hat{j} \leq \hat{k} \star \hat{z}) \tag{13}$$

$$(\forall \hat{j}, \hat{k}, \hat{z} \in \mathfrak{U}) ((\hat{j} \star \hat{k}) \star \hat{z} \leq \hat{j} \star (\hat{k} \star \hat{z})) \tag{14}$$

Definition 2.2. [11] Let \mathfrak{S} be a nonempty subset of \mathfrak{U} . Then \mathfrak{S} is said to be a BCC-subalgebra (BCCS) of \mathfrak{U} whenever

$$(\forall \hat{j}, \hat{k} \in \mathfrak{S}) (\hat{j} \star \hat{k} \in \mathfrak{S}). \tag{15}$$

Definition 2.3. [18] Given a nonempty set \mathfrak{U} , a neutrosophic set (NS) on \mathfrak{U} is described by the collection

$$\mathfrak{N} := \{(\hat{j}, \mathfrak{N}_{tr}(\hat{j}), \mathfrak{N}_{ind}(\hat{j}), \mathfrak{N}_{fa}(\hat{j})) : \hat{j} \in \mathfrak{U}\}, \tag{16}$$

where $\mathfrak{N}_{tr} : \mathfrak{U} \rightarrow [0, 1]$ represents the truth-membership function, $\mathfrak{N}_{ind} : \mathfrak{U} \rightarrow [0, 1]$ denotes the indeterminacy-membership function, and $\mathfrak{N}_{fa} : \mathfrak{U} \rightarrow [0, 1]$ corresponds to the falsity-membership function. For brevity, the NS in (16) will be denoted by $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$.

3. Neutrosophic BCC-subalgebras of several types

Given an NS $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ on a universe \mathfrak{U} , with parameters $\lambda, \mu \in (0, 1]$ and $\nu \in [0, 1)$, we introduce the following associated collections:

$$\begin{aligned} \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{tr}(\check{j}) \geq \lambda\}, \\ \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{ind}(\check{j}) \geq \mu\}, \\ \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{fa}(\check{j}) \leq \nu\}, \\ \mathfrak{Tr}_q(\mathfrak{N}, \lambda) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{tr}(\check{j}) + \lambda > 1\}, \\ \mathfrak{Ind}_q(\mathfrak{N}, \mu) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{ind}(\check{j}) + \mu > 1\}, \\ \mathfrak{Fa}_q(\mathfrak{N}, \nu) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{fa}(\check{j}) + \nu < 1\}, \\ \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{tr}(\check{j}) \geq \lambda \text{ or } \mathfrak{N}_{tr}(\check{j}) + \lambda > 1\}, \\ \mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \mu) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{ind}(\check{j}) \geq \mu \text{ or } \mathfrak{N}_{ind}(\check{j}) + \mu > 1\}, \\ \mathfrak{Fa}_{\in \vee q}(\mathfrak{N}, \nu) &= \{\check{j} \in \mathfrak{U} : \mathfrak{N}_{fa}(\check{j}) \leq \nu \text{ or } \mathfrak{N}_{fa}(\check{j}) + \nu < 1\}. \end{aligned}$$

The sets $\mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_{\in}(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_{\in}(\mathfrak{N}, \nu)$ are referred to as neutrosophic \in -subsets of \mathfrak{U} , while $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ are called neutrosophic q -subsets. Likewise, $\mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_{\in \vee q}(\mathfrak{N}, \nu)$ are identified as neutrosophic $\in \vee q$ -subsets. For $\Phi \in \{\in, q, \in \vee q\}$, any element of $\mathfrak{Tr}_{\Phi}(\mathfrak{N}, \lambda)$ (resp., $\mathfrak{Ind}_{\Phi}(\mathfrak{N}, \mu)$, $\mathfrak{Fa}_{\Phi}(\mathfrak{N}, \nu)$) is designated as a neutrosophic \mathfrak{Tr}_{Φ} -point (resp., \mathfrak{Ind}_{Φ} -point, \mathfrak{Fa}_{Φ} -point) with threshold value λ (resp., μ, ν).

It directly follows that

$$\begin{aligned} \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda) &= \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda) \cup \mathfrak{Tr}_q(\mathfrak{N}, \lambda), \\ \mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \mu) &= \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu) \cup \mathfrak{Ind}_q(\mathfrak{N}, \mu), \\ \mathfrak{Fa}_{\in \vee q}(\mathfrak{N}, \nu) &= \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu) \cup \mathfrak{Fa}_q(\mathfrak{N}, \nu). \end{aligned}$$

Definition 3.1. Let $\Phi, \Psi \in \{\in, q, \in \vee q\}$. An NS $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ on \mathfrak{U} is said to form a (Φ, Ψ) -neutrosophic BCC-subalgebra ((Φ, Ψ)-NBCCS) of \mathfrak{U} whenever the following conditions are satisfied:

$$(\forall \check{j}, \check{k} \in \mathfrak{U}) \left(\begin{array}{l} \check{j} \in \mathfrak{Tr}_{\Phi}(\mathfrak{N}, \lambda_{\check{j}}), \check{k} \in \mathfrak{Tr}_{\Phi}(\mathfrak{N}, \lambda_{\check{k}}) \Rightarrow \check{j} * \check{k} \in \mathfrak{Tr}_{\Psi}(\mathfrak{N}, \lambda_{\check{j}} \wedge \lambda_{\check{k}}), \\ \check{j} \in \mathfrak{Ind}_{\Phi}(\mathfrak{N}, \mu_{\check{j}}), \check{k} \in \mathfrak{Ind}_{\Phi}(\mathfrak{N}, \mu_{\check{k}}) \Rightarrow \check{j} * \check{k} \in \mathfrak{Ind}_{\Psi}(\mathfrak{N}, \mu_{\check{j}} \wedge \mu_{\check{k}}), \\ \check{j} \in \mathfrak{Fa}_{\Phi}(\mathfrak{N}, \nu_{\check{j}}), \check{k} \in \mathfrak{Fa}_{\Phi}(\mathfrak{N}, \nu_{\check{k}}) \Rightarrow \check{j} * \check{k} \in \mathfrak{Fa}_{\Psi}(\mathfrak{N}, \nu_{\check{j}} \vee \nu_{\check{k}}), \end{array} \right) \quad (17)$$

for all $\lambda_{\check{j}}, \lambda_{\check{k}}, \mu_{\check{j}}, \mu_{\check{k}} \in (0, 1]$ and $\nu_{\check{j}}, \nu_{\check{k}} \in [0, 1)$.

Theorem 3.2. *Let $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ be an NS on \mathfrak{U} . Then \mathfrak{N} constitutes a (\in, \in) -NBCCS of \mathfrak{U} precisely when the following inequalities hold:*

$$(\forall \check{j}, \check{\ddot{j}} \in \mathfrak{U}) \left(\begin{array}{l} \mathfrak{N}_{tr}(\check{j} \star \check{\ddot{j}}) \geq \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{\ddot{j}}), \\ \mathfrak{N}_{ind}(\check{j} \star \check{\ddot{j}}) \geq \mathfrak{N}_{ind}(\check{j}) \wedge \mathfrak{N}_{ind}(\check{\ddot{j}}), \\ \mathfrak{N}_{fa}(\check{j} \star \check{\ddot{j}}) \leq \mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{\ddot{j}}). \end{array} \right) \tag{18}$$

Proof. Suppose that $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (\in, \in) -NBCCS of \mathfrak{U} . If there exist $\check{j}, \check{\ddot{j}} \in \mathfrak{U}$ such that $\mathfrak{N}_{tr}(\check{j} \star \check{\ddot{j}}) < \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{\ddot{j}})$, then one can choose $\lambda_{\check{s}} \in (0, 1]$ with $\mathfrak{N}_{tr}(\check{j} \star \check{\ddot{j}}) < \lambda_{\check{s}} \leq \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{\ddot{j}})$. This would imply $\check{j}, \check{\ddot{j}} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\check{s}})$ but $\check{j} \star \check{\ddot{j}} \notin \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\check{s}})$, contradicting the assumption. Hence, the inequality $\mathfrak{N}_{tr}(\check{j} \star \check{\ddot{j}}) \geq \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{\ddot{j}})$ must hold for all $\check{j}, \check{\ddot{j}} \in \mathfrak{U}$. By a parallel argument, one also obtains $\mathfrak{N}_{ind}(\check{j} \star \check{\ddot{j}}) \geq \mathfrak{N}_{ind}(\check{j}) \wedge \mathfrak{N}_{ind}(\check{\ddot{j}})$ for every $\check{j}, \check{\ddot{j}} \in \mathfrak{U}$. Finally, if there exist $\check{a}, \check{b} \in \mathfrak{U}$ and $\nu_{\check{s}} \in [0, 1)$ such that $\mathfrak{N}_{fa}(\check{a} \star \check{b}) > \nu_{\check{s}} \geq \mathfrak{N}_{fa}(\check{a}) \vee \mathfrak{N}_{fa}(\check{b})$, then $\check{a}, \check{b} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\check{s}})$ but $\check{a} \star \check{b} \notin \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\check{s}})$, again a contradiction. Thus, $\mathfrak{N}_{fa}(\check{j} \star \check{\ddot{j}}) \leq \mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{\ddot{j}})$ for all $\check{j}, \check{\ddot{j}} \in \mathfrak{U}$.

Conversely, assume that $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ satisfies condition (18). If $\check{j}, \check{\ddot{j}} \in \mathfrak{U}$ with $\check{j} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\check{j}})$ and $\check{\ddot{j}} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\check{\ddot{j}}})$, then $\mathfrak{N}_{tr}(\check{j}) \geq \lambda_{\check{j}}$ and $\mathfrak{N}_{tr}(\check{\ddot{j}}) \geq \lambda_{\check{\ddot{j}}}$, which yields $\mathfrak{N}_{tr}(\check{j} \star \check{\ddot{j}}) \geq \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{\ddot{j}}) \geq \lambda_{\check{j}} \wedge \lambda_{\check{\ddot{j}}}$, so that $\check{j} \star \check{\ddot{j}} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\check{j}} \wedge \lambda_{\check{\ddot{j}}})$. A similar reasoning applies to the indeterminacy component, showing that $\check{j} \star \check{\ddot{j}} \in \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu_{\check{j}} \wedge \mu_{\check{\ddot{j}}})$ whenever $\check{j} \in \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu_{\check{j}})$ and $\check{\ddot{j}} \in \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu_{\check{\ddot{j}}})$. Finally, if $\check{j} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\check{j}})$ and $\check{\ddot{j}} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\check{\ddot{j}}})$, then $\mathfrak{N}_{fa}(\check{j}) \leq \nu_{\check{j}}$ and $\mathfrak{N}_{fa}(\check{\ddot{j}}) \leq \nu_{\check{\ddot{j}}}$, so that $\mathfrak{N}_{fa}(\check{j} \star \check{\ddot{j}}) \leq \mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{\ddot{j}}) \leq \nu_{\check{j}} \vee \nu_{\check{\ddot{j}}}$, implying $\check{j} \star \check{\ddot{j}} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\check{j}} \vee \nu_{\check{\ddot{j}}})$. Therefore, \mathfrak{N} is indeed a (\in, \in) -NBCCS of \mathfrak{U} . \square

Theorem 3.3. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (\in, \in) -NBCCS of \mathfrak{U} , then for every $\lambda, \mu \in (0, 1]$ and $\nu \in [0, 1)$, the sets $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ (whenever nonempty) form BCCSs of \mathfrak{U} .*

Proof. Take $\check{j}, \check{\ddot{j}} \in \mathfrak{Tr}_q(\mathfrak{N}, \lambda)$. By definition, $\mathfrak{N}_{tr}(\check{j}) + \lambda > 1$ and $\mathfrak{N}_{tr}(\check{\ddot{j}}) + \lambda > 1$. Since \mathfrak{N} is a (\in, \in) -NBCCS, we obtain

$$\mathfrak{N}_{tr}(\check{j} \star \check{\ddot{j}}) + \lambda \geq (\mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{\ddot{j}})) + \lambda = (\mathfrak{N}_{tr}(\check{j}) + \lambda) \wedge (\mathfrak{N}_{tr}(\check{\ddot{j}}) + \lambda) > 1,$$

which shows $\check{j} \star \check{\ddot{j}} \in \mathfrak{Tr}_q(\mathfrak{N}, \lambda)$. Thus, $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$ is closed under the operation and hence a BCCS of \mathfrak{U} .

An analogous argument applies to the indeterminacy case: if $\check{j}, \check{\ddot{j}} \in \mathfrak{Ind}_q(\mathfrak{N}, \mu)$, then $\check{j} \star \check{\ddot{j}} \in \mathfrak{Ind}_q(\mathfrak{N}, \mu)$, so $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$ is also a BCCS.

Finally, let $\check{j}, \check{\ddot{j}} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu)$. Then $\mathfrak{N}_{fa}(\check{j}) + \nu < 1$ and $\mathfrak{N}_{fa}(\check{\ddot{j}}) + \nu < 1$. Consequently,

$$\mathfrak{N}_{fa}(\check{j} \star \check{\ddot{j}}) + \nu \leq (\mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{\ddot{j}})) + \nu = (\mathfrak{N}_{fa}(\check{j}) + \nu) \vee (\mathfrak{N}_{fa}(\check{\ddot{j}}) + \nu) < 1,$$

which ensures $\check{j} \star \check{\ddot{j}} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu)$. Therefore, $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ is a BCCS of \mathfrak{U} . \square

Theorem 3.4. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a $(q, \in \vee q)$ -NBCCS of \mathfrak{U} , then the neutrosophic q -subsets $\mathfrak{T}r_q(\mathfrak{N}, \lambda)$, $\mathfrak{I}nd_q(\mathfrak{N}, \mu)$, and $\mathfrak{F}a_q(\mathfrak{N}, \nu)$ are BCCSs of \mathfrak{U} for all $\lambda, \mu \in (0.5, 1]$ and $\nu \in [0, 0.5)$, whenever nonempty.*

Proof. Consider $\check{j}, \check{j} \in \mathfrak{T}r_q(\mathfrak{N}, \lambda)$. By assumption, $\check{j} \star \check{j} \in \mathfrak{T}r_{\in \vee q}(\mathfrak{N}, \lambda)$, which means $\check{j} \star \check{j} \in \mathfrak{T}r_{\in}(\mathfrak{N}, \lambda)$ or $\check{j} \star \check{j} \in \mathfrak{T}r_q(\mathfrak{N}, \lambda)$. If the first case holds, then $\mathfrak{N}_{tr}(\check{j} \star \check{j}) \geq \lambda > 1 - \lambda$ since $\lambda > 0.5$, implying $\check{j} \star \check{j} \in \mathfrak{T}r_q(\mathfrak{N}, \lambda)$. Thus, $\mathfrak{T}r_q(\mathfrak{N}, \lambda)$ is closed under \star and therefore a BCCS of \mathfrak{U} .

The argument for $\mathfrak{I}nd_q(\mathfrak{N}, \mu)$ proceeds in the same way: any $\check{j}, \check{j} \in \mathfrak{I}nd_q(\mathfrak{N}, \mu)$ yield $\check{j} \star \check{j} \in \mathfrak{I}nd_q(\mathfrak{N}, \mu)$, so $\mathfrak{I}nd_q(\mathfrak{N}, \mu)$ is also a BCCS.

For the falsity component, take $\check{j}, \check{j} \in \mathfrak{F}a_q(\mathfrak{N}, \nu)$. Then $\check{j} \star \check{j} \in \mathfrak{F}a_{\in \vee q}(\mathfrak{N}, \nu)$, hence either $\check{j} \star \check{j} \in \mathfrak{F}a_{\in}(\mathfrak{N}, \nu)$ or $\check{j} \star \check{j} \in \mathfrak{F}a_q(\mathfrak{N}, \nu)$. If the former occurs, then $\mathfrak{N}_{fa}(\check{j} \star \check{j}) \leq \nu < 1 - \nu$ (since $\nu < 0.5$), which forces $\check{j} \star \check{j} \in \mathfrak{F}a_q(\mathfrak{N}, \nu)$. Consequently, $\mathfrak{F}a_q(\mathfrak{N}, \nu)$ is a BCCS of \mathfrak{U} . \square

Theorem 3.5. *Let $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ be an NS on \mathfrak{U} . Then \mathfrak{N} constitutes a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} precisely when the following inequalities are fulfilled:*

$$(\forall \check{j}, \check{j} \in \mathfrak{U}) \left(\begin{array}{l} \mathfrak{N}_{tr}(\check{j} \star \check{j}) \geq \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\check{j}), 0.5 \}, \\ \mathfrak{N}_{ind}(\check{j} \star \check{j}) \geq \bigwedge \{ \mathfrak{N}_{ind}(\check{j}), \mathfrak{N}_{ind}(\check{j}), 0.5 \}, \\ \mathfrak{N}_{fa}(\check{j} \star \check{j}) \leq \bigvee \{ \mathfrak{N}_{fa}(\check{j}), \mathfrak{N}_{fa}(\check{j}), 0.5 \}. \end{array} \right) \tag{19}$$

Proof. Assume that $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} and let $\check{j}, \check{j} \in \mathfrak{U}$.

For the truth component, suppose first that $\mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{j}) < 0.5$. If $\mathfrak{N}_{tr}(\check{j} \star \check{j}) < \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{j})$, choose $\lambda_{\check{s}}$ with $\mathfrak{N}_{tr}(\check{j} \star \check{j}) < \lambda_{\check{s}} < \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{j})$. Then $\check{j}, \check{j} \in \mathfrak{T}r_{\in}(\mathfrak{N}, \lambda_{\check{s}})$ but $\check{j} \star \check{j} \notin \mathfrak{T}r_{\in}(\mathfrak{N}, \lambda_{\check{s}})$. Moreover, $\mathfrak{N}_{tr}(\check{j} \star \check{j}) + \lambda_{\check{s}} < 1$, i.e. $\check{j} \star \check{j} \in \mathfrak{T}r_q(\mathfrak{N}, \lambda_{\check{s}})$, which contradicts closure in $\mathfrak{T}r_{\in \vee q}(\mathfrak{N}, \lambda_{\check{s}})$. Hence, $\mathfrak{N}_{tr}(\check{j} \star \check{j}) \geq \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\check{j}), 0.5 \}$ whenever $\mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{j}) < 0.5$.

If instead $\mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\check{j}) \geq 0.5$, then $\check{j}, \check{j} \in \mathfrak{T}r_{\in}(\mathfrak{N}, 0.5)$ and thus $\check{j} \star \check{j} \in \mathfrak{T}r_{\in \vee q}(\mathfrak{N}, 0.5)$. This implies $\mathfrak{N}_{tr}(\check{j} \star \check{j}) \geq 0.5$, since otherwise $\mathfrak{N}_{tr}(\check{j} \star \check{j}) + 0.5 < 1$, a contradiction. Consequently, $\mathfrak{N}_{tr}(\check{j} \star \check{j}) \geq \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\check{j}), 0.5 \}$ for all $\check{j}, \check{j} \in \mathfrak{U}$.

A parallel reasoning shows $\mathfrak{N}_{ind}(\check{j} \star \check{j}) \geq \bigwedge \{ \mathfrak{N}_{ind}(\check{j}), \mathfrak{N}_{ind}(\check{j}), 0.5 \}$ for every $\check{j}, \check{j} \in \mathfrak{U}$.

For the falsity part, suppose $\mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{j}) > 0.5$. If $\mathfrak{N}_{fa}(\check{j} \star \check{j}) > \nu_{\check{s}} := \mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{j})$, then $\check{j}, \check{j} \in \mathfrak{F}a_{\in}(\mathfrak{N}, \nu_{\check{s}})$ while $\check{j} \star \check{j} \notin \mathfrak{F}a_{\in}(\mathfrak{N}, \nu_{\check{s}})$, and moreover $\mathfrak{N}_{fa}(\check{j} \star \check{j}) + \nu_{\check{s}} > 1$, so $\check{j} \star \check{j} \notin \mathfrak{F}a_q(\mathfrak{N}, \nu_{\check{s}})$ which is a contradiction. Thus, $\mathfrak{N}_{fa}(\check{j} \star \check{j}) \leq \bigvee \{ \mathfrak{N}_{fa}(\check{j}), \mathfrak{N}_{fa}(\check{j}), 0.5 \}$ whenever $\mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{j}) > 0.5$.

If $\mathfrak{N}_{fa}(\check{j}) \vee \mathfrak{N}_{fa}(\check{j}) \leq 0.5$, then $\check{j}, \check{j} \in \mathfrak{F}a_{\in}(\mathfrak{N}, 0.5)$ and hence $\check{j} \star \check{j} \in \mathfrak{F}a_{\in \vee q}(\mathfrak{N}, 0.5)$. Therefore, $\mathfrak{N}_{fa}(\check{j} \star \check{j}) \leq 0.5$, since otherwise $\mathfrak{N}_{fa}(\check{j} \star \check{j}) + 0.5 > 1$, a contradiction. Altogether, $\mathfrak{N}_{fa}(\check{j} \star \check{j}) \leq \bigvee \{ \mathfrak{N}_{fa}(\check{j}), \mathfrak{N}_{fa}(\check{j}), 0.5 \}$ for all $\check{j}, \check{j} \in \mathfrak{U}$.

Conversely, assume that \mathfrak{N} satisfies condition (19). Let $\check{j}, \check{j} \in \mathfrak{U}$ and choose thresholds $\lambda_{\check{j}}, \lambda_{\check{j}}, \mu_{\check{j}}, \mu_{\check{j}}, \nu_{\check{j}}, \nu_{\check{j}} \in [0, 1]$.

If $\check{j} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\check{j}})$ and $\ddot{j} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda_{\ddot{j}})$, then $\mathfrak{N}_{tr}(\check{j}) \geq \lambda_{\check{j}}$ and $\mathfrak{N}_{tr}(\ddot{j}) \geq \lambda_{\ddot{j}}$. Using (19), we deduce $\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) \geq \mathfrak{N}_{tr}(\check{j}) \wedge \mathfrak{N}_{tr}(\ddot{j}) \wedge 0.5 \geq \lambda_{\check{j}} \wedge \lambda_{\ddot{j}}$, so $\check{j} \star \ddot{j} \in \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda_{\check{j}} \wedge \lambda_{\ddot{j}})$.

A similar reasoning applies to the indeterminacy component, giving $\check{j} \star \ddot{j} \in \mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \mu_{\check{j}} \wedge \mu_{\ddot{j}})$ whenever $\check{j} \in \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu_{\check{j}})$ and $\ddot{j} \in \mathfrak{Ind}_{\in}(\mathfrak{N}, \mu_{\ddot{j}})$.

Finally, if $\check{j} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\check{j}})$ and $\ddot{j} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\ddot{j}})$, then $\mathfrak{N}_{fa}(\check{j}) \leq \nu_{\check{j}}$ and $\mathfrak{N}_{fa}(\ddot{j}) \leq \nu_{\ddot{j}}$. If $\mathfrak{N}_{fa}(\check{j} \star \ddot{j}) > \nu_{\check{j}} \vee \nu_{\ddot{j}}$, then condition (19) would be violated, giving a contradiction. Hence, $\mathfrak{N}_{fa}(\check{j} \star \ddot{j}) \leq \nu_{\check{j}} \vee \nu_{\ddot{j}}$, so $\check{j} \star \ddot{j} \in \mathfrak{Fa}_{\in \vee q}(\mathfrak{N}, \nu_{\check{j}} \vee \nu_{\ddot{j}})$.

Therefore, \mathfrak{N} is indeed a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} . \square

Theorem 3.6. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} , then the neutrosophic q -subsets $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ form BCCSs of \mathfrak{U} for all $\lambda, \mu \in (0.5, 1]$ and $\nu \in [0, 0.5)$, whenever they are nonempty.*

Proof. Suppose that $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ are nonempty for some $\lambda, \mu \in (0.5, 1]$ and $\nu \in [0, 0.5)$.

Take $\check{j}, \ddot{j} \in \mathfrak{Tr}_q(\mathfrak{N}, \lambda)$. Since $\mathfrak{N}_{tr}(\check{j}) + \lambda > 1$ and $\mathfrak{N}_{tr}(\ddot{j}) + \lambda > 1$, Theorem 3.5 ensures

$$\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) + \lambda \geq \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\ddot{j}), 0.5 \} + \lambda = \bigwedge \{ \mathfrak{N}_{tr}(\check{j}) + \lambda, \mathfrak{N}_{tr}(\ddot{j}) + \lambda, 0.5 + \lambda \} > 1.$$

Thus, $\check{j} \star \ddot{j} \in \mathfrak{Tr}_q(\mathfrak{N}, \lambda)$, showing that $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$ is closed under \star and therefore a BCCS of \mathfrak{U} .

The same reasoning applies to the indeterminacy case: for any $\check{j}, \ddot{j} \in \mathfrak{Ind}_q(\mathfrak{N}, \mu)$, Theorem 3.5 guarantees $\check{j} \star \ddot{j} \in \mathfrak{Ind}_q(\mathfrak{N}, \mu)$. Hence, $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$ also forms a BCCS.

Finally, consider $\check{j}, \ddot{j} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu)$. Then $\mathfrak{N}_{fa}(\check{j}) + \nu < 1$ and $\mathfrak{N}_{fa}(\ddot{j}) + \nu < 1$. By Theorem 3.5,

$$\mathfrak{N}_{fa}(\check{j} \star \ddot{j}) + \nu \leq \bigvee \{ \mathfrak{N}_{fa}(\check{j}), \mathfrak{N}_{fa}(\ddot{j}), 0.5 \} + \nu = \bigvee \{ \mathfrak{N}_{fa}(\check{j}) + \nu, \mathfrak{N}_{fa}(\ddot{j}) + \nu, 0.5 + \nu \} < 1,$$

which yields $\check{j} \star \ddot{j} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu)$. Consequently, $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ is a BCCS of \mathfrak{U} . \square

Theorem 3.7. *For an NS $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ in \mathfrak{U} , if the nonempty neutrosophic $\in \vee q$ -subsets $\mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_{\in \vee q}(\mathfrak{N}, \nu)$ are BCCSs of \mathfrak{U} for all $\lambda, \mu \in (0, 1]$ and $\nu \in [0, 1)$, then \mathfrak{N} is a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} .*

Proof. Assume that $\mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda)$ is a BCCS of \mathfrak{U} and suppose, for contradiction, that $\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) < \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\ddot{j}), 0.5 \}$ for some $\check{j}, \ddot{j} \in \mathfrak{U}$. Then there exists $\lambda \in (0, 0.5]$ with $\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) < \lambda \leq \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\ddot{j}), 0.5 \}$. This forces $\check{j}, \ddot{j} \in \mathfrak{Tr}_{\in}(\mathfrak{N}, \lambda) \subseteq \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda)$, and hence $\check{j} \star \ddot{j} \in \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda)$. Consequently, $\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) \geq \lambda$ or $\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) + \lambda > 1$, contradicting the choice of λ . Therefore, $\mathfrak{N}_{tr}(\check{j} \star \ddot{j}) \geq \bigwedge \{ \mathfrak{N}_{tr}(\check{j}), \mathfrak{N}_{tr}(\ddot{j}), 0.5 \}$ for all $\check{j}, \ddot{j} \in \mathfrak{U}$.

A parallel reasoning applies to the indeterminacy component, giving $\mathfrak{N}_{ind}(\check{j} \star \ddot{j}) \geq \bigwedge \{ \mathfrak{N}_{ind}(\check{j}), \mathfrak{N}_{ind}(\ddot{j}), 0.5 \}$.

Next, let $\mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$ be a BCCS of \mathfrak{U} and assume that $\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) > \bigvee \{\mathfrak{N}_{fa}(\mathfrak{j}), \mathfrak{N}_{fa}(\mathfrak{k}), 0.5\}$ for some $\mathfrak{j}, \mathfrak{k} \in \mathfrak{U}$. Then there exists $\nu \in [0.5, 1)$ such that

$$\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) > \nu \geq \bigvee \{\mathfrak{N}_{fa}(\mathfrak{j}), \mathfrak{N}_{fa}(\mathfrak{k}), 0.5\}. \tag{20}$$

Thus, $\mathfrak{j}, \mathfrak{k} \in \mathfrak{A}_{\in}(\mathfrak{N}, \nu) \subseteq \mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$, which forces $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$. However, by (20) we would have $\mathfrak{j} \star \mathfrak{k} \notin \mathfrak{A}_{\in}(\mathfrak{N}, \nu)$ and simultaneously $\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) + \nu > 1$, so $\mathfrak{j} \star \mathfrak{k} \notin \mathfrak{A}_q(\mathfrak{N}, \nu)$, a contradiction. Hence, $\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) \leq \bigvee \{\mathfrak{N}_{fa}(\mathfrak{j}), \mathfrak{N}_{fa}(\mathfrak{k}), 0.5\}$ for all $\mathfrak{j}, \mathfrak{k} \in \mathfrak{U}$.

Combining the above with Theorem 3.5, it follows that $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is indeed a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} . \square

Theorem 3.8. *Let $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ be a $(\in, \in \vee q)$ -NBCCS of \mathfrak{U} . Then, for every $\lambda, \mu \in (0, 0.5]$ and $\nu \in [0.5, 1)$, the nonempty neutrosophic $\in \vee q$ -subsets $\mathfrak{I}r_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{I}nd_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$ form BCCSs of \mathfrak{U} .*

Proof. Assume that $\mathfrak{I}r_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{I}nd_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$ are nonempty for all $\lambda, \mu \in (0, 0.5]$ and $\nu \in [0.5, 1)$.

Take $\mathfrak{j}, \mathfrak{k} \in \mathfrak{I}nd_{\in \vee q}(\mathfrak{N}, \mu)$. Each of them lies either in $\mathfrak{I}nd_{\in}(\mathfrak{N}, \mu)$ or in $\mathfrak{I}nd_q(\mathfrak{N}, \mu)$, so four possibilities occur:

- (i) $\mathfrak{j}, \mathfrak{k} \in \mathfrak{I}nd_{\in}(\mathfrak{N}, \mu)$,
- (ii) $\mathfrak{j} \in \mathfrak{I}nd_{\in}(\mathfrak{N}, \mu)$ and $\mathfrak{k} \in \mathfrak{I}nd_q(\mathfrak{N}, \mu)$,
- (iii) $\mathfrak{j} \in \mathfrak{I}nd_q(\mathfrak{N}, \mu)$ and $\mathfrak{k} \in \mathfrak{I}nd_{\in}(\mathfrak{N}, \mu)$,
- (iv) $\mathfrak{j}, \mathfrak{k} \in \mathfrak{I}nd_q(\mathfrak{N}, \mu)$.

In case (i), closure in $\mathfrak{I}nd_{\in \vee q}(\mathfrak{N}, \mu)$ is immediate. In case (ii), $\mathfrak{k} \in \mathfrak{I}nd_q(\mathfrak{N}, \mu)$ implies $\mathfrak{N}_{ind}(\mathfrak{k}) > 1 - \mu \geq \mu$, hence $\mathfrak{k} \in \mathfrak{I}nd_{\in}(\mathfrak{N}, \mu)$ and the pair reduces to case (i). Case (iii) is symmetric. In case (iv), both $\mathfrak{N}_{ind}(\mathfrak{j})$ and $\mathfrak{N}_{ind}(\mathfrak{k})$ exceed $1 - \mu \geq \mu$, so $\mathfrak{j}, \mathfrak{k} \in \mathfrak{I}nd_{\in}(\mathfrak{N}, \mu)$ and again closure follows. Therefore, $\mathfrak{I}nd_{\in \vee q}(\mathfrak{N}, \mu)$ forms a BCCS of \mathfrak{U} for all $\mu \in (0, 0.5]$.

An analogous argument applies to $\mathfrak{I}r_{\in \vee q}(\mathfrak{N}, \lambda)$, showing it is a BCCS for each $\lambda \in (0, 0.5]$.

Now consider $\mathfrak{j}, \mathfrak{k} \in \mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$. Then either $\mathfrak{N}_{fa}(\mathfrak{j}) \leq \nu$ or $\mathfrak{N}_{fa}(\mathfrak{j}) + \nu < 1$, and the same for \mathfrak{k} . We inspect the cases:

If both $\mathfrak{N}_{fa}(\mathfrak{j}), \mathfrak{N}_{fa}(\mathfrak{k}) \leq \nu$, then by Theorem 3.5,

$$\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) \leq \bigvee \{\mathfrak{N}_{fa}(\mathfrak{j}), \mathfrak{N}_{fa}(\mathfrak{k}), 0.5\} \leq \nu,$$

so $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{A}_{\in}(\mathfrak{N}, \nu) \subseteq \mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$.

If $\mathfrak{N}_{fa}(\mathfrak{j}) \leq \nu$ and $\mathfrak{N}_{fa}(\mathfrak{k}) + \nu < 1$, then Theorem 3.5 gives

$$\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) \leq \bigvee \{\nu, 1 - \nu, 0.5\} = \nu,$$

hence $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{A}_{\in \vee q}(\mathfrak{N}, \nu)$. The case $\mathfrak{N}_{fa}(\mathfrak{j}) + \nu < 1$ and $\mathfrak{N}_{fa}(\mathfrak{k}) \leq \nu$ is symmetric.

If both $\mathfrak{N}_{fa}(\mathfrak{j}) + \nu < 1$ and $\mathfrak{N}_{fa}(\mathfrak{k}) + \nu < 1$, then

$$\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) \leq \bigvee \{1 - \nu, 0.5\} = 0.5 < \nu,$$

so again $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \nu) \subseteq \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$.

Thus, $\mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$ is a BCCS of \mathfrak{U} for all $\nu \in [0.5, 1)$. \square

Theorem 3.9. *Let $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ be a $(q, \in \vee q)$ -NBCCS of \mathfrak{U} . Then, for every $\lambda, \mu \in (0, 0.5]$ and $\nu \in [0.5, 1)$, the nonempty neutrosophic $\in \vee q$ -subsets $\mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{I}\mathfrak{n}\mathfrak{d}_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$ constitute BCCSs of \mathfrak{U} .*

Proof. Assume that $\mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{I}\mathfrak{n}\mathfrak{d}_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$ are nonempty for all $\lambda, \mu \in (0, 0.5]$ and $\nu \in [0.5, 1)$.

Let $\mathfrak{j}, \mathfrak{k} \in \mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda)$. Then $\mathfrak{j} \in \mathfrak{T}\mathfrak{r}_{\in}(\mathfrak{N}, \lambda)$ or $\mathfrak{j} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda)$, and likewise $\mathfrak{k} \in \mathfrak{T}\mathfrak{r}_{\in}(\mathfrak{N}, \lambda)$ or $\mathfrak{k} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda)$. If both belong to $\mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda)$, closure is clear. If $\mathfrak{j} \in \mathfrak{T}\mathfrak{r}_{\in}(\mathfrak{N}, \lambda)$ and $\mathfrak{k} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda)$, then $\mathfrak{N}_{tr}(\mathfrak{j}) + \lambda \geq 2\lambda > 1$, so $\mathfrak{j} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda)$, and hence $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda)$. The other cases are symmetric, showing $\mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda)$ is a BCCS.

A parallel reasoning applies to $\mathfrak{I}\mathfrak{n}\mathfrak{d}_{\in \vee q}(\mathfrak{N}, \mu)$ for $\mu \in (0, 0.5]$.

Now, let $\mathfrak{j}, \mathfrak{k} \in \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$. Then $\mathfrak{j} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \nu)$ or $\mathfrak{j} \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \nu)$, and similarly for \mathfrak{k} . If both are in $\mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \nu)$, then by Theorem 3.5, $\mathfrak{N}_{fa}(\mathfrak{j} \star \mathfrak{k}) \leq \bigvee \{\mathfrak{N}_{fa}(\mathfrak{j}), \mathfrak{N}_{fa}(\mathfrak{k}), 0.5\} \leq \nu$, so $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \nu) \subseteq \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$. If one element lies in $\mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \nu)$ and the other in $\mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \nu)$, then again $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$. Finally, if $\mathfrak{j}, \mathfrak{k} \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \nu)$, then $\mathfrak{N}_{fa}(\mathfrak{j}) + \nu < 1$ and $\mathfrak{N}_{fa}(\mathfrak{k}) + \nu < 1$, implying $\mathfrak{j}, \mathfrak{k} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \nu)$ as well, so closure holds.

Therefore, $\mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda)$, $\mathfrak{I}\mathfrak{n}\mathfrak{d}_{\in \vee q}(\mathfrak{N}, \mu)$, and $\mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu)$ are BCCSs of \mathfrak{U} for all $\lambda, \mu \in (0, 0.5]$ and $\nu \in [0.5, 1)$. \square

Given an NS $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ on a universe \mathfrak{U} , we introduce the subset

$$\mathfrak{U}_0^1 = \{\mathfrak{j} \in \mathfrak{U} : \mathfrak{N}_{tr}(\mathfrak{j}) > 0, \mathfrak{N}_{ind}(\mathfrak{j}) > 0, \mathfrak{N}_{fa}(\mathfrak{j}) < 1\}.$$

Theorem 3.10. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (\in, \in) -NBCCS of \mathfrak{U} , then the set \mathfrak{U}_0^1 is itself a BCCS of \mathfrak{U} .*

Proof. Take $\mathfrak{j}, \mathfrak{k} \in \mathfrak{U}_0^1$. By definition,

$$\mathfrak{N}_{tr}(\mathfrak{j}) > 0, \mathfrak{N}_{ind}(\mathfrak{j}) > 0, \mathfrak{N}_{fa}(\mathfrak{j}) < 1,$$

and likewise $\mathfrak{N}_{tr}(\mathfrak{k}) > 0, \mathfrak{N}_{ind}(\mathfrak{k}) > 0, \mathfrak{N}_{fa}(\mathfrak{k}) < 1$.

Assume for contradiction that $\mathfrak{N}_{tr}(\mathfrak{j} \star \mathfrak{k}) = 0$. Since $\mathfrak{j} \in \mathfrak{T}\mathfrak{r}_{\in}(\mathfrak{N}, \mathfrak{N}_{tr}(\mathfrak{j}))$ and $\mathfrak{k} \in \mathfrak{T}\mathfrak{r}_{\in}(\mathfrak{N}, \mathfrak{N}_{tr}(\mathfrak{k}))$, closure of $\mathfrak{T}\mathfrak{r}_{\in}$ would require $\mathfrak{j} \star \mathfrak{k} \in \mathfrak{T}\mathfrak{r}_{\in}(\mathfrak{N}, \mathfrak{N}_{tr}(\mathfrak{j}) \wedge \mathfrak{N}_{tr}(\mathfrak{k}))$. But this fails because $\mathfrak{N}_{tr}(\mathfrak{j} \star \mathfrak{k}) = 0 < \mathfrak{N}_{tr}(\mathfrak{j}) \wedge \mathfrak{N}_{tr}(\mathfrak{k})$, a contradiction. Hence, $\mathfrak{N}_{tr}(\mathfrak{j} \star \mathfrak{k}) > 0$.

A parallel argument shows $\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) > 0$.

For the falsity part, note that $\dot{j} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \mathfrak{N}_{fa}(\dot{j}))$ and $\ddot{j} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \mathfrak{N}_{fa}(\ddot{j}))$. If $\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) = 1$, then $\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) > \mathfrak{N}_{fa}(\dot{j}) \vee \mathfrak{N}_{fa}(\ddot{j})$, contradicting closure in $\mathfrak{F}\mathfrak{a}_{\in}$. Therefore, $\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) < 1$.

Altogether, $\dot{j} \star \ddot{j} \in \mathfrak{U}_0^1$, establishing that \mathfrak{U}_0^1 is a BCCS of \mathfrak{U} . \square

Theorem 3.11. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (\in, q) -NBCCS of \mathfrak{U} , then the set \mathfrak{U}_0^1 forms a BCCS of \mathfrak{U} .*

Proof. Take $\dot{j}, \ddot{j} \in \mathfrak{U}_0^1$. By definition,

$$\mathfrak{N}_{tr}(\dot{j}) > 0, \mathfrak{N}_{ind}(\dot{j}) > 0, \mathfrak{N}_{fa}(\dot{j}) < 1,$$

and similarly for \ddot{j} .

Suppose $\mathfrak{N}_{tr}(\dot{j} \star \ddot{j}) = 0$. Then

$$\mathfrak{N}_{tr}(\dot{j} \star \ddot{j}) + (\mathfrak{N}_{tr}(\dot{j}) \wedge \mathfrak{N}_{tr}(\ddot{j})) = \mathfrak{N}_{tr}(\dot{j}) \wedge \mathfrak{N}_{tr}(\ddot{j}) \leq 1,$$

which would force $\dot{j} \star \ddot{j} \notin \mathfrak{I}\mathfrak{r}_q(\mathfrak{N}, \mathfrak{N}_{tr}(\dot{j}) \wedge \mathfrak{N}_{tr}(\ddot{j}))$. But since $\dot{j} \in \mathfrak{I}\mathfrak{r}_{\in}(\mathfrak{N}, \mathfrak{N}_{tr}(\dot{j}))$ and $\ddot{j} \in \mathfrak{I}\mathfrak{r}_{\in}(\mathfrak{N}, \mathfrak{N}_{tr}(\ddot{j}))$, closure requires $\dot{j} \star \ddot{j} \in \mathfrak{I}\mathfrak{r}_q(\mathfrak{N}, \mathfrak{N}_{tr}(\dot{j}) \wedge \mathfrak{N}_{tr}(\ddot{j}))$ which is a contradiction. Therefore, $\mathfrak{N}_{tr}(\dot{j} \star \ddot{j}) > 0$.

An analogous reasoning shows $\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) > 0$.

For the falsity part, assume $\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) = 1$. Then

$$\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) + (\mathfrak{N}_{fa}(\dot{j}) \vee \mathfrak{N}_{fa}(\ddot{j})) = 1 + \mathfrak{N}_{fa}(\dot{j}) \vee \mathfrak{N}_{fa}(\ddot{j}) \geq 1,$$

so $\dot{j} \star \ddot{j} \notin \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \mathfrak{N}_{fa}(\dot{j}) \vee \mathfrak{N}_{fa}(\ddot{j}))$. Yet $\dot{j} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \mathfrak{N}_{fa}(\dot{j}))$ and $\ddot{j} \in \mathfrak{F}\mathfrak{a}_{\in}(\mathfrak{N}, \mathfrak{N}_{fa}(\ddot{j}))$, which contradicts closure. Thus, $\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) < 1$.

Hence, $\dot{j} \star \ddot{j} \in \mathfrak{U}_0^1$, proving that \mathfrak{U}_0^1 is a BCCS of \mathfrak{U} . \square

Theorem 3.12. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (q, \in) -NBCCS of \mathfrak{U} , then the set \mathfrak{U}_0^1 is a BCCS of \mathfrak{U} .*

Proof. Take $\dot{j}, \ddot{j} \in \mathfrak{U}_0^1$. Then

$$\mathfrak{N}_{tr}(\dot{j}) > 0, \mathfrak{N}_{ind}(\dot{j}) > 0, \mathfrak{N}_{fa}(\dot{j}) < 1,$$

and likewise for \ddot{j} . This implies

$$\mathfrak{N}_{tr}(\dot{j}) + 1 > 1, \mathfrak{N}_{tr}(\ddot{j}) + 1 > 1, \mathfrak{N}_{ind}(\dot{j}) + 1 > 1, \mathfrak{N}_{ind}(\ddot{j}) + 1 > 1, \mathfrak{N}_{fa}(\dot{j}) + 0 < 1, \mathfrak{N}_{fa}(\ddot{j}) + 0 < 1,$$

so that $\dot{j}, \ddot{j} \in \mathfrak{I}\mathfrak{r}_q(\mathfrak{N}, 1) \cap \mathfrak{I}\mathfrak{n}\mathfrak{d}_q(\mathfrak{N}, 1) \cap \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0)$.

Suppose $\mathfrak{N}_{tr}(\dot{j} \star \ddot{j}) = 0$ or $\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) = 0$. Then $\mathfrak{N}_{tr}(\dot{j} \star \ddot{j}) < 1 = 1 \wedge 1$ or $\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) < 1 = 1 \wedge 1$, which means $\dot{j} \star \ddot{j} \notin \mathfrak{I}\mathfrak{r}_q(\mathfrak{N}, 1 \wedge 1)$ or $\dot{j} \star \ddot{j} \notin \mathfrak{I}\mathfrak{n}\mathfrak{d}_q(\mathfrak{N}, 1 \wedge 1)$ which is a contradiction. Hence, $\mathfrak{N}_{tr}(\dot{j} \star \ddot{j}) > 0$ and $\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) > 0$.

Now assume $\mathfrak{N}_{fa}(\check{\mathfrak{j}} \star \check{\mathfrak{j}}) = 1$. Then $\check{\mathfrak{j}} \star \check{\mathfrak{j}} \notin \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0 \vee 0)$, contradicting closure. Therefore, $\mathfrak{N}_{fa}(\check{\mathfrak{j}} \star \check{\mathfrak{j}}) < 1$.

Thus, $\check{\mathfrak{j}} \star \check{\mathfrak{j}} \in \mathfrak{U}_0^1$, and so \mathfrak{U}_0^1 is a BCCS of \mathfrak{U} . \square

Theorem 3.13. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (q, q) -NBCCS of \mathfrak{U} , then the set \mathfrak{U}_0^1 is a BCCS of \mathfrak{U} .*

Proof. Let $\check{\mathfrak{j}}, \check{\check{\mathfrak{j}}} \in \mathfrak{U}_0^1$. Then

$$\mathfrak{N}_{tr}(\check{\mathfrak{j}}) > 0, \mathfrak{N}_{ind}(\check{\mathfrak{j}}) > 0, \mathfrak{N}_{fa}(\check{\mathfrak{j}}) < 1,$$

and similarly for $\check{\check{\mathfrak{j}}}$. This ensures

$$\mathfrak{N}_{tr}(\check{\mathfrak{j}}) + 1 > 1, \mathfrak{N}_{tr}(\check{\check{\mathfrak{j}}}) + 1 > 1, \mathfrak{N}_{ind}(\check{\mathfrak{j}}) + 1 > 1, \mathfrak{N}_{ind}(\check{\check{\mathfrak{j}}}) + 1 > 1, \mathfrak{N}_{fa}(\check{\mathfrak{j}}) + 0 < 1, \mathfrak{N}_{fa}(\check{\check{\mathfrak{j}}}) + 0 < 1,$$

so $\check{\mathfrak{j}}, \check{\check{\mathfrak{j}}} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, 1) \cap \mathfrak{I}\mathfrak{n}\mathfrak{d}_q(\mathfrak{N}, 1) \cap \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0)$.

Suppose $\mathfrak{N}_{tr}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) = 0$ or $\mathfrak{N}_{ind}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) = 0$. Then

$$\mathfrak{N}_{tr}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) + 1 = 1 \text{ or } \mathfrak{N}_{ind}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) + 1 = 1,$$

so $\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}} \notin \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, 1 \wedge 1)$ or $\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}} \notin \mathfrak{I}\mathfrak{n}\mathfrak{d}_q(\mathfrak{N}, 1 \wedge 1)$, a contradiction. Thus, $\mathfrak{N}_{tr}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) > 0$ and $\mathfrak{N}_{ind}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) > 0$.

Now assume $\mathfrak{N}_{fa}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) = 1$. Then

$$\mathfrak{N}_{fa}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) + 0 \vee 0 = 1,$$

which means $\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}} \notin \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0 \vee 0)$, again a contradiction. Hence, $\mathfrak{N}_{fa}(\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}}) < 1$.

Therefore, $\check{\mathfrak{j}} \star \check{\check{\mathfrak{j}}} \in \mathfrak{U}_0^1$, showing that \mathfrak{U}_0^1 is a BCCS of \mathfrak{U} . \square

Theorem 3.14. *If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a (q, q) -NBCCS of \mathfrak{U} , then \mathfrak{N} is neutrosophic constant on \mathfrak{U}_0^1 , i.e., the functions \mathfrak{N}_{tr} , \mathfrak{N}_{ind} , and \mathfrak{N}_{fa} take constant values on \mathfrak{U}_0^1 .*

Proof. Assume first that \mathfrak{N}_{tr} is not constant on \mathfrak{U}_0^1 . Then there exists $\check{\mathfrak{j}} \in \mathfrak{U}_0^1$ such that $\lambda_{\check{\mathfrak{j}}} = \mathfrak{N}_{tr}(\check{\mathfrak{j}}) \neq \mathfrak{N}_{tr}(0) = \lambda_0$. Hence, either $\lambda_{\check{\mathfrak{j}}} > \lambda_0$ or $\lambda_{\check{\mathfrak{j}}} < \lambda_0$.

Suppose $\lambda_{\check{\mathfrak{j}}} < \lambda_0$. Choose $\lambda_1, \lambda_2 \in (0, 1]$ with $1 - \lambda_0 < \lambda_1 \leq 1 - \lambda_{\check{\mathfrak{j}}} < \lambda_2$. Then

$$\mathfrak{N}_{tr}(0) + \lambda_1 = \lambda_0 + \lambda_1 > 1, \mathfrak{N}_{tr}(\check{\mathfrak{j}}) + \lambda_2 = \lambda_{\check{\mathfrak{j}}} + \lambda_2 > 1,$$

so $0 \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda_1)$ and $\check{\mathfrak{j}} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda_2)$. However,

$$\mathfrak{N}_{tr}(\check{\mathfrak{j}} \star 0) + \lambda_1 \wedge \lambda_2 = \mathfrak{N}_{tr}(\check{\mathfrak{j}}) + \lambda_1 = \lambda_{\check{\mathfrak{j}}} + \lambda_1 \leq 1,$$

which implies $\check{\mathfrak{j}} \star 0 \notin \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda_1 \wedge \lambda_2)$, a contradiction.

Now suppose $\lambda_{\check{\mathfrak{j}}} > \lambda_0$. Then

$$\mathfrak{N}_{tr}(\check{\mathfrak{j}}) + (1 - \lambda_0) = \lambda_{\check{\mathfrak{j}}} + 1 - \lambda_0 > 1,$$

so $\ddot{j} \in \mathfrak{I}r_q(\mathfrak{N}, 1 - \lambda_0)$. But

$$\mathfrak{N}_{tr}(\ddot{j} \star \ddot{j}) + (1 - \lambda_0) = \mathfrak{N}_{tr}(0) + 1 - \lambda_0 = \lambda_0 + 1 - \lambda_0 = 1,$$

hence $\ddot{j} \star \ddot{j} \notin \mathfrak{I}r_q(\mathfrak{N}, (1 - \lambda_0) \wedge (1 - \lambda_0))$, a contradiction. Therefore, \mathfrak{N}_{tr} is constant on \mathfrak{U}_0^1 . A similar argument shows that \mathfrak{N}_{ind} is constant on \mathfrak{U}_0^1 .

Finally, suppose \mathfrak{N}_{fa} is not constant on \mathfrak{U}_0^1 . Then for some $\ddot{j} \in \mathfrak{U}_0^1$ we have $\nu_{\ddot{j}} = \mathfrak{N}_{fa}(\ddot{j}) \neq \mathfrak{N}_{fa}(0) = \nu_0$. Consider two cases:

If $\nu_{\ddot{j}} < \nu_0$, then

$$\mathfrak{N}_{fa}(\ddot{j}) + 1 - \nu_0 = \nu_{\ddot{j}} + 1 - \nu_0 < 1,$$

so $\ddot{j} \in \mathfrak{F}a_q(\mathfrak{N}, 1 - \nu_0)$. Hence, $\ddot{j} \star \ddot{j} \in \mathfrak{F}a_q(\mathfrak{N}, 1 - \nu_0)$, implying $0 \in \mathfrak{F}a_q(\mathfrak{N}, 1 - \nu_0)$, a contradiction since $\mathfrak{N}_{fa}(0) + 1 - \nu_0 = 1$.

If $\nu_{\ddot{j}} > \nu_0$, choose $\nu_1, \nu_2 \in (0, 1)$ such that $1 - \nu_0 > \nu_1 > 1 - \nu_{\ddot{j}} > \nu_2$. Then

$$\mathfrak{N}_{fa}(\ddot{j}) + \nu_2 = \nu_{\ddot{j}} + \nu_2 < 1 \Rightarrow \ddot{j} \in \mathfrak{F}a_q(\mathfrak{N}, \nu_2),$$

and

$$\mathfrak{N}_{fa}(0) + \nu_1 = \nu_0 + \nu_1 < 1 \Rightarrow 0 \in \mathfrak{F}a_q(\mathfrak{N}, \nu_1).$$

Thus, $\ddot{j} \star 0 \in \mathfrak{F}a_q(\mathfrak{N}, \nu_1 \vee \nu_2)$. However,

$$\mathfrak{N}_{fa}(\ddot{j} \star 0) + \nu_1 \vee \nu_2 = \mathfrak{N}_{fa}(\ddot{j}) + \nu_1 = \nu_{\ddot{j}} + \nu_1 > 1,$$

which contradicts membership in $\mathfrak{F}a_q(\mathfrak{N}, \nu_1 \vee \nu_2)$.

Therefore, \mathfrak{N}_{fa} must also be constant on \mathfrak{U}_0^1 . This completes the proof. \square

To further extend the structural analysis, we present a set of conditions that guarantees when an NS can be regarded as a $(q, \in \vee q)$ -NBCCS.

Theorem 3.15. *Let \mathfrak{S} be a BCCS of \mathfrak{U} , and let $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ be an NS on \mathfrak{U} satisfying:*

$$(\forall \dot{j} \in \mathfrak{S})(\mathfrak{N}_{tr}(\dot{j}) \geq 0.5, \mathfrak{N}_{ind}(\dot{j}) \geq 0.5, \mathfrak{N}_{fa}(\dot{j}) \leq 0.5), \tag{21}$$

$$(\forall \dot{j} \in \mathfrak{U} \setminus \mathfrak{S})(\mathfrak{N}_{tr}(\dot{j}) = 0, \mathfrak{N}_{ind}(\dot{j}) = 0, \mathfrak{N}_{fa}(\dot{j}) = 1). \tag{22}$$

Then $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a $(q, \in \vee q)$ -NBCCS of \mathfrak{U} .

Proof. Assume that $\dot{j} \in \mathfrak{I}nd_q(\mathfrak{N}, \lambda_{\dot{j}})$ and $\ddot{j} \in \mathfrak{I}nd_q(\mathfrak{N}, \lambda_{\ddot{j}})$ for some $\dot{j}, \ddot{j} \in \mathfrak{U}$ and $\lambda_{\dot{j}}, \lambda_{\ddot{j}} \in [0, 1]$. Then $\mathfrak{N}_{ind}(\dot{j}) + \lambda_{\dot{j}} > 1$ and $\mathfrak{N}_{ind}(\ddot{j}) + \lambda_{\ddot{j}} > 1$. If $\dot{j} \star \ddot{j} \notin \mathfrak{S}$, then either $\dot{j} \in \mathfrak{U} \setminus \mathfrak{S}$ or $\ddot{j} \in \mathfrak{U} \setminus \mathfrak{S}$, since \mathfrak{S} is a BCCS of \mathfrak{U} . This would force $\mathfrak{N}_{ind}(\dot{j}) = 0$ or $\mathfrak{N}_{ind}(\ddot{j}) = 0$, implying $\lambda_{\dot{j}} > 1$ or $\lambda_{\ddot{j}} > 1$, a contradiction. Hence, $\dot{j} \star \ddot{j} \in \mathfrak{S}$.

If $\lambda_{\dot{j}} \wedge \lambda_{\ddot{j}} > 0.5$, then

$$\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) + (\lambda_{\dot{j}} \wedge \lambda_{\ddot{j}}) > 1,$$

so $\dot{j} \star \ddot{j} \in \mathfrak{Ind}_q(\mathfrak{N}, \lambda_{\dot{j}} \wedge \lambda_{\ddot{j}})$. If $\lambda_{\dot{j}} \wedge \lambda_{\ddot{j}} \leq 0.5$, then

$$\mathfrak{N}_{ind}(\dot{j} \star \ddot{j}) \geq 0.5 \geq \lambda_{\dot{j}} \wedge \lambda_{\ddot{j}},$$

so $\dot{j} \star \ddot{j} \in \mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \lambda_{\dot{j}} \wedge \lambda_{\ddot{j}})$. In either case, $\dot{j} \star \ddot{j} \in \mathfrak{Ind}_{\in \vee q}(\mathfrak{N}, \lambda_{\dot{j}} \wedge \lambda_{\ddot{j}})$.

A similar argument shows that if $\dot{j} \in \mathfrak{Tr}_q(\mathfrak{N}, \lambda_{\dot{j}})$ and $\ddot{j} \in \mathfrak{Tr}_q(\mathfrak{N}, \lambda_{\ddot{j}})$, then $\dot{j} \star \ddot{j} \in \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \lambda_{\dot{j}} \wedge \lambda_{\ddot{j}})$.

Now let $\dot{j} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu_{\dot{j}})$ and $\ddot{j} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu_{\ddot{j}})$ for some $\nu_{\dot{j}}, \nu_{\ddot{j}} \in [0, 1]$. Then $\mathfrak{N}_{fa}(\dot{j}) + \nu_{\dot{j}} < 1$ and $\mathfrak{N}_{fa}(\ddot{j}) + \nu_{\ddot{j}} < 1$. If $\dot{j} \star \ddot{j} \notin \mathfrak{S}$, then either $\dot{j} \in \mathfrak{U} \setminus \mathfrak{S}$ or $\ddot{j} \in \mathfrak{U} \setminus \mathfrak{S}$, and hence $\mathfrak{N}_{fa}(\dot{j}) = 1$ or $\mathfrak{N}_{fa}(\ddot{j}) = 1$, which implies $\nu_{\dot{j}} < 0$ or $\nu_{\ddot{j}} < 0$, a contradiction. Thus, $\dot{j} \star \ddot{j} \in \mathfrak{S}$.

If $\nu_{\dot{j}} \vee \nu_{\ddot{j}} \geq 0.5$, then

$$\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) \leq 0.5 \leq \nu_{\dot{j}} \vee \nu_{\ddot{j}},$$

so $\dot{j} \star \ddot{j} \in \mathfrak{Fa}_{\in}(\mathfrak{N}, \nu_{\dot{j}} \vee \nu_{\ddot{j}})$. If $\nu_{\dot{j}} \vee \nu_{\ddot{j}} < 0.5$, then

$$\mathfrak{N}_{fa}(\dot{j} \star \ddot{j}) + (\nu_{\dot{j}} \vee \nu_{\ddot{j}}) < 1,$$

so $\dot{j} \star \ddot{j} \in \mathfrak{Fa}_q(\mathfrak{N}, \nu_{\dot{j}} \vee \nu_{\ddot{j}})$. In either case, $\dot{j} \star \ddot{j} \in \mathfrak{Fa}_{\in \vee q}(\mathfrak{N}, \nu_{\dot{j}} \vee \nu_{\ddot{j}})$.

Therefore, $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is a $(q, \in \vee q)$ -NBCCS of \mathfrak{U} . \square

The interplay between Theorems 3.4 and 3.15 leads directly to the following corollary.

Corollary 3.16. *Let \mathfrak{S} be a BCCS of \mathfrak{U} . If $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ is an NS on \mathfrak{U} satisfying conditions (21) and (22), then the subsets $\mathfrak{Tr}_q(\mathfrak{N}, \lambda)$, $\mathfrak{Ind}_q(\mathfrak{N}, \mu)$, and $\mathfrak{Fa}_q(\mathfrak{N}, \nu)$ are BCCSs of \mathfrak{U} for all $\lambda, \mu \in (0.5, 1]$ and $\nu \in [0, 0.5)$, whenever they are nonempty.*

Theorem 3.17. *Let $\mathfrak{N} = (\mathfrak{U}, \mathfrak{N}_{tr}, \mathfrak{N}_{ind}, \mathfrak{N}_{fa})$ be a $(q, \in \vee q)$ -neutrosophic BCCS of \mathfrak{U} such that \mathfrak{N}_{tr} , \mathfrak{N}_{ind} , and \mathfrak{N}_{fa} are not constant on \mathfrak{U}_0^1 . Then there exist $\dot{j}, \ddot{j}, \hat{j} \in \mathfrak{U}$ with $\mathfrak{N}_{tr}(\dot{j}) \geq 0.5$, $\mathfrak{N}_{ind}(\ddot{j}) \geq 0.5$, and $\mathfrak{N}_{fa}(\hat{j}) \leq 0.5$. In particular, these inequalities hold for all $\dot{j}, \ddot{j}, \hat{j} \in \mathfrak{U}_0^1$.*

Proof. Assume first that $\mathfrak{N}_{tr}(\dot{j}) < 0.5$ for all $\dot{j} \in \mathfrak{U}$. Since \mathfrak{N}_{tr} is not constant on \mathfrak{U}_0^1 , there exists $\dot{a} \in \mathfrak{U}_0^1$ such that $\lambda_{\dot{a}} = \mathfrak{N}_{tr}(\dot{a}) \neq \mathfrak{N}_{tr}(0) = \lambda_0$. Thus, either $\lambda_{\dot{a}} > \lambda_0$ or $\lambda_{\dot{a}} < \lambda_0$.

If $\lambda_{\dot{a}} > \lambda_0$, choose $\delta > 0.5$ such that $\lambda_0 + \delta < 1 < \lambda_{\dot{a}} + \delta$. Then $a \in \mathfrak{Tr}_q(\mathfrak{N}, \delta)$ but

$$\mathfrak{N}_{tr}(\dot{a} \star \dot{a}) + \delta = \mathfrak{N}_{tr}(0) + \delta = \lambda_0 + \delta < 1,$$

and also $\mathfrak{N}_{tr}(\dot{a} \star \dot{a}) = \lambda_0 < \delta$, so $\dot{a} \star \dot{a} \notin \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \delta)$, a contradiction.

If $\lambda_{\dot{a}} < \lambda_0$, choose $\delta > 0.5$ such that $\lambda_{\dot{a}} + \delta < 1 < \lambda_0 + \delta$. Then $0 \in \mathfrak{Tr}_q(\mathfrak{N}, \delta)$ and $a \in \mathfrak{Tr}_q(\mathfrak{N}, 1)$, but

$$\mathfrak{N}_{tr}(\dot{a} \star 0) + \delta = \lambda_{\dot{a}} + \delta < 1, \mathfrak{N}_{tr}(\dot{a} \star 0) = \lambda_{\dot{a}} < \delta,$$

so $\dot{a} \star 0 \notin \mathfrak{Tr}_{\in \vee q}(\mathfrak{N}, \delta)$, a contradiction. Hence, $\mathfrak{N}_{tr}(\dot{j}) \geq 0.5$ for some $\dot{j} \in \mathfrak{U}$.

By the same reasoning, there exists $\ddot{j} \in \mathfrak{U}$ with $\mathfrak{N}_{ind}(\ddot{j}) \geq 0.5$.

Now suppose $\mathfrak{N}_{fa}(\hat{\mathfrak{j}}) > 0.5$ for all $\hat{\mathfrak{j}} \in \mathfrak{U}$. Since \mathfrak{N}_{fa} is not constant on \mathfrak{U}_0^1 , there exists $\hat{\mathfrak{c}} \in \mathfrak{U}_0^1$ with $\nu_{\hat{\mathfrak{c}}} = \mathfrak{N}_{fa}(\hat{\mathfrak{c}}) \neq \mathfrak{N}_{fa}(0) = \nu_0$. Consider two cases:

If $\nu_{\hat{\mathfrak{c}}} < \nu_0$, then for some $\varepsilon \in [0, 0.5)$ we have $\nu_0 + \varepsilon > 1 > \nu_{\hat{\mathfrak{c}}} + \varepsilon$, so $\hat{\mathfrak{c}} \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \varepsilon)$. But then

$$\mathfrak{N}_{fa}(\hat{\mathfrak{c}} \star \hat{\mathfrak{c}}) + \varepsilon = \nu_0 + \varepsilon > 1, \mathfrak{N}_{fa}(\hat{\mathfrak{c}} \star \hat{\mathfrak{c}}) = \nu_0 > \varepsilon,$$

which implies $\hat{\mathfrak{c}} \star \hat{\mathfrak{c}} \notin \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \varepsilon)$, a contradiction.

If $\nu_{\hat{\mathfrak{c}}} > \nu_0$, choose $\varepsilon \in [0, 0.5)$ such that $\nu_0 + \varepsilon < 1 < \nu_{\hat{\mathfrak{c}}} + \varepsilon$. Then $0 \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \varepsilon)$ and $\hat{\mathfrak{c}} \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0)$. But

$$\mathfrak{N}_{fa}(\hat{\mathfrak{c}} \star 0) + \varepsilon = \nu_{\hat{\mathfrak{c}}} + \varepsilon > 1,$$

so $\hat{\mathfrak{c}} \star 0 \notin \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \varepsilon)$, again a contradiction.

Thus, $\mathfrak{N}_{fa}(\hat{\mathfrak{j}}) \leq 0.5$ for some $\hat{\mathfrak{j}} \in \mathfrak{U}$.

Next, we show that $\mathfrak{N}_{tr}(0) \geq 0.5$, $\mathfrak{N}_{ind}(0) \geq 0.5$, and $\mathfrak{N}_{fa}(0) \leq 0.5$. Suppose $\mathfrak{N}_{tr}(0) = \lambda_0 < 0.5$ while $\mathfrak{N}_{tr}(\hat{\mathfrak{j}}) = \lambda_{\hat{\mathfrak{j}}} \geq 0.5$ for some $\hat{\mathfrak{j}} \in \mathfrak{U}$. Choose λ_1 with $\lambda_0 < \lambda_1$ and $\lambda_0 + \lambda_1 < 1 < \lambda_{\hat{\mathfrak{j}}} + \lambda_1$. Then $\hat{\mathfrak{j}} \in \mathfrak{T}\mathfrak{r}_q(\mathfrak{N}, \lambda_1)$ but

$$\mathfrak{N}_{tr}(\hat{\mathfrak{j}} \star \hat{\mathfrak{j}}) + \lambda_1 = \lambda_0 + \lambda_1 < 1, \mathfrak{N}_{tr}(\hat{\mathfrak{j}} \star \hat{\mathfrak{j}}) = \lambda_0 < \lambda_1,$$

so $\hat{\mathfrak{j}} \star \hat{\mathfrak{j}} \notin \mathfrak{T}\mathfrak{r}_{\in \vee q}(\mathfrak{N}, \lambda_1)$, a contradiction. Hence, $\mathfrak{N}_{tr}(0) \geq 0.5$. Similarly, $\mathfrak{N}_{ind}(0) \geq 0.5$.

If $\mathfrak{N}_{fa}(0) = \nu_0 > 0.5$, but there exists $\hat{\mathfrak{j}}$ with $\mathfrak{N}_{fa}(\hat{\mathfrak{j}}) = \nu_{\hat{\mathfrak{j}}} \leq 0.5$, then $\nu_{\hat{\mathfrak{j}}} < \nu_0$. Choosing $\nu_1 < \nu_0$ with $\nu_0 + \nu_1 > 1 > \nu_{\hat{\mathfrak{j}}} + \nu_1$, we obtain $\hat{\mathfrak{j}} \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, \nu_1)$ but

$$\mathfrak{N}_{fa}(\hat{\mathfrak{j}} \star \hat{\mathfrak{j}}) + \nu_1 = \nu_0 + \nu_1 > 1, \mathfrak{N}_{fa}(\hat{\mathfrak{j}} \star \hat{\mathfrak{j}}) = \nu_0 > \nu_1,$$

so $\hat{\mathfrak{j}} \star \hat{\mathfrak{j}} \notin \mathfrak{F}\mathfrak{a}_{\in \vee q}(\mathfrak{N}, \nu_1)$, a contradiction. Thus, $\mathfrak{N}_{fa}(0) \leq 0.5$.

Finally, assume $\mathfrak{N}_{ind}(\hat{\mathfrak{j}}) = \mu_{\hat{\mathfrak{j}}} < 0.5$ for some $\hat{\mathfrak{j}} \in \mathfrak{U}_0^1$. Take $\mu > 0$ with $\mu_{\hat{\mathfrak{j}}} + \mu < 0.5$. Then $\hat{\mathfrak{j}} \in \mathfrak{I}\mathfrak{n}\mathfrak{d}_q(\mathfrak{N}, \mu)$ and $0 \in \mathfrak{I}\mathfrak{n}\mathfrak{d}_q(\mathfrak{N}, \mu + 0.5)$, but

$$\mathfrak{N}_{ind}(\hat{\mathfrak{j}} \star 0) + (\mu + 0.5) = \mu_{\hat{\mathfrak{j}}} + \mu + 0.5 < 1, \mathfrak{N}_{ind}(\hat{\mathfrak{j}} \star 0) = \mu_{\hat{\mathfrak{j}}} < \mu + 0.5,$$

so $\hat{\mathfrak{j}} \star 0 \notin \mathfrak{I}\mathfrak{n}\mathfrak{d}_{\in \vee q}(\mathfrak{N}, \mu + 0.5)$, a contradiction. Hence, $\mathfrak{N}_{ind}(\hat{\mathfrak{j}}) \geq 0.5$ for all $\hat{\mathfrak{j}} \in \mathfrak{U}_0^1$. By the same reasoning, $\mathfrak{N}_{tr}(\hat{\mathfrak{j}}) \geq 0.5$ for all $\hat{\mathfrak{j}} \in \mathfrak{U}_0^1$.

If $\mathfrak{N}_{fa}(\hat{\mathfrak{j}}) > 0.5$ for some $\hat{\mathfrak{j}} \in \mathfrak{U}_0^1$, then for some $\nu \in (0, 0.5)$ we have $0 \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0.5 - \nu)$ and $\hat{\mathfrak{j}} \in \mathfrak{F}\mathfrak{a}_q(\mathfrak{N}, 0)$, but $\mathfrak{N}_{fa}(\hat{\mathfrak{j}} \star 0) + 0.5 - \nu > 1$, a contradiction. Therefore, $\mathfrak{N}_{fa}(\hat{\mathfrak{j}}) \leq 0.5$ for all $\hat{\mathfrak{j}} \in \mathfrak{U}_0^1$. \square

4. Conclusion

This study explores the theoretical framework of NBCCSs within BCC-algebras, introducing a systematic classification of their types, including (\in, \in) -neutrosophic and $(q, \in \vee q)$ -neutrosophic subalgebras. By integrating NS theory, which generalizes FSs and IFSSs, we

establish threshold-based conditions (λ, μ, ν) for neutrosophic \in -subsets, q -subsets, and $\in \vee q$ -subsets to qualify as BCCSs. Key results demonstrate that these subsets exhibit subalgebraic properties under specific constraints (e.g., $\lambda, \mu > 0.5$; $\nu < 0.5$), extending classical fuzzy algebraic results. The work also highlights the role of the set \mathcal{U}_0^1 (elements with $T > 0$, $I > 0$, $F < 1$) in preserving subalgebraic structures and provides a method to construct NBCCSs from classical ones. These findings advance the interplay between neutrosophic logic and abstract algebra, offering a foundation for modeling uncertainty in logical systems and decision-making. Future research could explore dynamic neutrosophic systems or applications in computational intelligence.

Acknowledgments: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026: FF69).

Conflicts of Interest: The authors declare no conflict of interest.

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Received: April 26, 2025. Accepted: Sep 26, 2025