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Multi-Valued Interval Neutrosophic Soft Sets and Their Aggregation Operators

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Abstract: Recent studies have increasingly focused on aggregation within the neutrosophic environment due to its capability to handle ambiguity and uncertainty. However, to the authors' information, no current study has investigated aggregation operators for multi-valued interval neutrosophic soft numbers (MVINSSs) in the context of alternative ranking for decision-making problems. This paper proposes two novel aggregation operators: the multi-valued interval neutrosophic soft-weighted geometric averaging (MVINSWGA) and the multi-valued interval neutrosophic soft-weighted arithmetic averaging (MVINSWAA) operators under the MVINSS framework. The fundamental properties of the proposed operators, including idempotency, monotonicity, and boundedness, are established. In addition, a structured multi-criteria group decision-making (MCGDM) procedure incorporating the proposed operators is introduced. A numerical example involving software selection is provided to illustrate the applicability of the suggested approach. Comparative analysis confirms the consistency of ranking results, indicating that the MVINSWGA and MVINSWAA operators are robust and effective in addressing MCGDM problems within the MVINSS environment.

Keywords: arithmetic aggregation; geometric aggregation; multi-valued neutrosophic set; soft set; decision-making

1. Introduction

Classical set theory effectively models problems characterized by determinacy and precision. However, it lacks the capability to manage the uncertainty and imprecision frequently encountered in real-world situations. To address this shortcoming, several mathematical models have been introduced, including as fuzzy sets [1], interval-valued fuzzy sets [2], intuitionistic fuzzy sets (IFS) [3], vague sets [4] and rough sets [5]. Despite their contributions, these models are often limited by insufficient parameterization [6].

To overcome this, Molodtsov proposed the soft set (SS) theory [6], which emphasizes parameterization in decision approximation rather than relying solely on membership functions. Since its inception, SS theory has been applied to diverse areas such as integration [7], optimization [8], game theory [9], [10], lattice theory [11-13], algebraic structures [14], [15], topology [16-18], data analysis and operations research [19-22], medical diagnosis [23], and decision-making under uncertainty [24-28].

In parallel, Zadeh's fuzzy set theory [1] introduced the concept of fuzziness, enabling the handling of imprecise information. Maji et al. [29] later integrated FS with SS to propose the fuzzy soft set (FSS), which offers a framework to represent fuzzy information with parameterization. FSS has been widely explored [30-32] and applied in areas such as forecasting [33], medicine [34], and flood prediction [35].

To enhance FS further, Atanassov introduced IFS [3], which incorporates dual membership functions—truth and falsity—allowing simultaneous representation of membership and non-membership degrees. This led to the development of the intuitionistic fuzzy soft set (IFSS) by integrating IFS and SS [36], with several studies following [37-41]. However, IFS is constrained by the dependency between membership values, where the sum of truth and falsity is less than 1.

To address this, Smarandache [42] introduced neutrosophic set (NS) theory, which the symbols τ, δ and λ are used to represent truth-membership function, indeterminacy-membership function and falsity-membership function respectively, with each membership function ranging within the non-standard interval $]^{-}0, 1^{+}[$. This generalization allows NS to better interpret the ambiguous and confusing data that frequently arises in actual decision-making.

It can be said that the NS is a new set that overcomes the limitation of IFS. The dual memberships of IFSs are unable to cater for the indefinite and ambiguous information in which this kind of information always exists in belief systems and decision-making processes. The NS which consists of three independent memberships of truth, indeterminacy and falsity become an enhancement to the dual memberships of IFSs. Fundamentally, it is the generalization to the typical interval in IFS [3] which is $[0,1]$.

Recent years have seen active development in the study of neutrosophic set (NS) theory [43-47]. Recognizing the limitations of classical SS in uncertain contexts, researchers integrated NS with SS, forming the neutrosophic soft set (NSS) [48]. Numerous scholars have since worked on this concept. [49-52]. This framework was later extended into the interval-valued neutrosophic soft set (IVNSS) [53], enabling interval-based uncertainty modeling. The interval-valued neutrosophic set (IVNS) proposed by Wang et al. [54] supports more expressive modeling of imprecise, inadequate, and inconsistent data and has gained attention in various studies [55-57].

Meanwhile, Wang and Li extended NS into the multi-valued neutrosophic set (MVNS) [58], where the T, I, and F memberships are not limited to single values [59-63]. Alkhazaleh [64] further combined MVNS with SS to form the multi-valued neutrosophic soft set (MVNSS), suitable for problems involving multiple uncertain values [65-68].

Despite these developments, challenges remain when decision-makers (DMs) are faced with complex problems and are hesitant to provide single-valued or non-interval assessments. To accommodate such scenarios, Broumi et al. [69] proposed the multi-valued interval neutrosophic set (MVINS), which allows DMs to provide evaluations in the form of multi-valued interval memberships. This model has been discussed in several works [69-72].

Building on this, Mohd Kamal et al. [73] introduced the multi-valued interval neutrosophic soft set (MVIN-SS) by integrating SS and MVINS. This model is able to be used to multi-criteria group decision-making (MCGDM) cases and defines fundamental operations like intersection, union, complement, AND, and OR.

In MCGDM, aggregation is a critical step, where evaluations from multiple DMs are combined into a consensus decision. The weighted arithmetic average [74] and weighted geometric average [75] are foundational aggregation operators, widely applied across various domains. Extensions and variants include the trapezoidal intuitionistic fuzzy prioritized weighted averaging and geometric operators [76], aggregation under triangular intuitionistic fuzzy environments [77], single-valued neutrosophic weighted averaging (SVNWA) [78], and interval neutrosophic weighted operators [79-82].

Peng and Wang [62] focused on aggregation in multi-valued neutrosophic environments, while Ye [83] introduced trapezoidal neutrosophic number-based operators. Khan et al. [84] explored hesitant fuzzy aggregation using logarithmic spherical functions. Gao et al. [85] developed a linguistic aggregation framework, and Cagman et al. [86] proposed fuzzy soft aggregation operators. Saqlain et al. [49] and Jana and Pal [87] introduced aggregation techniques for neutrosophic hypersoft sets and single-valued neutrosophic soft sets, respectively.

Despite this progress, most existing aggregation techniques are restricted to the IFS, SVNS, IVNS, and SVNSS domains. There is a clear gap in exploring aggregation operators within the MVIN-SS framework, especially in MCGDM contexts involving interval-based and multi-valued evaluations.

To address this, we propose two novel aggregation operators for MVIN-SS: the multi-valued interval neutrosophic soft-weighted arithmetic averaging (MVINSWAA) and geometric averaging (MVINSWGA) operators. These operators effectively aggregate information characterized by uncertainty, vagueness, and indeterminacy, and accommodate interval and multi-valued inputs.

To ensure the mathematical rigor and reliability of the proposed operators, key properties such as idempotency, monotonicity, and boundedness are established through algebraic proofs. A numerical example focused on software selection demonstrates the practical application of these operators within an MCGDM framework.

The contributions of this paper are threefold: (1) we propose two novel aggregation operators—MVINSWAA and MVINSWGA—within the MVIN-SS framework; (2) we mathematically prove essential aggregation properties including idempotency, monotonicity, and boundedness; (3) we demonstrate the effectiveness of the proposed approach through a real-world case study involving software selection, using score functions to rank alternatives.

This paper has the following structure: The fundamental terms and ideas associated with MVIN-SS are reviewed in Section 2. Section 3 introduces MVINSWAA and MVINSWGA operators along with their mathematical properties. Section 4 presents an MCGDM framework incorporating the proposed operators. Section 5 provides an illustrative example. Section 6 offers a comparative analysis with existing methods. Section 7 wraps up the work and suggests areas for further research.

2. Preliminaries

In this section, we present some definitions and properties which are related to NS and MVIN-SS.

2.1. Neutrosophic Set

Definition 2.1 [42]

Let U be a universe of discourse, then NS A can be defined as

$$A = \{<\tau_A(y),\, \delta_A(y),\, \lambda_A(y)>/\, y, y \in U\}$$

where $\tau, \delta, \lambda : U \to \,]^-0, 1^+[$ define the degree of truth-membership $\tau_A(y)$, degree of indeterminacy $\delta_A(y)$ and degree of falsity $\lambda_A(y)$ respectively and there is no restriction on the sum of $\tau_A(y), \delta_A(y)$ and $\lambda_A(y)$, so ${}^-0 \le \tau_A(y) + \delta_A(y) + \lambda_A(y) \le 3^+$.

According to philosophical perspective, the NS derives its value from actual standard or non-standard subsets of $]^-0, 1^+[$. However, in real implementations, particularly in scientific and engineering domains, it is more appropriate to adopt the closed interval $[0, 1]$, as the use of $]^-0, 1^+[$ presents difficulties in real-world implementations.

2.2. Multi-Valued Interval Neutrosophic Set

Definition 2.2 [69]

Let U be a space of points (objects), with a generic element in U denoted by y. An MVINS $\dddot{A}$ over U can be defined as

$$\dddot{A} = \{< \dddot{\tau}^{l}_{\dddot{A}} \le (y), \dddot{\delta}^{m}_{\dddot{A}}(y), \dddot{\lambda}^{n}_{\dddot{A}}(y) > /y,\, y \in U\}$$

where

$\dddot{\tau}^{l}_{\dddot{A}}(y) = [\dddot{\tau}^{1}_{\dddot{A}}{}^{-}(y), \dddot{\tau}^{1}_{\dddot{A}}{}^{+}(y)], [\dddot{\tau}^{2}_{\dddot{A}}{}^{-}(y), \dddot{\tau}^{2}_{\dddot{A}}{}^{+}(y)], \ldots, [\dddot{\tau}^{q}_{\dddot{A}}{}^{-}(y), \dddot{\tau}^{q}_{\dddot{A}}{}^{+}(y)]$, $\dddot{\delta}^{m}_{\dddot{A}}(y) = [\dddot{\delta}^{1}_{\dddot{A}}{}^{-}(y), \dddot{\delta}^{1}_{\dddot{A}}{}^{+}(y)], [\dddot{\delta}^{2}_{\dddot{A}}{}^{-}(y), \dddot{\delta}^{2}_{\dddot{A}}{}^{+}(y)], \ldots, [\dddot{\delta}^{r}_{\dddot{A}}{}^{-}(y), \dddot{\delta}^{r}_{\dddot{A}}{}^{+}(y)]$, $\dddot{\lambda}^{n}_{\dddot{A}}(y) = [\dddot{\lambda}^{1}_{\dddot{A}}{}^{-}(y), \dddot{\lambda}^{1}_{\dddot{A}}{}^{+}(y)], [\dddot{\lambda}^{2}_{\dddot{A}}{}^{-}(y), \dddot{\lambda}^{2}_{\dddot{A}}{}^{+}(y)], \ldots, [\dddot{\lambda}^{s}_{\dddot{A}}{}^{-}(y), \dddot{\lambda}^{s}_{\dddot{A}}{}^{+}(y)] \in U\}$ such that $0 \le \tilde{\tau}^{l+}_{\tilde{A}}(y), \tilde{\delta}^{m+}_{\tilde{A}}(y), \tilde{\lambda}^{n+}_{\tilde{A}}(y) \le 3$, for all $l = 1, 2, \ldots, q,\ m = 1, 2, \ldots, r,\ n = 1, 2, \ldots, s.$

In this research, the interval truth-membership sequence $\ddot{\tau}^{l}_{\ddot{A}}(y)$, interval indeterminacy-membership sequence $\ddot{\delta}^{m}_{\ddot{A}}(y)$, and interval falsity-membership sequence $\ddot{\lambda}^{n}_{\ddot{A}}(y)$ of an element y are assumed to be equal, where $q=r=s$, respectively. The symbols l, m, n represent the dimensions of the MVINS A.. Clearly, upon equalizing the lower and upper bounds of $\ddot{\tau}^{l}_{\ddot{A}}(y), \ddot{\delta}^{m}_{\ddot{A}}(y), \ddot{\lambda}^{n}_{\ddot{A}}(y)$, the MVINS reduces to a MVNS.

Definition 2.3 [69]

Let $\ddot{A}$ and $\ddot{B}$ be two MVINS. Then some operations for MVINS are given as follows:

1) Difference

$$
\begin{aligned}
\ddot{A} \setminus \ddot{B} = \{< (& [\ddot{\tau}_{\ddot{A}}^{1\,-}(y) \wedge \ddot{\lambda}_{\ddot{B}}^{1\,-}(y), \ddot{\tau}_{\ddot{A}}^{1\,+}(y) \wedge \ddot{\lambda}_{\ddot{B}}^{1\,+}(y)], [\ddot{\tau}_{\ddot{A}}^{2\,-}(y) \wedge \ddot{\lambda}_{\ddot{B}}^{2\,-}(y), \ddot{\tau}_{\ddot{A}}^{2\,+}(y) \wedge \ddot{\lambda}_{\ddot{B}}^{2\,+}(y)], \ldots, \\
& [\ddot{\tau}_{\ddot{A}}^{q\,-}(y) \wedge \ddot{\lambda}_{\ddot{B}}^{q\,-}(y), \ddot{\tau}_{\ddot{A}}^{q\,+}(y) \wedge \ddot{\lambda}_{\ddot{B}}^{q\,+}(y)]), ([\dddot{\delta}_{\ddot{A}}^{1\,-}(y) \vee (1 - \dddot{\delta}_{\ddot{B}}^{1+}(y)), \dddot{\delta}_{\ddot{A}}^{1\,+}(y) \vee (1 - \dddot{\delta}_{\ddot{B}}^{1\,-}(y))], [\\
& \dddot{\delta}_{\ddot{A}}^{2\,-}(y) \vee (1 - \dddot{\delta}_{\ddot{B}}^{2\,+}(y)), \dddot{\delta}_{\ddot{A}}^{2\,+}(y) \vee (1 - \dddot{\delta}_{\ddot{B}}^{2\,-}(y))], \ldots, [\dddot{\delta}_{\ddot{A}}^{r\,-}(y) \vee (1 - \dddot{\delta}_{\ddot{B}}^{r\,+}(y)), \\
& \dddot{\delta}_{\ddot{A}}^{r\,+}(y) \vee (1 - \dddot{\delta}_{\ddot{B}}^{r\,-}(y))]), ([\ddot{\lambda}_{\ddot{A}}^{1\,-}(y) \vee \ddot{\tau}_{\ddot{B}}^{1-}(y), \ddot{\lambda}_{\ddot{A}}^{1\,+}(y) \vee \ddot{\tau}_{\ddot{B}}^{1+}(y)], [\ddot{\lambda}_{\ddot{A}}^{2\,-}(y) \vee \ddot{\tau}_{\ddot{B}}^{2\,-}(y), \ddot{\lambda}_{\ddot{A}}^{2\,+}(y) \vee \ddot{\tau}_{\ddot{B}}^{2\,+}(y)], \ldots, \\
& [\ddot{\lambda}_{\ddot{A}}^{s\,-}(y) \vee \ddot{\tau}_{\ddot{B}}^{s\,-}(y), \ddot{\lambda}_{\ddot{A}}^{s\,+}(y) \vee \ddot{\tau}_{\ddot{B}}^{s\,+}(y)]) > / y, \; y \in U.
\end{aligned}
$$

2) Addition

$$
\begin{aligned}
\ddot{A} + \ddot{B} = \{< (& [(\ddot{\tau}_{\ddot{A}}^{1\,-}(y) + \ddot{\tau}_{\ddot{B}}^{1\,-}(y)) \wedge 1, (\ddot{\tau}_{\ddot{A}}^{1\,+}(y) + \ddot{\tau}_{\ddot{B}}^{1\,+}(y)) \wedge 1], [(\ddot{\tau}_{\ddot{A}}^{2\,-}(y) + \ddot{\tau}_{\ddot{B}}^{2\,-}(y)) \wedge 1, (\ddot{\tau}_{\ddot{A}}^{2\,+}(y) + \ddot{\tau}_{\ddot{B}}^{2\,+}(y)) \wedge 1], \ldots, \\
& [(\ddot{\tau}_{\ddot{A}}^{q\,-}(y) + \ddot{\tau}_{\ddot{B}}^{q\,-}(y)) \wedge 1, (\ddot{\tau}_{\ddot{A}}^{q\,+}(y) + \ddot{\tau}_{\ddot{B}}^{q\,+}(y)) \wedge 1]), ([(\dddot{\delta}_{\ddot{A}}^{1\,-}(y) + \dddot{\delta}_{\ddot{B}}^{1-}(y)) \wedge 1, (\dddot{\delta}_{\ddot{A}}^{1\,+}(y) + \dddot{\delta}_{\ddot{B}}^{1\,+}(y)) \wedge 1], \\
& [(\dddot{\delta}_{\ddot{A}}^{2\,-}(y) + \dddot{\delta}_{\ddot{B}}^{2\,-}(y)) \wedge 1, (\dddot{\delta}_{\ddot{A}}^{2\,+}(y) + \dddot{\delta}_{\ddot{B}}^{2\,+}(y)) \wedge 1], [(\dddot{\delta}_{\ddot{A}}^{r\,-}(y) + \dddot{\delta}_{\ddot{B}}^{r\,-}(y)) \wedge 1, (\dddot{\delta}_{\ddot{A}}^{r\,+}(y) + \dddot{\delta}_{\ddot{B}}^{r\,+}(y)) \wedge 1]), \\
& \ldots, ([(\ddot{\lambda}_{\ddot{A}}^{1\,-}(y) + \ddot{\lambda}_{\ddot{B}}^{1-}(y)) \wedge 1, (\ddot{\lambda}_{\ddot{A}}^{1\,+}(y) + \ddot{\lambda}_{\ddot{B}}^{1\,+}(y)) \wedge 1], [(\ddot{\lambda}_{\ddot{A}}^{2\,-}(y) + \ddot{\lambda}_{\ddot{B}}^{2\,-}(y)) \wedge 1, (\ddot{\lambda}_{\ddot{A}}^{2\,+}(y) + \ddot{\lambda}_{\ddot{B}}^{2\,+}(y)) \wedge 1], \\
& \ldots, [(\ddot{\lambda}_{\ddot{A}}^{s\,-}(y) + \ddot{\lambda}_{\ddot{B}}^{s\,-}(y)) \wedge 1, (\ddot{\lambda}_{\ddot{A}}^{s\,+}(y) + \ddot{\lambda}_{\ddot{B}}^{s\,+}(y)) \wedge 1]) > / y, \; y \in U.
\end{aligned}
$$

3) Scalar Multiplication

$$
\begin{aligned}
\psi \square \ddot{A} = \{< (& [(\psi \square \ddot{\tau}_{\ddot{A}}^{1\,-}(y)) \wedge 1, (\psi \square \ddot{\tau}_{\ddot{A}}^{1\,+}(y)) \wedge 1)], [(\psi \square \ddot{\tau}_{\ddot{A}}^{2\,-}(y)) \wedge 1, (\psi \square \ddot{\tau}_{\ddot{A}}^{2\,+}(y)) \wedge 1)], \\
& \ldots, [(\psi \square \ddot{\tau}_{\ddot{A}}^{q\,-}(y)) \wedge 1, (\psi \square \ddot{\tau}_{\ddot{A}}^{q\,+}(y)) \wedge 1)]), ([(\psi \square \dddot{\delta}_{\ddot{A}}^{1\,-}(y)) \wedge 1, (\psi \square \dddot{\delta}_{\ddot{A}}^{1\,+}(y)) \wedge 1)], \\
& [(\psi \square \dddot{\delta}_{\ddot{A}}^{2\,-}(y)) \wedge 1, (\psi \square \dddot{\delta}_{\ddot{A}}^{2\,+}(y)) \wedge 1)], \ldots, [(\psi \square \dddot{\delta}_{\ddot{A}}^{r\,-}(y)) \wedge 1, (\psi \square \dddot{\delta}_{\ddot{A}}^{r\,+}(y)) \wedge 1)]), \\
& ([(\psi \square \ddot{\lambda}_{\ddot{A}}^{1\,-}(y)) \wedge 1, (\psi \square \ddot{\lambda}_{\ddot{A}}^{1\,+}(y)) \wedge 1)], [(\psi \square \ddot{\lambda}_{\ddot{A}}^{2\,-}(y)) \wedge 1, (\psi \square \ddot{\lambda}_{\ddot{A}}^{2\,+}(y)) \wedge 1)], \\
& \ldots, [(\psi \square \ddot{\lambda}_{\ddot{A}}^{s\,-}(y)) \wedge 1, (\psi \square \ddot{\lambda}_{\ddot{A}}^{s\,+}(y)) \wedge 1)]) / y, \; y \in U, \psi \in R^{+}\}.
\end{aligned}
$$

4) Scalar Division

$$
\begin{aligned}
\ddot{A} / \psi = \{< (& [(\ddot{\tau}_{\ddot{A}}^{1\,-}(y) / \psi) \wedge 1, (\ddot{\tau}_{\ddot{A}}^{1\,+}(y) / \psi) \wedge 1)], [\ddot{\tau}_{\ddot{A}}^{2\,-}(y) / \psi) \wedge 1, (\ddot{\tau}_{\ddot{A}}^{2\,+}(y) / \psi) \wedge 1)], \\
& \ldots, [\ddot{\tau}_{\ddot{A}}^{q\,-}(y) / \psi) \wedge 1, (\ddot{\tau}_{\ddot{A}}^{q\,+}(y) / \psi) \wedge 1)]), ([(\dddot{\delta}_{\ddot{A}}^{1\,-}(y) / \psi) \wedge 1, (\dddot{\delta}_{\ddot{A}}^{1\,+}(y) / \psi) \wedge 1)], \\
& [\dddot{\delta}_{\ddot{A}}^{2\,-}(y) / \psi) \wedge 1, (\dddot{\delta}_{\ddot{A}}^{2\,+}(y) / \psi) \wedge 1)], \ldots, [\dddot{\delta}_{\ddot{A}}^{r\,-}(y) / \psi) \wedge 1, (\dddot{\delta}_{\ddot{A}}^{r\,+}(y) / \psi) \wedge 1)]), \\
& ([(\ddot{\lambda}_{\ddot{A}}^{1\,-}(y) / \psi) \wedge 1, (\ddot{\lambda}_{\ddot{A}}^{1\,+}(y) / \psi) \wedge 1)], [(\ddot{\lambda}_{\ddot{A}}^{2\,-}(y) / \psi) \wedge 1, (\ddot{\lambda}_{\ddot{A}}^{2\,+}(y) / \psi) \wedge 1)], \\
& \ldots, [\ddot{\lambda}_{\ddot{A}}^{s\,-}(y) / \psi) \wedge 1, (\ddot{\lambda}_{\ddot{A}}^{s\,+}(y) / \psi) \wedge 1)]) / y, y \in U, \psi \in R^{+}\}.
\end{aligned}
$$

Definition 2.4 [72]

Let $\ddot{L} = \left\{ < \ddot{\tau}^{q}(y),\ \dddot{\delta}^{r}(y),\ \ddot{\lambda}^{s}(y) > / y;\ y \in U \right\}$ be an MVINS. Then,

$$
s(\ddot{L}) = \frac{1}{3}\left[\frac{1}{2q} \sum_{l=1}^{q} (\ddot{\tau}_{A_l}^{-} + \ddot{\tau}_{A_l}^{+}) + \frac{1}{2r} \sum_{m=1}^{r} (2 - \dddot{\delta}_{A_m}^{-} - \dddot{\delta}_{A_m}^{+}) + \frac{1}{2s} \sum_{n=1}^{s} (2 - \ddot{\lambda}_{A_n}^{-} - \ddot{\lambda}_{A_n}^{+}) \right] \tag{1}
$$

is called the score function for $\ddot{L}$ where l, m, n are the numbers of multi-valued interval values in $\ddot{\tau}^{q}(y), \dddot{\delta}^{r}(y), \ddot{\lambda}^{s}(y)$.

2.3. Soft Set

Definition 2.5 [6]

Let U be an initial universe set and E be a set of parameters. Consider $A \subset E.$ Let $P\,(U)$ denotes the power SS of $U.$ A pair (L, A) is called an SS over U and the function L is a mapping defined by $L : A \to P\,(U)$ such that $L(\varepsilon)(y) = \varphi$ if $y \notin U.$

Here, $L(\varepsilon)$ is called the approximate function of the soft set $(L, A),$ and the value $L(\varepsilon)(y)$ is a set called *x-element* of the SS for all $y \in U.$ The sets can be random, empty, or have non-empty intersections.

2.4. Multi-Valued Interval Neutrosophic Soft Set

Definition 2.6 [73]

The pair $(\ddot{L}, A)$ is called an MVIN-SS over $\ddot{P}(U),$ where $\ddot{P}$ is a mapping given by $\ddot{L} : A \to \ddot{P}(U).$ $\ddot{P}(U)$ denotes the set of all MVIN-SS of U with parameters from A and the function $\ddot{L}(\varepsilon)$ is a mapping defined by

$$\ddot{L} : A \to \ddot{P}(U) \text{ such that } \ddot{L}(\varepsilon)(y) = \varphi \text{ if } y \notin U.$$

$(\ddot{L}, A)$ is characterized by $\ddot{\tau}_{\ddot{L}(\varepsilon)}(y), \ddot{\delta}_{\ddot{L}(\varepsilon)}(y)$ and $\ddot{\lambda}_{\ddot{L}(\varepsilon)}(y)$ in the form of a subset of $[0,1]$ and can be defined as follows:

$$(\ddot{L},\, A) = \left\{ < \ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}(y),\ \ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}(y),\ \ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}(y) > /y;\ \forall \varepsilon \in A,\ y \in U \right\}$$

where

$\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}(y)=[\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],\ [\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)],\ \ldots,\ [\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}(y)=[\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],$ $[\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)],\ \ldots,\ [\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y)]$ and $\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}(y)=[\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],[\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)],$ $\ldots,\ [\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y)]$ are the interval truth-membership sequence, interval indeterminacy-membership sequence and interval falsity-membership sequence respectively that object y holds on parameter $\varepsilon.$

3. Aggregation Based on Multi-Valued Interval Neutrosophic Soft Set

In this part, we introduce the aggregation based on MVIN-SS which are the multi-valued interval neutrosophic soft-weighted geometric average (MVINSWGA) and multi-valued interval

neutrosophic soft-weighted arithmetic average (MVINSWAA) operators to aggregate the attributes and alternatives respectively.
We define the MVINSWGA and give proof of its properties.

Definition 3.1

Let $(\ddot{L},\, A) = \left\{ < \dddot{\tau}^{q}_{\ddot{L}(\varepsilon)}(y),\ \dddot{\delta}^{r}_{\ddot{L}(\varepsilon)}(y),\ \dddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}(y) > /\, \mathrm{y};\ \forall \varepsilon \in A,\ y \in U \right\}$ be an MVIN-SS. A mapping $MVINSWGA : \ddot{L}_n \to \ddot{L}$ is called a multi-valued interval neutrosophic soft weighted geometric averaging (MVINSWGA) operator if it satisfies

$$MVINSWGA\ (A_1,\, A_2,\, \ldots,\, A_n) =$$

$$\begin{gathered}\Bigg\langle \Bigg(\left[\prod_{l=1}^{q} (\dddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l},\ \prod_{l=1}^{q} (\dddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right], \left[\prod_{l=1}^{q} (\dddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l},\ \prod_{l=1}^{q} (\dddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right], \ldots, \left[\prod_{l=1}^{q} (\dddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l},\ \prod_{l=1}^{q} (\dddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right] \Bigg), \\ \Bigg(\left[1 - \prod_{m=1}^{r} (1 - \dddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m},\ 1 - \prod_{m=1}^{r} (1 - \dddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right], \left[1 - \prod_{m=1}^{r} (1 - \dddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m},\ 1 - \prod_{m=1}^{r} (1 - \dddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right], \ldots, \left[1 - \prod_{m=1}^{r} (1 - \dddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m},\ 1 - \prod_{m=1}^{r} (1 - \dddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right] \Bigg), \\ \Bigg(\left[1 - \prod_{n=1}^{s} (1 - \dddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n},\ 1 - \prod_{n=1}^{s} (1 - \dddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right], \left[1 - \prod_{n=1}^{s} (1 - \dddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n},\ 1 - \prod_{n=1}^{s} (1 - \dddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right], \ldots, \left[1 - \prod_{n=1}^{s} (1 - \dddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n},\ 1 - \prod_{n=1}^{s} (1 - \dddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right] \Bigg) \Bigg\rangle\end{gathered} \tag{2}$$

for all $\varepsilon \in A,\ y \in U.$

Theorem 1

Let $(\ddot{L},\, A) = \left\{ < \dddot{\tau}^{q}_{\ddot{L}(\varepsilon)}(y),\ \dddot{\delta}^{r}_{\ddot{L}(\varepsilon)}(y),\ \dddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}(y) > /\, y;\ \forall \varepsilon \in A,\ y \in U \right\}$ be an MVIN-SS. Then,

(1) Idempotency

If $\ddot{L}_j = \ddot{L}$ for all $j = 1, 2, \ldots, t,$ then $MVINSWGA\ (\ddot{L}_1,\, \ddot{L}_2,\, \ldots,\, \ddot{L}_t) = \ddot{L}.$

(2) Monotonicity

If $\ddot{L}_j \le \ddot{L}^{*}_j$ for all $j = 1, 2, \ldots, t,$ then

$MVINSWGA\ (\ddot{L}_1,\, \ddot{L}_2,\, \ldots,\, \ddot{L}_t) \le MVINSWGA^{*}\ (\ddot{L}^{*}_1,\, \ddot{L}^{*}_2,\, \ldots,\, \ddot{L}^{*}_t).$

(3) Boundedness

$\min\limits_{j=1,2,\ldots,\mathrm{t}} \{\ddot{L}_j\} \le MVINSWGA\ (\ddot{L}_1,\, \ddot{L}_2,\, \ldots,\, \ddot{L}_j) \le \max\limits_{j=1,2,\ldots,q} \{\ddot{L}_j\}.$

Proof (1) Idempotency:

Since

$$\ddot{L}_j = \ddot{L} = \Big\langle \Big([\dddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],\ [\dddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)],\ \ldots,\ [\dddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \Big),\ \Big([\dddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],$$

$$[\dddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ldots, [\dddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \Big),\ \Big([\dddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],\ [\dddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)],$$

$$\ldots, [\dddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y),\ \dddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \Big) \Big\rangle \text{ for all } j,$$

we have $MVINSWGA_{\omega}(\ddot{L}_j) = \prod_{j=1}^{t} (\ddot{L}_j)^{\omega_j}$

$$\Rightarrow \left\langle \left(\left[\prod_{l=1}^{q} (\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_l}, \prod_{l=1}^{q} (\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_l} \right], \left[\prod_{l=1}^{q} (\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_l}, \prod_{l=1}^{q} (\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_l} \right], \ldots, \left[\prod_{l=1}^{q} (\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_l}, \prod_{l=1}^{q} (\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_l} \right] \right), \right.$$
$$\left(\left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_m}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_m} \right], \left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_m}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_m} \right], \ldots, \left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_m}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_m} \right] \right),$$
$$\left. \left(\left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_n}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_n} \right], \left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_n}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_n} \right], \ldots, \left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_n}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_n} \right] \right) \right\rangle$$

$$\Rightarrow \left\langle \left(\left[(\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_l^q \omega_l}, (\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_l^q \omega_l} \right], \left[(\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_l^q \omega_l}, (\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_l^q \omega_l} \right], \ldots, \left[(\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_l^q \omega_l}, (\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_l^q \omega_l} \right] \right), \right.$$
$$\left(\left[1-(1-\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_m^r \omega_m}, 1-(1-\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_m^r \omega_m} \right], \left[1-(1-\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_m^r \omega_m}, 1-(1-\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_m^r \omega_m} \right], \ldots, \left[1-(1-\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_m^r \omega_m}, 1-(1-\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_m^r \omega_m} \right] \right),$$
$$\left. \left(\left[1-(1-\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_n^s \omega_n}, 1-(1-\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_n^s \omega_n} \right], \left[1-(1-\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_n^s \omega_n}, 1-(1-\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_n^s \omega_n} \right], \ldots, \left[1-(1-\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^-(y))^{\sum_n^s \omega_n}, 1-(1-\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^+(y))^{\sum_n^s \omega_n} \right] \right) \right\rangle$$

Since $\sum_l^q \omega_l = 1, \sum_m^r \omega_m = 1, \sum_n^s \omega_n = 1,$ we have

$$\Rightarrow \left\langle \left(\left[(\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^-(y)), (\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^+(y)) \right], \left[(\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^-(y)), (\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^+(y)) \right], \ldots, \left[(\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^-(y)), (\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^+(y)) \right] \right), \right.$$
$$\left(\left[1-(1-\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^-(y)), 1-(1-\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^+(y)) \right], \left[1-(1-\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^-(y)), 1-(1-\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^+(y)) \right], \ldots, \left[1-(1-\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^-(y)), 1-(1-\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^+(y)) \right] \right),$$
$$\left. \left(\left[1-(1-\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^-(y)), 1-(1-\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^+(y)) \right], \left[1-(1-\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^-(y)), 1-(1-\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^+(y)) \right], \ldots, \left[1-(1-\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^-(y)), 1-(1-\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^+(y)) \right] \right) \right\rangle$$

$$\Rightarrow \left\langle \left([\ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\tau}^1_{\ddot{L}(\varepsilon)}{}^+(y)], [\ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\tau}^2_{\ddot{L}(\varepsilon)}{}^+(y)], \ldots, [\ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\tau}^q_{\ddot{L}(\varepsilon)}{}^+(y)] \right), \left([\ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\delta}^1_{\ddot{L}(\varepsilon)}{}^+(y)], [\ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\delta}^2_{\ddot{L}(\varepsilon)}{}^+(y)], \right. \right.$$
$$\left. \left. \ldots, [\ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\delta}^r_{\ddot{L}(\varepsilon)}{}^+(y)] \right), \left([\ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\lambda}^1_{\ddot{L}(\varepsilon)}{}^+(y)], [\ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\lambda}^2_{\ddot{L}(\varepsilon)}{}^+(y)], \ldots, [\ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^-(y), \ddot{\lambda}^s_{\ddot{L}(\varepsilon)}{}^+(y)] \right\rangle \right)$$

$\Rightarrow \left\langle \ddot{\tau}^l_{\ddot{L}(\varepsilon)}(y), \ddot{\delta}^m_{\ddot{L}(\varepsilon)}(y), \ddot{\lambda}^n_{\ddot{L}(\varepsilon)}(y) \right\rangle = \ddot{L}$ which proves the Theorem 1 (1).

Proof (2) Monotonicity:

Since $\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^-(y) \geq \ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^{-*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^-(y) \leq 1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^{-*}(y)$$

$$\Rightarrow \prod_{l=1}^{q} (1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^-(y))^{\omega_l} \leq \prod_{l=1}^{q} (1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^{-*}(y))^{\omega_l}$$

Since $\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^+(y) \geq \ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^{+*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^+(y) \leq 1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^{+*}(y)$$

$$\Rightarrow \prod_{l=1}^{q} (1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^+(y))^{\omega_l} \leq \prod_{l=1}^{q} (1-\ddot{\tau}^l_{\ddot{L}(\varepsilon)}{}^{+*}(y))^{\omega_l}$$

Since $\ddot{\delta}^m_{\ddot{L}(\varepsilon)}{}^-(y) \leq \ddot{\delta}^m_{\ddot{L}(\varepsilon)}{}^{-*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\delta}^m_{\ddot{L}(\varepsilon)}{}^-(y) \geq 1-\ddot{\delta}^m_{\ddot{L}(\varepsilon)}{}^{-*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{-}(y)) \geq \prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{-*}(y))$$

$$\Rightarrow 1-\prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{-}(y)) \leq 1-\prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{-*}(y))$$

Since $\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+}(y) \leq \ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+}(y) \geq 1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+}(y)) \geq \prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+*}(y))$$

$$\Rightarrow 1-\prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+}(y)) \leq 1-\prod_{m}^{r}(1-\ddot{\delta}_{\ddot{L}(\varepsilon)}^{m}{}^{+*}(y))$$

Since $\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-}(y) \leq \ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-}(y) \geq 1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-}(y)) \geq \prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-*}(y))$$

$$\Rightarrow 1-\prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-}(y)) \leq 1-\prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{-*}(y))$$

Since $\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+}(y) \leq \ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+}(y) \geq 1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+}(y)) \geq \prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+*}(y))$$

$$\Rightarrow 1-\prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+}(y)) \leq 1-\prod_{m}^{r}(1-\ddot{\lambda}_{\ddot{L}(\varepsilon)}^{n}{}^{+*}(y))$$

In the proof above, we have $MVINSWGA\ (\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_t) \leq MVINSWGA^*\ (\ddot{L}_1^*, \ddot{L}_2^*, \ldots, \ddot{L}_t^*)$ which proves the Theorem 1 (2).

Proof (3) Boundedness:

Based on Property 1, we can get

$$MVINSWGA\ (\min_{j=1,2,..,t}\{\ddot{L}_j\}, \min_{j=1,2,..,t}\{\ddot{L}_j\}, \ldots, \min_{j=1,2,..,t}\{\ddot{L}_j\}) = \min_{j=1,2,..,t}\{\ddot{L}_j\}.$$

$$MVINSWGA\ (\max_{j=1,2,..,t}\{\ddot{L}_j\}, \max_{j=1,2,..,t}\{\ddot{L}_j\}, \ldots, \max_{j=1,2,..,t}\{\ddot{L}_j\}) = \max_{j=1,2,..,t}\{\ddot{L}_j\}.$$

Since $\min_{j=1,2,..,t}\{\ddot{L}_j\} \leq \ddot{L}_j \leq \max_{j=1,2,..,t}\{\ddot{L}_j\}$, then, based on Property 2, we have

$\min_{j=1,2,..,t}\{\ddot{L}_j\} \leq MVINSWGA\ (\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_j) \leq \max_{j=1,2,..,t}\{\ddot{L}_j\}$ which proves the Theorem 1 (3).

We will next define the MVINSWAA and provide proof of its properties.

Definition 3.2

Let $(\ddot{L}, A) = \left\{ < \ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}(y),\ \ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}(y),\ \ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}(y) > / \text{y};\ \forall \varepsilon \in A,\ y \in U \right\}$ be an MVIN-SS. A mapping $MVINSWAA : \ddot{L}_n \rightarrow \ddot{L}$ is called a multi-valued interval neutrosophic soft weighted arithmetic averaging (MVINSWAA) operator if it satisfies

$MVINSWAA\ (\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_n) =$

$$\left\langle \left(\left[1-\prod_{l=1}^{q}(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l}, 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right], \left[1-\prod_{l=1}^{q}(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l}, 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right], \ldots, \left[1-\prod_{l=1}^{q}(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l}, 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right] \right), \right.$$
$$\left(\left[\prod_{m=1}^{r}(\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m}, \prod_{m=1}^{r}(\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right], \left[\prod_{m=1}^{r}(\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m}, \prod_{m=1}^{r}(\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right], \ldots, \left[\prod_{m=1}^{r}(\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m}, \prod_{m=1}^{r}(\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right] \right),$$
$$\left. \left(\left[\prod_{n=1}^{s}(\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n}, \prod_{n=1}^{s}(\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right], \left[\prod_{n=1}^{s}(\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n}, \prod_{n=1}^{s}(\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right], \ldots, \left[\prod_{n=1}^{s}(\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n}, \prod_{n=1}^{s}(\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right] \right) \right\rangle \quad (3)$$

for all $\varepsilon \in A,\ y \in U.$

Theorem 2

Let $(\ddot{L}, A) = \left\{ < \ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}(y),\ \ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}(y),\ \ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}(y) > / \text{y};\ \forall \varepsilon \in A,\ y \in U \right\}$ be an MVIN-SS. Then,

1. Idempotency

   If $\ddot{L}_j = \ddot{L}$ for all $j = 1, 2, \ldots, t,$ then $MVINSWAA\ (\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_t) = \ddot{L}.$

2. Monotonicity

   If $\ddot{L}_j \le \ddot{L}^{*}_j$ for all $j = 1, 2, \ldots, t,$ then $MVINSWAA\ (\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_t) \le MVINSWAA^{*}\ (\ddot{L}^{*}_1, \ddot{L}^{*}_2, \ldots, \ddot{L}^{*}_t).$

3. Boundedness

   $$\min_{j=1,2,\ldots,t} \{\ddot{L}_j\} \le MVINSWAA\ (\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_t) \le \max_{j=1,2,\ldots,t} \{\ddot{L}_j\}$$

Proof (1) Idempotency:

Since

$\ddot{L}_j = \ddot{L} = \Big\langle \Big([\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)], [\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ldots, [\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \Big), \Big([\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)],$ $[\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ldots, [\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \Big), \Big([\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)], [\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ldots,$ $[\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \Big) \Big\rangle$ for all j, we have $MVINSWAA_{\omega}(\ddot{L}_j) = \prod_{j=1}^{t} (\ddot{L}_j)^{\omega_j}$

$$\Rightarrow \left\langle \left(\left[1-\prod_{l=1}^{q}(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l}, 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right], \left[1-\prod_{l=1}^{q}(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l}, 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right], \ldots, \left[1-\prod_{l=1}^{q}(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l}, 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \right] \right), \right.$$
$$\left(\left[\prod_{m=1}^{r}(\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m}, \prod_{m=1}^{r}(\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right], \left[\prod_{m=1}^{r}(\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m}, \prod_{m=1}^{r}(\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right], \ldots, \left[\prod_{m=1}^{r}(\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_m}, \prod_{m=1}^{r}(\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_m} \right] \right),$$
$$\left. \left(\left[\prod_{n=1}^{s}(\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n}, \prod_{n=1}^{s}(\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right], \left[\prod_{n=1}^{s}(\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n}, \prod_{n=1}^{s}(\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right], \ldots, \left[\prod_{n=1}^{s}(\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_n}, \prod_{n=1}^{s}(\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_n} \right] \right) \right\rangle$$

$$\Rightarrow \left\langle \left(\left[1-(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_l^q \omega_l}, 1-(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_l^q \omega_l} \right], \left[1-(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_l^q \omega_l}, 1-(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_l^q \omega_l} \right], \ldots, \left[1-(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_l^q \omega_l}, 1-(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_l^q \omega_l} \right] \right), \right.$$
$$\left(\left[(\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_m^r \omega_m}, (\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_m^r \omega_m} \right], \left[(\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_m^r \omega_m}, (\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_m^r \omega_m} \right], \ldots, \left[(\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_m^r \omega_m}, (\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_m^r \omega_m} \right] \right),$$
$$\left. \left(\left[(\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_n^s \omega_n}, (\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_n^s \omega_n} \right], \left[(\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_n^s \omega_n}, (\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_n^s \omega_n} \right], \ldots, \left[(\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\sum_n^s \omega_n}, (\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\sum_n^s \omega_n} \right] \right) \right\rangle$$

Since $\sum_{l}^{q}\omega_l = 1, \sum_{m}^{r}\omega_m = 1, \sum_{n}^{s}\omega_n = 1,$ we have

$$\Rightarrow \left\langle \left(\left[1-(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y)), 1-(1-\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)) \right], \left[1-(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y)), 1-(1-\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)) \right], \ldots, \left[1-(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y)), 1-(1-\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y)) \right] \right), \right.$$
$$\left. \left(\left[\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y) \right], \left[\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y) \right], \ldots, \left[\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y) \right] \right), \left(\left[\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y) \right], \left[\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y) \right], \ldots, \left[\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y) \right] \right) \right\rangle$$

$$\Rightarrow \left\langle \left([\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)], [\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ldots, [\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \right), \left([\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)], [\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \right. \right.$$
$$\left. \left. \ldots, [\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \right), \left([\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y)], [\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y)], \ldots, [\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y), \ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y)] \right) \right\rangle$$

$\Rightarrow \left\langle \ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}(y), \ \ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}(y), \ \ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}(y) \right\rangle = \ddot{L}$ which prove the Theorem 2 (1).

Proof (2) Monotonicity:

Since $\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-}(y) \le \ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-}(y) \ge 1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-*}(y)$$

$$\Rightarrow \prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l} \ge \prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-*}(y))^{\omega_l}$$

$$\Rightarrow 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega_l} \le 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{-*}(y))^{\omega_l}$$

Since $\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+}(y) \le \ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+}(y) \ge 1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+*}(y)$$

$$\Rightarrow \prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \ge \prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+*}(y))^{\omega_l}$$

$$\Rightarrow 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega_l} \le 1-\prod_{l=1}^{q}(1-\ddot{\tau}^{l}_{\ddot{L}(\varepsilon)}{}^{+*}(y))^{\omega_l}$$

Since $\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{-}(y) \geq \ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{-*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{-}(y) \leq 1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{-*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{-}(y)) \leq \prod_{m}^{r}(1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{-*}(y))$$

Since $\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{+}(y) \geq \ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{+*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{+}(y) \leq 1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{+*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{+}(y)) \leq \prod_{m}^{r}(1-\ddot{\delta}^{m}_{\ddot{L}(\varepsilon)}{}^{+*}(y))$$

Since $\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{-}(y) \geq \ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{-*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{-}(y) \leq 1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{-*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{-}(y)) \leq \prod_{m}^{r}(1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{-*}(y))$$

Since $\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{+}(y) \geq \ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{+*}(y)$ for all j, then we have

$$\Rightarrow 1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{+}(y) \leq 1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{+*}(y)$$

$$\Rightarrow \prod_{m}^{r}(1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{+}(y)) \leq \prod_{m}^{r}(1-\ddot{\lambda}^{n}_{\ddot{L}(\varepsilon)}{}^{+*}(y))$$

Based on the proof above, we have $MVINSWAA\,(\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_t) \leq MVINSWAA^{*}\,(\ddot{L}_1^{*}, \ddot{L}_2^{*}, \ldots, \ddot{L}_t^{*})$ which proves the Theorem 2 (2).

Proof (3) Boundedness:

Based on Property 1, we can get

$$MVINSWAA\,(\min_{j=1,2,..,t}\{\ddot{L}_j\}, \min_{j=1,2,..,t}\{\ddot{L}_j\}, \ldots, \min_{j=1,2,..,t}\{\ddot{L}_j\}) = \min_{j=1,2,..,t}\{\ddot{L}_j\}.$$

$$MVINSWAA\,(\max_{j=1,2,..,t}\{\ddot{L}_j\}, \max_{j=1,2,..,t}\{\ddot{L}_j\}, \ldots, \max_{j=1,2,..,t}\{\ddot{L}_j\}) = \max_{j=1,2,..,t}\{\ddot{L}_j\}.$$

Since $\min_{j=1,2,..,t}\{\ddot{L}_j\} \leq \ddot{L}_j \leq \max_{j=1,2,..,t}\{\ddot{L}_j\}$, then, based on Property 2, we have

$\min_{j=1,2,..,t}\{\ddot{L}_j\} \leq MVINSWAA\,(\ddot{L}_1, \ddot{L}_2, \ldots, \ddot{L}_j) \leq \max_{j=1,2,..,t}\{\ddot{L}_j\}$ which proves the Theorem 2 (3).

4. MCDM Method Based on The MVINSWGA and MVINSWAA Operators

In this section, we apply the MVINSWAA and MVINSWGA operators to solve MCDM problem under MVIN-SS environment. Some adaptations from [79] are made to meet this purpose. Let $U = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_n\}$ be a set of alternatives and $(\ddot{P}, A)$ be a set of parameters where $A = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$. The general procedure for MCDM method based on MVINSWGA and MVINSWAA operators is constructed in Figure 1:

Step 1: Construct the standard decision matrix.

If ς_j is a benefit attribute, then $(\ddot{B}, Q) = (\ddot{A}, P)$

If ς_j is a cost attribute, then $(\ddot{B}, Q) = (\ddot{A}, P^c)$

Then we get the standard decision matrix $D = [\ddot{B}, Q]_{m \times}$.

Step 2: Aggregate the attributes and alternatives

Proposed MVINSWGA Operator (Definition 3.1)

$$\left\langle \left(\left[\prod_{l=1}^{q} (\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, \prod_{l=1}^{q} (\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \left[\prod_{l=1}^{q} (\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, \prod_{l=1}^{q} (\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \ldots, \left[\prod_{l=1}^{q} (\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, \prod_{l=1}^{q} (\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right] \right), \right.$$
$$\left(\left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \ldots, \left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right] \right),$$
$$\left. \left(\left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \ldots, \left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right] \right) \right\rangle$$

Proposed MVINSWAA Operator (Definition 3.2)

$$\left\langle \left(\left[\prod_{l=1}^{q} (\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, \prod_{l=1}^{q} (\ddot{\tau}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \left[\prod_{l=1}^{q} (\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, \prod_{l=1}^{q} (\ddot{\tau}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \ldots, \left[\prod_{l=1}^{q} (\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, \prod_{l=1}^{q} (\ddot{\tau}^{q}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right] \right), \right.$$
$$\left(\left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \ldots, \left[1-\prod_{m=1}^{r} (1-\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{m=1}^{r} (1-\ddot{\delta}^{r}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right] \right),$$
$$\left. \left(\left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{1}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{2}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right], \ldots, \left[1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{-}(y))^{\omega}, 1-\prod_{n=1}^{s} (1-\ddot{\lambda}^{s}_{\ddot{L}(\varepsilon)}{}^{+}(y))^{\omega} \right] \right) \right\rangle$$

Step 3: Determine the multi-valued interval neutrosophic reference for positive-ideal solution (MVINRPIS, $\dddot{A}^{+}$) and negative-ideal solution (MVINRNIS, $\dddot{A}^{-}$) respectively

The $\dddot{A}^{+}$ and $\dddot{A}^{-}$ are obtained as follows:

$\dddot{A}^{*+} = \left([\ddot{\tau}^{-}_{\ddot{A}_l}, \ddot{\tau}^{+}_{\ddot{A}_l}], [\ddot{\delta}^{-}_{\ddot{A}_m}, \ddot{\delta}^{+}_{\ddot{A}_m}], [\ddot{\lambda}^{-}_{\ddot{A}_n}, \ddot{\lambda}^{+}_{\ddot{A}_n}] \right)$ where $\ddot{\tau}^{-}_{\ddot{A}_l} = \ddot{\tau}^{+}_{\ddot{A}_l} = 1$, $\ddot{\delta}^{-}_{\ddot{A}_m} = \ddot{\delta}^{+}_{\ddot{A}_m} = \ddot{\lambda}^{-}_{\ddot{A}_n} = \ddot{\lambda}^{+}_{\ddot{A}_n} = 0$,

$\dddot{A}^{*-} = \left([\ddot{\tau}^{-}_{\ddot{A}_l}, \ddot{\tau}^{+}_{\ddot{A}_l}], [\ddot{\delta}^{-}_{\ddot{A}_m}, \ddot{\delta}^{+}_{\ddot{A}_m}], [\ddot{\lambda}^{-}_{\ddot{A}_n}, \ddot{\lambda}^{+}_{\ddot{A}_n}] \right)$ where $\ddot{\tau}^{-}_{\ddot{A}_l} = \ddot{\tau}^{+}_{\ddot{A}_l} = 0$, $\ddot{\delta}^{-}_{\ddot{A}_m} = \ddot{\delta}^{+}_{\ddot{A}_m} = \ddot{\lambda}^{-}_{\ddot{A}_n} = \ddot{\lambda}^{+}_{\ddot{A}_n} = 1$,

$l = 1, 2, \ldots, q$, $m = 1, 2, \ldots, r$, $n = 1, 2, \ldots, s$.

The equation for distance of each alternative $\ddot{A}$ from $\dddot{A}^{*+}$ and $\dddot{A}^{*-}$ is given as follows:

$$d^{+} = \left(\frac{1}{3}\left(\frac{1}{2q} \sum_{l=1}^{q} \left((\ddot{\tau}^{-}_{\ddot{A}_l^{*}} - \ddot{\tau}^{-}_{\ddot{A}_l})^2 + (\ddot{\tau}^{+}_{\ddot{A}_l^{*}} - \ddot{\tau}^{+}_{\ddot{A}_l})^2 \right) \right) + \left(\frac{1}{2r} \sum_{m=1}^{r} \left((\ddot{\delta}^{-}_{\ddot{A}_m^{*}} - \ddot{\delta}^{-}_{\ddot{A}_m})^2 + (\ddot{\delta}^{+}_{\ddot{A}_m^{*}} - \ddot{\delta}^{+}_{\ddot{A}_m})^2 \right) \right) + \left(\frac{1}{2s} \sum_{n=1}^{s} \left((\ddot{\lambda}^{-}_{\ddot{A}_n^{*}} - \ddot{\lambda}^{-}_{\ddot{A}_n})^2 + (\ddot{\lambda}^{+}_{\ddot{A}_n^{*}} - \ddot{\lambda}^{+}_{\ddot{A}_n})^2 \right) \right) \right)^{\frac{1}{2}} \tag{4}$$

$$d^{-} = \left(\frac{1}{3}\left(\frac{1}{2q} \sum_{l=1}^{q} \left((\ddot{\tau}^{-}_{\ddot{A}_l^{*}} - \ddot{\tau}^{-}_{\ddot{A}_l})^2 + (\ddot{\tau}^{+}_{\ddot{A}_l^{*}} - \ddot{\tau}^{+}_{\ddot{A}_l})^2 \right) \right) + \left(\frac{1}{2r} \sum_{m=1}^{r} \left((\ddot{\delta}^{-}_{\ddot{A}_m^{*}} - \ddot{\delta}^{-}_{\ddot{A}_m})^2 + (\ddot{\delta}^{+}_{\ddot{A}_m^{*}} - \ddot{\delta}^{+}_{\ddot{A}_m})^2 \right) \right) + \left(\frac{1}{2s} \sum_{n=1}^{s} \left((\ddot{\lambda}^{-}_{\ddot{A}_n^{*}} - \ddot{\lambda}^{-}_{\ddot{A}_n})^2 + (\ddot{\lambda}^{+}_{\ddot{A}_n^{*}} - \ddot{\lambda}^{+}_{\ddot{A}_n})^2 \right) \right) \right)^{\frac{1}{2}} \tag{5}$$

where d^{+} : shortest distance of $\dddot{A}_n$;

d^{-} : farthest distance of $\dddot{A}_n$

Then, the closeness coefficient for all alternatives is calculated as follows.

$$CC_i = \frac{d_i^{-}}{d_i^{+} + d_i^{-}} \tag{6}$$

Step 4: Rank the alternatives according to the value of closeness coefficient. The greater the value of closeness coefficient, the higher the ranking position of the alternative.

Figure 1 The general procedure for MCDM method based on MVINSWGA and MVINSAA operators

5. Illustrative Example

This section provides an example of a software selection problem which adapted from [83], in which MCDM problem consists of MVINS environment. The proposed set is applied to a decision-making approach in order to illustrate its effectiveness and feasibility.

Suppose that an investment company wants to choose the best software. Let $U = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}$ be the universe set consisting of five software systems (alternatives). There are three decision-makers (D_1, D_2, D_3) from different departments who are involved in obtaining their opinions and selecting best software. Let us consider the soft set $(\ddot{P}, A)$ describes the "quality of software" where $A = \{\varepsilon_1 = (\text{contribution to organization performance}),$ $\varepsilon_2 = (\text{effort to transform from current system}),$ $\varepsilon_3 = (\text{hardware/software investment cost}),$ $\varepsilon_4 = (\text{out sourcing software developer reliability})\}$. The weights of parameters are $\omega = (0.15, 0.40, 0.35, 0.10)^T$.

Step 1: The standard decision matrix D_i is constructed and shown in Table 1.

Table 1 Standard decision matrix, D

	U	ε_1 = contribution to organization performance	ε_2 = effort to transform from current system	ε_3 = hardware/software investment cost	ε_4 = out sourcing software developer reliability
D_1	ς_1	$\langle([0.8,0.9],[0.6,0.9]),([0.3,0.6],[0.1,0.5]),([0.6,0.8],[0.7,0.8])\rangle$	$\langle([0.2,0.5],[0.2,0.5]),([0.4,0.7],[0.2,0.7]),([0.2,0.5],[0.1,0.7])\rangle$	$\langle([0.2,0.5],[0.6,0.9]),([0.1,0.3],[0.2,0.6]),([0.4,0.7],[0.4,0.6])\rangle$	$\langle([0.4,0.6],[0.5,0.8]),([0.2,0.4],[0.1,0.4]),([0.5,0.8],[0.5,0.7])\rangle$
	ς_2	$\langle([0.2,0.4],[0.5,0.8]),([0.2,0.8],[0.5,0.9]),([0.1,0.4],[0.1,0.5])\rangle$	$\langle([0.4,0.7],[0.4,0.8]),([0.6,0.9],[0.1,0.3]),([0.6,0.7],[0.2,0.5])\rangle$	$\langle([0.5,0.8],[0.2,0.7]),([0.5,0.8],[0.7,0.9]),([0.4,0.7],[0.3,0.5])\rangle$	$\langle([0.2,0.3],[0.5,0.7]),([0.6,0.8],[0.3,0.6]),([0.2,0.6],[0.6,0.9])\rangle$
	ς_3	$\langle([0.4,0.7],[0.4,0.8]),([0.3,0.6],[0.4,0.6]),([0.6,0.9],[0.4,0.7])\rangle$	$\langle([0.2,0.8],[0.1,0.4]),([0.5,0.7],[0.2,0.6]),([0.3,0.6],[0.3,0.6])\rangle$	$\langle([0.3,0.5],[0.2,0.5]),([0.5,0.9],[0.3,0.7]),([0.7,0.9],[0.7,0.8])\rangle$	$\langle([0.7,0.9],[0.6,0.9]),([0.4,0.6],[0.4,0.5]),([0.2,0.3],[0.1,0.3])\rangle$
	ς_4	$\langle([0.1,0.5],[0.2,0.4]),([0.2,0.5],[0.2,0.4]),([0.3,0.4],[0.6,0.9])\rangle$	$\langle([0.5,0.8],[0.3,0.8]),([0.1,0.7],[0.2,0.5]),([0.7,0.8],[0.4,0.6])\rangle$	$\langle([0.2,0.5],[0.2,0.6]),([0.3,0.5],[0.3,0.7]),([0.5,0.9],[0.7,0.9])\rangle$	$\langle([0.2,0.6],[0.1,0.5]),([0.1,0.4],[0.5,0.8]),([0.6,0.7],[0.1,0.3])\rangle$
	ς_5	$\langle([0.1,0.7],[0.3,0.4]),([0.2,0.5],[0.2,0.7]),([0.2,0.3],[0.6,0.9])\rangle$	$\langle([0.1,0.2],[0.5,0.7]),([0.2,0.5],[0.2,0.5]),([0.4,0.7],[0.4,0.8])\rangle$	$\langle([0.6,0.9],[0.2,0.4]),([0.1,0.4],[0.5,0.7]),([0.2,0.5],[0.3,0.7])\rangle$	$\langle([0.4,0.5],[0.2,0.5]),([0.1,0.5],[0.6,0.8]),([0.4,0.9],[0.2,0.6])\rangle$

	U	$\varepsilon_1 =$ contribution to organization performance	$\varepsilon_2 =$ effort to transform from current system
D_2	ς_1	$\langle([0.4,0.7],[0.2,0.5]),([0.8,0.9],[0.2,0.6]),([0.7,0.9],[0.4,0.7])\rangle$	$\langle([0.2,0.5],[0.2,0.6]),([0.4,0.8],[0.2,0.7]),([0.1,0.5],[0.8,0.9])\rangle$
	ς_2	$\langle([0.3,0.6],[0.2,0.5]),([0.7,0.9],[0.4,0.6]),([0.6,0.9],[0.1,0.3])\rangle$	$\langle([0.2,0.6],[0.6,0.8]),([0.1,0.4],[0.2,0.5]),([0.3,0.6],[0.7,0.9])\rangle$
	ς_3	$\langle([0.2,0.5],[0.3,0.5]),([0.5,0.7],[0.8,0.9]),([0.4,0.7],[0.1,0.4])\rangle$	$\langle([0.1,0.5],[0.2,0.6]),([0.2,0.4],[0.5,0.8]),([0.1,0.4],[0.7,0.9])\rangle$
	ς_4	$\langle([0.4,0.6],[0.3,0.7]),([0.5,0.9],[0.1,0.4]),([0.2,0.6],[0.3,0.4])\rangle$	$\langle([0.2,0.4],[0.1,0.5]),([0.5,0.7],[0.1,0.4]),([0.6,0.9],[0.1,0.4])\rangle$
	ς_5	$\langle([0.2,0.4],[0.1,0.5]),([0.4,0.8],[0.6,0.9]),([0.4,0.8],[0.4,0.6])\rangle$	$\langle([0.1,0.5],[0.2,0.5]),([0.6,0.9],[0.3,0.7]),([0.3,0.8],[0.6,0.9])\rangle$

	U	$\varepsilon_3 =$ hardware/software investment cost	$\varepsilon_4 =$ outsourcing software developer reliability
D_2	ς_1	$\langle([0.3,0.7],[0.1,0.4]),([0.2,0.6],[0.2,0.5]),([0.4,0.7],[0.3,0.6])\rangle$	$\langle([0.5,0.7],[0.3,0.7]),([0.4,0.8],[0.2,0.7]),([0.4,0.7],[0.5,0.9])\rangle$
	ς_2	$\langle([0.4,0.6],[0.2,0.5]),([0.7,0.9],[0.3,0.6]),([0.2,0.5],[0.8,0.9])\rangle$	$\langle([0.6,0.7],[0.2,0.5]),([0.5,0.8],[0.1,0.4]),([0.1,0.6],[0.5,0.8])\rangle$
	ς_3	$\langle([0.1,0.4],[0.2,0.6]),([0.1,0.5],[0.8,0.9]),([0.5,0.9],[0.4,0.9])\rangle$	$\langle([0.1,0.6],[0.5,0.8]),([0.1,0.4],[0.8,0.9]),([0.5,0.7],[0.7,0.8])\rangle$
	ς_4	$\langle([0.1,0.4],[0.5,0.7]),([0.2,0.4],[0.6,0.8]),([0.6,0.9],[0.3,0.6])\rangle$	$\langle([0.3,0.5],[0.2,0.5]),([0.2,0.4],[0.6,0.9]),([0.3,0.5],[0.1,0.4])\rangle$
	ς_5	$\langle([0.4,0.7],[0.1,0.5]),([0.2,0.7],[0.7,0.9]),([0.2,0.7],[0.2,0.7])\rangle$	$\langle([0.2,0.6],[0.2,0.6]),([0.4,0.7],[0.2,0.6]),([0.7,0.9],[0.1,0.4])\rangle$

	U	$\varepsilon_1 =$ contribution to organization performance	$\varepsilon_2 =$ effort to transform from current system
D_3	ς_1	$\langle([0.1,0.4],[0.2,0.7]),([0.2,0.7],[0.5,0.8]),([0.2,0.7],[0.3,0.6])\rangle$	$\langle([0.7,0.9],[0.2,0.5]),([0.4,0.7],[0.5,0.7]),([0.8,0.9],[0.2,0.5])\rangle$
	ς_2	$\langle([0.2,0.5],[0.3,0.8]),([0.4,0.8],[0.2,0.7]),([0.1,0.7],[0.2,0.4])\rangle$	$\langle([0.7,0.9],[0.1,0.4]),([0.4,0.7],[0.3,0.5]),([0.8,0.9],[0.2,0.4])\rangle$
	ς_3	$\langle([0.3,0.4],[0.1,0.7]),([0.4,0.5],[0.3,0.5]),([0.7,0.9],[0.2,0.5])\rangle$	$\langle([0.5,0.7],[0.3,0.5]),([0.3,0.6],[0.1,0.4]),([0.6,0.9],[0.7,0.9])\rangle$
	ς_4	$\langle([0.2,0.4],[0.1,0.5]),([0.3,0.7],[0.3,0.6]),([0.3,0.7],[0.4,0.8])\rangle$	$\langle([0.6,0.9],[0.2,0.6]),([0.3,0.8],[0.4,0.8]),([0.3,0.7],[0.1,0.3])\rangle$
	ς_5	$\langle([0.3,0.7],[0.3,0.6]),([0.1,0.4],[0.2,0.8]),([0.1,0.4],[0.2,0.6])\rangle$	$\langle([0.4,0.7],[0.2,0.5]),([0.1,0.7],[0.2,0.5]),([0.5,0.9],[0.2,0.7])\rangle$

	U	$\varepsilon_3 =$ hardware/software investment cost	$\varepsilon_4 =$ outsourcing software developer reliability
D_3	ς_1	$\langle([0.1,0.3],[0.4,0.7]),([0.2,0.6],[0.2,0.4]),([0.1,0.8],[0.6,0.8])\rangle$	$\langle([0.2,0.6],[0.2,0.5]),([0.3,0.5],[0.2,0.4]),([0.7,0.9],[0.2,0.6])\rangle$
	ς_2	$\langle([0.5,0.8],[0.5,0.8]),([0.3,0.6],[0.4,0.7]),([0.3,0.6],[0.4,0.7])\rangle$	$\langle([0.3,0.7],[0.1,0.4]),([0.6,0.8],[0.3,0.7]),([0.2,0.6],[0.4,0.8])\rangle$
	ς_3	$\langle([0.3,0.6],[0.4,0.7]),([0.3,0.7],[0.3,0.7]),([0.8,0.9],[0.5,0.8])\rangle$	$\langle([0.2,0.7],[0.2,0.6]),([0.4,0.7],[0.5,0.9]),([0.7,0.8],[0.4,0.8])\rangle$
	ς_4	$\langle([0.3,0.5],[0.6,0.8]),([0.1,0.4],[0.2,0.6]),([0.3,0.6],[0.4,0.7])\rangle$	$\langle([0.7,0.9],[0.3,0.6]),([0.4,0.7],[0.4,0.7]),([0.1,0.5],[0.7,0.9])\rangle$
	ς_5	$\langle([0.6,0.9],[0.3,0.6]),([0.7,0.8],[0.2,0.6]),([0.4,0.7],[0.8,0.9])\rangle$	$\langle([0.4,0.7],[0.4,0.5]),([0.6,0.7],[0.2,0.5]),([0.1,0.3],[0.4,0.5])\rangle$

Step 2: Aggregate the attributes and alternatives using MVINSWGA and MVINSWAA operator.

By applying the equation in Definition 3.1, the aggregated attributes are presented in Table 2.

Table 2 Aggregated Attributes

	U	ε_1 = contribution to organization performance	ε_2 = effort to transform from current system
	ς_1	$\langle([0.18, 0.50], [0.15, 0.56]), ([0.67, 0.89], [0.40, 0.80]), ([0.69, 0.92], [0.65, 0.85])\rangle$	$\langle([0.17, 0.47], [0.09, 0.39]), ([0.54, 0.87], [0.43, 0.84]), ([0.62, 0.84], [0.62, 0.88])\rangle$
	ς_2	$\langle([0.11, 0.35], [0.17, 0.57]), ([0.62, 0.94], [0.51, 0.89]), ([0.43, 0.87], [0.20, 0.54])\rangle$	$\langle([0.24, 0.61], [0.15, 0.51]), ([0.541, 0.87], [0.29, 0.58]), ([0.76, 0.89], [0.56, 0.83])\rangle$
D	ς_3	$\langle([0.15, 0.37], [0.11, 0.53]), ([0.54, 0.76], [0.71, 0.86]), ([0.73, 0.95], [0.34, 0.70])\rangle$	$\langle([0.10, 0.53], [0.08, 0.35]), ([0.47, 0.73], [0.40, 0.78]), ([0.50, 0.85], [0.75, 0.94])\rangle$
	ς_4	$\langle([0.09, 0.35], [0.08, 0.37]), ([0.47, 0.88], [0.29, 0.62]), ([0.37, 0.73], [0.59, 0.89])\rangle$	$\langle([0.24, 0.54], [0.08, 0.49]), ([0.44, 0.87], [0.34, 0.76]), ([0.71, 0.92], [0.30, 0.59])\rangle$
	ς_5	$\langle([0.08, 0.44], [0.09, 0.35]), ([0.34, 0.76], [0.49, 0.92]), ([0.34, 0.71], [0.56, 0.87])\rangle$	$\langle([0.06, 0.35], [0.14, 0.42]), ([0.46, 0.88], [0.33, 0.73]), ([0.54, 0.92], [0.56, 0.92])\rangle$

	U	ε_3 = hardware/software investment cost	ε_4 = outsourcing software developer reliability
D	ς_1	$\langle([0.08, 0.32], [0.15, 0.50]), ([0.24, 0.67], [0.28, 0.65]), ([0.43, 0.87], [0.59, 0.82])\rangle$	$\langle([0.20, 0.50], [0.17, 0.53]), ([0.42, 0.76], [0.24, 0.67]), ([0.70, 0.92], [0.55, 0.89])\rangle$
	ς_2	$\langle([0.32, 0.62], [0.14, 0.53]), ([0.68, 0.91], [0.65, 0.89]), ([0.42, 0.76], [0.71, 0.88])\rangle$	$\langle([0.19, 0.38], [0.10, 0.37]), ([0.72, 0.91], [0.34, 0.73]), ([0.24, 0.75], [0.65, 0.94])\rangle$
	ς_3	$\langle([0.09, 0.35], [0.13, 0.46]), ([0.44, 0.88], [0.69, 0.91]), ([0.83, 0.97], [0.70, 0.94])\rangle$	$\langle([0.12, 0.61], [0.24, 0.66]), ([0.43, 0.73], [0.76, 0.93]), ([0.65, 0.80], [0.60, 0.83])\rangle$
	ς_4	$\langle([0.08, 0.32], [0.24, 0.58]), ([0.29, 0.58], [0.53, 0.85]), ([0.63, 0.94], [0.65, 0.89])\rangle$	$\langle([0.20, 0.52], [0.08, 0.39]), ([0.34, 0.67], [0.65, 0.92]), ([0.50, 0.73], [0.51, 0.80])\rangle$
	ς_5	$\langle([0.38, 0.75], [0.08, 0.35]), ([0.54, 0.81], [0.65, 0.89]), ([0.38, 0.79], [0.67, 0.91])\rangle$	$\langle([0.18, 0.46], [0.13, 0.39]), ([0.54, 0.79], [0.49, 0.80]), ([0.60, 0.92], [0.34, 0.65])\rangle$

Refer to the equation in Definition 3.2, the aggregated alternatives are given in Table 3.

Table 3 Aggregated Alternatives

U	Aggregated Matrix
ς_1	$\langle([0.29, 0.70], [0.27, 0.75]), ([0.19, 0.62], [0.11, 0.54]), ([0.36, 0.79], [0.36, 0.74])\rangle$
ς_2	$\langle([0.39, 0.76], [0.27, 0.75]), ([0.40, 0.82], [0.18, 0.58]), ([0.18, 0.66], [0.23, 0.61])\rangle$
ς_3	$\langle([0.22, 0.73], [0.26, 0.76]), ([0.22, 0.60], [0.38, 0.75]), ([0.44, 0.78], [0.33, 0.72])\rangle$
ς_4	$\langle([0.29, 0.68], [0.23, 0.71]), ([0.14, 0.54], [0.18, 0.60]), ([0.29, 0.68], [0.24, 0.61])\rangle$
ς_5	$\langle([0.34, 0.77], [0.21, 0.61]), ([0.21, 0.65], [0.23, 0.69]), ([0.21, 0.69], [0.27, 0.69])\rangle$

Step 3: The multi-valued interval neutrosophic reference for positive-ideal solution (MVINRPIS, $\dddot{A}^{+}$) and negative-ideal solution (MVINRNIS, $\dddot{A}^{-}$) are identified respectively.

The $\dddot{A}^{+}$ and $\dddot{A}^{-}$ are determined as follows:

$$\dddot{A}^{+}=\left([1,1],[1,1],[0,0],[0,0],[0,0],[0,0]\right)$$

$$\dddot{A}^{-}=\left([0,0],[0,0],[1,1],[1,1],[1,1],[1,1]\right)$$

The Euclidean distance of each alternative from $\dddot{A}^{+}$ and $\dddot{A}^{-}$ is calculated by using eqn (4) and (5) as presented in Table 4.

Table 4 The distance of each alternative from $\dddot{A}^{+}$ and $\dddot{A}^{-}$

Alternatives	d^{+}	d^{-}
ς_1	0.5279	0.5734
ς_2	0.5098	0.5873
ς_3	0.5650	0.5269
ς_4	0.4969	0.5932
ς_5	0.5273	0.5705

Then, the closeness coefficient for each alternative is calculated by using eqn (6) as shown in Table 5.

Table 5 The closeness coefficients of each alternative

Alternatives	CC_n	Ranking
ς_1	0.5207	3
ς_2	0.5353	2
ς_3	0.4825	5
ς_4	0.5442	1
ς_5	0.5197	4

Step 4: According to the value of closeness coefficient in Table 5, we can rank the alternatives in descending order as

$$\varsigma_4 \succ \varsigma_2 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_3$$

where the symbol "$\succ$" refers to 'superior to'. So, it can be concluded that ς_4 is the best alternative. With similar computation, the value of score function (from eqn 1) is calculated and presented in Table 6.

Table 6 The score function of each alternative

Alternatives	Score Function	Ranking
ς_1	0.5251	3
ς_2	0.5418	2
ς_3	0.4794	5
ς_4	0.5522	1
ς_5	0.5237	4

It can be seen that the highest value of closeness coefficient is 0.5442 and the highest value of score function is 0.5522 (see Table 5 and Table 6 respectively). This obviously shows that ς_4 is the best alternative.

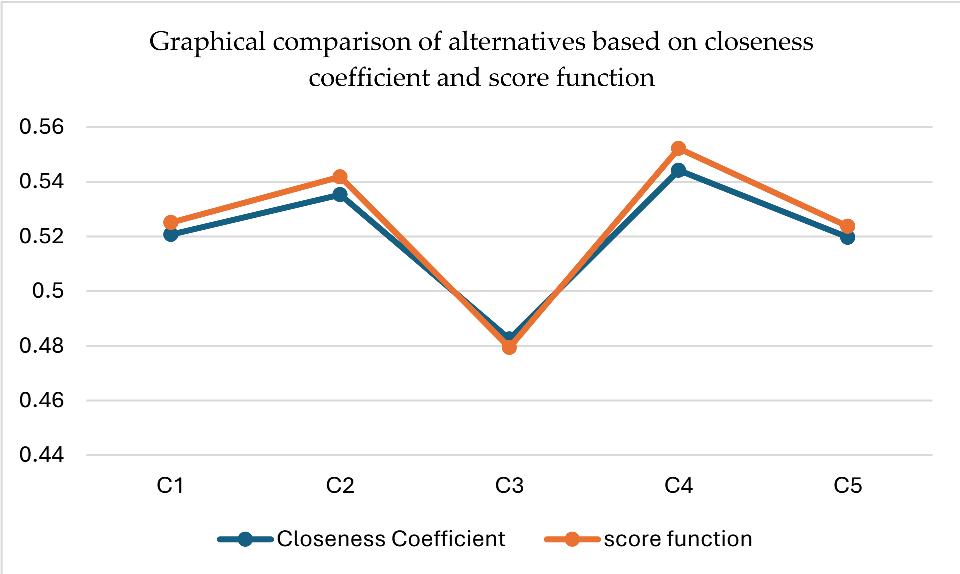

Figure 2 Graphical comparison of alternatives based on closeness coefficient and score function.

The alternatives can be ranked as $\varsigma_4 \succ \varsigma_2 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_3$ according to the score function value. Figure 2 presents a line chart comparing the closeness coefficients and score function values of each alternative. As observed, Alternative ς_4 has the highest values in both metrics, confirming it as the best option. This graphical representation facilitates an intuitive understanding of the ranking consistency across different evaluation measures. Not only the best choice, in fact, the order of preference of the proposed aggregation method under MVIN-SS information either using score function or closeness coefficient is consistent.

6. Comparative Analysis

By adopting the same case study of software selection problems, a comparative analysis with other methods is conducted to validate the efficacy and feasibility of the proposed decision-making approach based on MVINSWGA and MVINSWAA operators. The comparisons between the proposed methods with the existing methods are shown in Table 7.

Table 7 The comparison with the existing methods

Set	Aggregation Operator	Weight	Measurement Function	Ranking Order	Consideration of τ, δ, λ
Triangular intuitionistic fuzzy set [77]	TIFOWG	✓	Score function	$\varsigma_4 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_2 \succ \varsigma_3$	×
Trapezoidal intuitionistic fuzzy set [76]	TIFPWA	✓	Score function	$\varsigma_4 \succ \varsigma_1 \succ \varsigma_2 \succ \varsigma_5 \succ \varsigma_3$	×
	TIFPWG	✓	Score function	$\varsigma_4 \succ \varsigma_1 \succ \varsigma_2 \succ \varsigma_5 \succ \varsigma_3$	×
Trapezoidal neutrosophic set [83]	TNNWAA	✓	Score function	$\varsigma_4 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_2 \succ \varsigma_3$	✓
	TNNWGA	✓	Score function	$\varsigma_4 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_2 \succ \varsigma_3$	✓
MVIN-SS (Proposed set)	MVINSWAA & MVINSWGA	✓	Euclidean Distance	$\varsigma_4 \succ \varsigma_2 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_3$	✓
	MVINSWAA & MVINSWGA	✓	Score function	$\varsigma_4 \succ \varsigma_2 \succ \varsigma_1 \succ \varsigma_5 \succ \varsigma_3$	✓

It can be seen that there are different types of sets and aggregation operators used in order to obtain the final ranking order. As a result of the comparative study described above, two issues may be considered. First, the proposed method yields a different outcome than the current aggregation operators, which took into account different types of sets. Although multiple aggregation operators may be employed to handle the various relationships of the aggregated arguments, the number of operations and the size of the results will grow exponentially as more MVINSNs are engaged in the processes. Furthermore, various aggregating operators might produce disparate outcomes. These might be because the proposed operators consider truth, indeterminacy, and falsity memberships of the new set MVIN-SS while proposing geometric and arithmetic based aggregation operators. The deterioration brought on by these difficulties may limit the performance of aggregation operators. One of the shortcomings with the existing method is that the truth, indeterminacy, and falsity memberships were not considered. For example, Wang [77] used TIFOWG operator and score function using intuitionistic fuzzy set. However, the truth, indeterminacy, and falsity are not present as the author used triangular intuitionistic fuzzy set. The ranking order generated by using a triangular intuitionistic fuzzy set is comparable to the other sets. Second, it is also worth noting that some aggregation operators employed score functions without taking into consideration the three memberships, whereas the proposed aggregation operators used score function and Euclidean distance in measurement. More importantly, the proposed operators consider the truth, indeterminacy, and falsity memberships of MVIN-SS of which the intervals of three memberships are the distinct feature of the proposed operators and can handle uncertain and indeterminacy information especially when the assessments provided by decision-makers are given in multiple values, interval scale, and bifurcated.

However, the proposed approach using MVIN-SS varies from existing methods, which always entail operations whose influence on the final solution may be regarded as previously indicated, since the proposed method may overcome these drawbacks. It is possible to avoid loss and distortion of the given preference information, which improves the final findings' correspondence with genuine decision-making issues. Furthermore, the proposed method is favoured for solving issues when the number of criteria observably surpasses the number of alternatives. As a result, the proposed method can successfully deal with the preference information presented by MVIN-SS, which is meant to ensure the validity of the final rankings. In other words, the proposed method can deal with information that is characterized by fuzziness, indeterminacy, and uncertainty, thereby germane to solve complex MCDM problems.

7. Conclusions

In this paper, two novel aggregation operators—MVINSWAA and MVINSWGA—were proposed within the multi-valued interval neutrosophic soft set (MVIN-SS) framework to facilitate more robust multi-criteria group decision-making (MCGDM). Theoretical validation of these operators was established via proofs of idempotency, monotonicity, and boundedness. A structured decision-making procedure and a software selection case study demonstrated the practical applicability and consistency of the proposed methods. Comparative analysis further confirmed the superiority of our approach in handling uncertainty, indeterminacy, and multi-valued information.

Future research could explore several directions. First, the introduced aggregation operators can be enhanced to handle dynamic or time-dependent decision-making environments, where evaluation criteria may evolve over time. Second, integration with machine learning techniques could enable automated weighting or ranking of alternatives based on historical data or user feedback. Third, future work could adapt the MVIN-SS framework for distributed decision-making systems, particularly in contexts involving autonomous agents or decentralized systems. Finally, applying the proposed methods to domain-specific applications such as healthcare diagnostics, environmental risk assessments, and smart city infrastructure planning would further validate their utility in real-world scenarios.

Conflicts of Interest: The authors declare no conflict of interest.

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