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Neutrosophic α-Discounted Cognitive Mapping for Financial Distress Prediction: Evidence from Saudi Emerging Markets

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Abstract

This study proposes a new framework to predict corporate financial distress in emerging markets, with a focus on Saudi Arabia. Traditional models such as the Altman Z-score and Ohlson O-score assume that financial data are complete, precise, and reliable. However, in emerging markets, information is often missing, noisy, or conflicting, and many important factors are qualitative. To address this problem, the paper develops a Neutrosophic α-Discounted Cognitive Mapping (Nα-FDM) model that can represent both risk and uncertainty. Financial indicators are expressed as neutrosophic triples that measure the degrees of distress, indeterminacy, and financial health. Expert judgments about the relative importance of key criteria, such as liquidity, solvency, and profitability, are combined using the α-discounting method to obtain consistent weights even when initial preferences are inconsistent. These elements are integrated into a neutrosophic cognitive map, which produces a Neutrosophic Financial Distress Index (NFDI) and a scalar score for each firm. A case study on four Saudi non-financial firms, using the current ratio, debt-to-equity ratio, and return on assets, shows that the model can distinguish clearly distressed, clearly healthy, and grey-zone firms characterized by high uncertainty. The results suggest that the Nα-FDM framework is a useful decision-support tool for investors, managers, and regulators in uncertainty-rich environments.

Keywords: Neutrosophic sets; α-discounting; cognitive maps; financial distress prediction; emerging markets; Saudi Arabia.

1. Introduction

1.1 Background and Motivation

Predicting corporate financial distress is a critical aspect of risk management, investment decisions, and regulatory oversight, especially in volatile economic landscapes [1, 2]. Since the pioneering discriminant analysis for bankruptcy prediction [1], numerous models have been developed to forecast firm failure using financial ratios, market indicators, and macroeconomic factors [3–5]. These models help stakeholders identify early warning signals of insolvency, allowing for loss mitigation and efficient resource allocation [6]. However, in emerging markets characterized by data scarcity, regulatory inconsistencies, and external shocks, traditional models often underperform due to their dependence on precise and complete datasets [7–9].

Saudi Arabia exemplifies these challenges as a rapidly evolving emerging market. Driven by Vision 2030 reforms, diversification away from oil dependency, and growing foreign investment, the Saudi economy has seen increased corporate activity in non-financial sectors like petrochemicals, logistics, and utilities [10–12]. Yet, firms in these sectors face unique distress risks from global commodity price fluctuations, governance opacity, and sectoral concentration, particularly in regions such as the Eastern Province [13, 14]. Studies on Saudi firms indicate that conventional predictors like liquidity and leverage ratios are influenced by local elements, including Sharia-compliant financing and government subsidies, introducing ambiguities not adequately addressed by crisp statistical methods [15–17].

1.2 Limitations of Classical and Advanced Distress Prediction Models

Classical models, such as the Altman Z-score [1] and Ohlson O-score [2], rely on deterministic inputs and linear relationships, often leading to overconfident predictions in noisy data environments [18, 19]. Machine learning extensions, including support vector machines and neural networks, enhance accuracy but frequently lack interpretability and fail to model indeterminacy explicitly [3, 20–22]. In emerging markets, these shortcomings are exacerbated: panel data analyses reveal that missing values, reporting delays, and conflicting signals from qualitative factors (e.g., governance quality) result in high misclassification rates [23–25]. For example, research on Middle Eastern markets demonstrates that models calibrated on developed economies perform poorly in high macroeconomic volatility contexts [26, 27].

Moreover, interactions among distress drivers, such as feedback loops between profitability and solvency, are rarely modeled explicitly, treating indicators as independent [28, 29]. This limitation is particularly acute in Saudi emerging markets, where structural reforms and energy transitions create interdependent risks [30, 31]. Borderline cases, where firms show mixed signals, pose additional challenges, as classical scores do not distinguish between true ambiguity and simple intermediacy [32, 33].

1.3 Neutrosophic Sets for Handling Indeterminacy

To address these gaps, this study employs neutrosophic sets, which extend fuzzy and intuitionistic fuzzy sets by incorporating three independent components: truth (supporting evidence), indeterminacy (ambiguity or conflict), and falsity (contradicting evidence) [4, 34–36]. Neutrosophic theory allows memberships that do not sum to one, enabling representation of conflicting data without forced resolution [37, 38]. Applications in risk assessment and multi-criteria decision-making (MCDM) highlight its effectiveness in uncertain settings, such as supplier selection and project evaluation [39–41].

In financial domains, neutrosophic approaches have been applied to credit scoring and investment analysis, quantifying vagueness in qualitative assessments [42, 43]. However, their integration into corporate distress prediction, particularly in emerging markets, is limited [44, 45]. This paper extends neutrosophic representations to encode financial indicators as triples, capturing not only distress signals but also inherent uncertainties common in Saudi data [46].

1.4 Cognitive Maps and Causal Structures

Financial distress arises from networked interactions among factors like liquidity, profitability, and governance [47]. Cognitive maps, especially fuzzy cognitive maps (FCMs), model these causal relationships as directed graphs with weighted edges [6]. Neutrosophic cognitive maps (NCMs) enhance FCMs by assigning neutrosophic triples to edges and nodes, accommodating uncertain causal influences. Previous applications of NCMs include healthcare and energy planning, but rarely financial distress [47].

In this framework, nodes represent distress drivers, and edges propagate neutrosophic states, enabling dynamic simulation of shock effects [47]. This is especially pertinent for Saudi firms, where macroeconomic stress (e.g., oil price volatility) interacts with firm-level metrics.

1.5 α-Discounting in Multi-Criteria Decision Making

Assigning weights to criteria often involves inconsistent expert judgments, particularly in diverse markets [7]. The α-discounting method resolves this by perturbing linear preference relations to achieve consistency while minimizing deviation from original inputs [7]. When integrated with neutrosophic MCDM, it yields robust weights and an inconsistency measure. Applications in engineering and environmental decisions underscore its utility [7], yet its pairing with NCMs for financial prediction is novel.

1.6 Research Gaps and Contributions

Despite progress, gaps remain: (i) classical models overlook explicit indeterminacy in emerging markets ; (ii) few studies combine neutrosophic representations with cognitive maps for distress dynamics; (iii) α-discounting is underutilized in financial MCDM; and (iv) Saudi-specific applications are scarce, emphasizing aggregate rather than firm-level nuances.

This paper bridges these gaps by proposing the Nα-FDM model, contributing: a neutrosophic financial distress index separating truth, indeterminacy, and falsity; an integrated approach merging neutrosophic encoding, α-discounted weights, and NCM propagation; and a Saudi case study on non-financial firms, differentiating clear and ambiguous cases.

2. Theoretical Background and Notation

This section introduces the main mathematical tools used in the paper: neutrosophic sets and numbers, their use in representing financial distress information, cognitive maps (fuzzy and neutrosophic), and the α-discounting method for deriving criteria weights from expert preferences [3,7].

2.1 Neutrosophic sets and single-valued neutrosophic numbers

Let Ube a non-empty universe of discourse. A neutrosophic set Ain $U$is characterized by three functions, $T_A, I_A, F_A: U \to [0,1]$, where, for each element $x \in U$,

a. $T_A(x)$is the degree of truth that "x belongs to A",
b. $I_A(x)$is the degree of indeterminacy,
c. $F_A(x)$is the degree of falsity [34,35].

Classical and fuzzy sets are recovered as special cases when indeterminacy is forced to zero, and the sum $T_A(x) + F_A(x)$is constrained. In the neutrosophic setting, the three components $T_A(x)$, $I_A(x)$, and $F_A(x)$are allowed to vary independently in [0,1], and they are not required to sum to one. This additional flexibility makes it possible to express:

1. high truth and high falsity simultaneously (conflicting evidence),
2. high indeterminacy with low truth and low falsity (lack of information),
3. low indeterminacy with dominant truth or falsity (clear signal).

In this paper, we work with single-valued neutrosophic numbers (SVNNs). A single-valued neutrosophic number is a triple, $x = (T_x, I_x, F_x)$ with

$$T_x \in [0,1], I_x \in [0,1], F_x \in [0,1].$$

We denote the set of all SVNNs by

$$\mathcal{N} = \{(T, I, F) \mid T, I, F \in [0,1]\}.$$

For later use, we define a few basic operations on SVNNs. Let $a = (T_a, I_a, F_a)$and $b = (T_b, I_b, F_b)$be elements of $\mathcal{N}$, and let $\lambda \geq 0$be a scalar [35].

1. Scalar multiplication (bounded):
$$\lambda \otimes a := (\min\{1, \lambda T_a\}, \min\{1, \lambda I_a\}, \min\{1, \lambda F_a\}). \quad (1)$$
2. Weighted sum aggregation.
Let $a_k = (T_k, I_k, F_k) \in \mathcal{N}$for $k = 1, \ldots, m$, and let $\omega_1, \ldots, \omega_m \geq 0$be weights satisfying

$$\sum_{k=1}^{m} \omega_k = 1. \tag{2}$$

The weighted sum (or convex aggregation) of the $a_k$is defined as

$$\sum_{k=1}^{m} \omega_k \otimes a_k := (\sum_{k=1}^{m} \omega_k T_k, \sum_{k=1}^{m} \omega_k I_k, \sum_{k=1}^{m} \omega_k F_k). \tag{3}$$

Because each T_k, I_k, F_k lies in [0,1]and the weights sum to one, each component of (3) also lies in [0,1].

3. Complement.
   The neutrosophic complement of $a$is defined by swapping truth and falsity while keeping indeterminacy:

$$a^c := (F_a, I_a, T_a). \tag{4}$$

These operations are sufficient for the constructions used in this paper. More elaborate operations (such as neutrosophic union and intersection) are not required for the proposed model and are therefore omitted here.

2.2 Neutrosophic representation of financial distress information

We now connect neutrosophic numbers to the financial distress context. Suppose there are Nfirms, indexed by $j = 1, \dots, N$. For each firm j, consider the proposition:
P_j: "Firm j is financially distressed." Our goal is to represent the available information regarding the truth of P_j in neutrosophic form.
Assume that we observe $m$indicators for each firm, such as financial ratios or qualitative scores. Let

$$X_{jk} \in \mathbb{R}, j = 1, \dots, N,\ k = 1, \dots, m,$$

Denote the value of the k-th indicator for firm j. Examples of indicators include:

a. liquidity ratios (e.g., current ratio),
b. solvency ratios (e.g., debt-to-equity),
c. profitability ratios (e.g., return on assets),
d. governance indices,
e. market-based measures (e.g., volatility, credit spreads).

For each indicator k, we define three membership functions:

$$\mu_k^T: \mathbb{R} \to [0,1], \mu_k^I: \mathbb{R} \to [0,1], \mu_k^F: \mathbb{R} \to [0,1].$$

These functions are designed to encode, for each possible indicator value x,

a. $\mu_k^T(x)$: the degree to which $x$supports the truth of P_j(i.e., supports distress),
b. $\mu_k^I(x)$: the degree of indeterminacy associated with x,
c. $\mu_k^F(x)$: the degree to which $x$supports the falsity of P_j(i.e., supports financial health).

Given an observed value X_{jk}, we obtain the indicator-level neutrosophic number for firm $j$and indicator k:

$$x_{jk} = (T_{jk}, I_{jk}, F_{jk}) = (\mu_k^T(X_{jk}),\ \mu_k^I(X_{jk}),\ \mu_k^F(X_{jk})) \in \mathcal{N}. \tag{5}$$

The choice of membership functions μ_k^T, μ_k^I, and μ_k^F is part of the model design and depends on the economic interpretation of each indicator. For example, for a liquidity ratio where low values are associated with distress:

a. Small values of the ratio should yield high μ_k^T and low μ_k^F,
b. Large values should yield low μ_k^T and high μ_k^F,
c. Intermediate values should be associated with a higher μ_k^I, reflecting ambiguity.

The explicit shapes of these functions will be defined later in the methodology and case study sections. Ultimately, the neutrosophic financial distress index for firm j will be constructed by aggregating the indicator-level neutrosophic numbers $\{x_{jk}\}_{k=1}^{m}$ across indicators and across criteria, using weights derived from expert preferences via the α-discounting method.

2.3 Cognitive maps: fuzzy and neutrosophic

The previous subsection focused on single indicators. Financial distress, however, is the outcome of a system of interacting factors. To represent this interaction structure, we use cognitive maps.

2.3.1 Fuzzy cognitive maps (FCMs)

A cognitive map is a directed graph whose nodes represent concepts and whose edges represent causal influences between concepts. Let

$$\mathcal{C} = \{C_1, C_2, \dots, C_n\}$$

Denote the set of concepts. In the context of financial distress, important concepts may include:

1. C_1: short-term liquidity risk,
2. C_2: long-term solvency pressure,
3. C_3: profitability weakness,
4. C_4: cash-flow volatility,
5. C_5: governance quality,
6. C_6: macroeconomic stress,
7. C_7: market-based distress signals,
8. C_8: overall financial distress status.

In a fuzzy cognitive map (FCM), each directed edge from a concept C_i to concept C_j carries a real-valued weight $w_{ij} \in [-1,1]$, which represents the sign and strength of the causal influence:

a. $w_{ij} > 0$: an increase in C_i tends to increase C_j,
b. $w_{ij} < 0$: an increase in C_i tends to decrease C_j,
c. $w_{ij} = 0$: no direct influence.

The state of the system at discrete time t is represented by the vector

$$S^{(t)} = (s_1^{(t)}, s_2^{(t)}, \dots, s_n^{(t)})^\top, \qquad (6)$$

where $s_i^{(t)} \in [0,1]$denotes the activation level of the concept $C_i$at time t. The typical FCM update rule is

$$s_j^{(t+1)} = f(\sum_{i=1}^{n} s_i^{(t)} w_{ij}), \qquad (7)$$

Where $f: \mathbb{R} \to [0,1]$is an activation function (for example, a sigmoid or a threshold function). Under suitable conditions, iterating (7) can lead to a fixed point or a limit cycle that represents the system's long-run behaviour.

2.3.2 Neutrosophic cognitive maps (NCMs)

In a neutrosophic cognitive map (NCM), uncertainty and indeterminacy in causal relationships are modelled explicitly. Instead of a single real weight w_{ij}, each edge from $C_i$to $C_j$is assigned a neutrosophic weight

$$w_{ij} = (T_{ij}, I_{ij}, F_{ij}) \in \mathcal{N}, \qquad (8)$$

Where:

a. $T_{ij}$is the degree of truth of the statement "C_i has a positive causal influence on C_j",
b. $I_{ij}$is the degree of indeterminacy about this influence,
c. $F_{ij}$is the degree of falsity (evidence against the influence).

Similarly, the state of each node C_i at time $t$is represented by a neutrosophic number

$$x_i^{(t)} = (T_i^{(t)}, I_i^{(t)}, F_i^{(t)}) \in \mathcal{N}. \qquad (9)$$

To update the system over time, we need a rule that aggregates incoming neutrosophic influences. There are many possible choices. In this paper, we adopt a linear aggregation followed by component-wise bounding, which is simple, transparent, and consistent with the operations in Section 2.1. For a given time $t$and node C_j, define the intermediate (unbounded) values

$$\begin{aligned} \tilde{T}_j^{(t+1)} &= \sum_{i=1}^{n} (T_i^{(t)} T_{ij} - F_i^{(t)} F_{ij}), \\ \tilde{I}_j^{(t+1)} &= \sum_{i=1}^{n} (I_i^{(t)} T_{ij} + T_i^{(t)} I_{ij}), \qquad (10) \\ \tilde{F}_j^{(t+1)} &= \sum_{i=1}^{n} (F_i^{(t)} T_{ij} - T_i^{(t)} F_{ij}). \end{aligned}$$

The intuition is as follows:

1. Truth at node C_j increases when "true" antecedents transmit true positive influence ($T_i^{(t)} T_{ij}$), and decreases when "false" antecedents support the non-occurrence of the consequent ($F_i^{(t)} F_{ij}$);
2. indeterminacy at C_j accumulates when indeterminate or ambiguous nodes influence C_j($I_i^{(t)} T_{ij}$) or when true antecedents have uncertain influence ($T_i^{(t)} I_{ij}$);

3. falsity at C_j increases when false antecedents transmit influence and decreases when true antecedents support the absence of the effect.

We then apply a clipped linear activation to ensure that the updated components remain in [0,1]:

$$T_j^{(t+1)} = f_T(\tilde{T}_j^{(t+1)}), I_j^{(t+1)} = f_I(\tilde{I}_j^{(t+1)}), F_j^{(t+1)} = f_F(\tilde{F}_j^{(t+1)}), \quad (11)$$

where, for each component $X \in \{T, I, F\}$,

$$f_X(z) = \min\{1, \max\{0, z\}\}. \quad (12)$$

Equations (10)–(12) define a discrete-time dynamical system on $\mathcal{N}^n$. A state vector $(x_1^{(t)}, \ldots, x_n^{(t)})$is said to reach a fixed point if there exists $t_0$such that

$$x_i^{(t+1)} = x_i^{(t)} \text{for all } i \text{ and for all } t \geq t_0. \quad (13)$$

In the context of financial distress, one of the nodes, say C_8, will represent overall distress. The neutrosophic state of this node at a fixed point will be used as a dynamic version of the firm-level distress index.

2.4 The α-discounting method for multi-criteria decision making

The neutrosophic framework and cognitive maps describe how to represent and propagate information. We still need a principled way to assign weights to the high-level criteria (such as liquidity, solvency, profitability, and so on). Expert judgement is an important source of such information, but expert statements are often inconsistent.

The α-discounting method for multi-criteria decision making is a procedure for deriving a consistent weight vector from a set of possibly inconsistent linear relations between weights.

2.4.1 General formulation

Consider $m$criteria

$$K_1, K_2, \ldots, K_m,$$

with unknown weights

$$w = (w_1, w_2, w_3, \ldots, w_m)^{\top}, \quad (14)$$

satisfying

$$w_k \geq 0 \text{for all } k, \sum_{k=1}^{m} w_k = 1. \quad (15)$$

Experts express preferences in the form of linear relations, such as. "K_1 is twice as important as K_2" or "K_2 and K_3have the same importance". Each such statement can be written as a linear equation in the weights. Suppose that, in total, we obtain Lequations:

$$\ell_\ell(w_1, \ldots, w_m) = 0, \ell = 1, \ldots, L. \quad (16)$$

These can be written compactly as $Aw = 0,$ (17)

Where $A$is an $L \times m$matrix whose coefficients are determined by the expert statements. If the system (17) admits a non-trivial solution $w \neq 0$with non-negative components, we can normalise it to satisfy (15), and we are done. In practice, however, the system (17) may be inconsistent, so that the only solution is $w = 0$. The α-discounting method addresses this by introducing discounting parameters that adjust the equations just enough to regain consistency.

2.4.2 α-discounting and the fairness principle

For each equation in (16), consider a representation of the form

(left-hand side) = (right-hand side).

We introduce a positive scalar $\alpha_\ell > 0$and multiply one side (typically the right-hand side) by α_ℓ. The α-discounted system then has the form

$$A(\alpha)w = 0, \qquad (18)$$

Where $A(\alpha)$is an $L \times m$matrix whose entries depend linearly on the discounting parameters

$$\alpha = (\alpha_1, \dots, \alpha_L)^\top.$$

A pair $(\alpha^\star, w^\star)$is called a consistent α-discounted solution if:

1. $A(\alpha^\star)w^\star = 0$,
2. $w_k^\star \geq 0$for all k,
3. $\sum_{k=1}^{m} \quad w_k^\star = 1$,
4. $\alpha_\ell^\star > 0$for all ℓ.

There may be many such pairs. To select a particular solution, a natural rule is the fairness principle, which sets all discounting parameters equal:

$$\alpha_1 = \alpha_2 = \cdots = \alpha_L = \alpha. \qquad (19)$$

Under this principle, the original system is adjusted uniformly, and the single scalar αmeasures the overall inconsistency of the expert preference system: the further $\alpha$is from 1, the stronger the inconsistency that had to be corrected.

2.4.3 Worked example with three criteria

We now illustrate the α-discounting method step by step in a simple example with three criteria: K_1, K_2, K_3

and weights, w_1, w_2, w_3.

<u>Suppose experts state:</u>

1. "K_1 is twice as important as K_2":

$$w_1 = 2w_2. \qquad (20)$$

2. "K_2 is three times as important as K_3":

$$w_2 = 3w_3. \qquad (21)$$

3. "K_3 is as important as K_1": $w_3 = w_1$. (22)

Equations (20)–(22) are inconsistent. To see this, substitute $w_3 = w_1$from (22) into (21):

$$w_2 = 3w_3 = 3w_1. \quad (23)$$

Then substitute $w_1 = 2w_2$ from (20) into (23):

$$w_2 = 3(2w_2) = 6w_2. \quad (24)$$

Equation (24) implies

$$w_2 - 6w_2 = -5w_2 = 0 \Rightarrow w_2 = 0. \quad (25)$$

From (20) and (22), this forces

$$w_1 = 2w_2 = 0, w_3 = w_1 = 0, \quad (26)$$

So the only solution is $w_1 = w_2 = w_3 = 0$, which cannot be normalised to satisfy (15). The system is therefore inconsistent.

Step 1: Introduce α-discounting.

We introduce discounting parameters $\alpha_1, \alpha_2, \alpha_3 > 0$ and modify each equation:

From (20): $w_1 = 2\alpha_1 w_2.$ (27)

From (21): $w_2 = 3\alpha_2 w_3.$ (28)

From (22): $w_3 = \alpha_3 w_1.$ (29)

Step 2: Solve the α-discounted system in terms of w_3.

From (28): $w_2 = 3\alpha_2 w_3.$ (30)

Substitute this into (27): $w_1 = 2\alpha_1 w_2 = 2\alpha_1(3\alpha_2 w_3) = 6\alpha_1\alpha_2 w_3.$ (31)

Substitute w_1 from (31) into (29): $w_3 = \alpha_3 w_1 = \alpha_3(6\alpha_1\alpha_2 w_3) = 6\alpha_1\alpha_2\alpha_3 w_3.$ (32)

Assuming $w_3 \neq 0$, equation (32) implies

$$1 = 6\alpha_1\alpha_2\alpha_3. \quad (33)$$

Step 3: Apply the fairness principle.

Under the fairness principle (19), we set

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha. \quad (34)$$

Equation (33) becomes

$$1 = 6\alpha^3 \Rightarrow \alpha^3 = \frac{1}{6} \Rightarrow \alpha = (\frac{1}{6})^{1/3}. \quad (35)$$

Numerically, $\alpha \approx 0.5503.$ (36)

Step 4: Compute the weights up to normalization.

Using (30) and (31) with $\alpha_1 = \alpha_2 = \alpha$ and an arbitrary positive value of w_3:

$$w_2 = 3\alpha w_3, \quad (37)$$

$$w_1 = 6\alpha^2 w_3. \quad (38)$$

To obtain concrete numbers, set $w_3 = 1$. Then, using $\alpha \approx 0.5503$:

a. $w_2 = 3\alpha \approx 3 \times 0.5503 \approx 1.6510,$
b. $w_1 = 6\alpha^2 \approx 6 \times (0.5503)^2 \approx 6 \times 0.3029 \approx 1.8171,$
c. $w_3 = 1.0000.$

The unnormalized sum is

$$w_1 + w_2 + w_3 \approx 1.8171 + 1.6510 + 1.0000 = 4.4681. \quad (39)$$

Step 5: Normalize the weights.

The normalized weights are

$$w_1^* = \frac{w_1}{w_1 + w_2 + w_3}, w_2^* = \frac{w_2}{w_1 + w_2 + w_3}, w_3^* = \frac{w_3}{w_1 + w_2 + w_3}. \quad (40)$$

Using the values above:

$$w_1^* \approx \frac{1.8171}{4.4681} \approx 0.4067,$$

$$w_2^* \approx \frac{1.6510}{4.4681} \approx 0.3695,$$

$$w_3^* \approx \frac{1.0000}{4.4681} \approx 0.2238. \qquad (41)$$

These satisfy

$$w_1^* + w_2^* + w_3^* \approx 0.4067 + 0.3695 + 0.2238 = 1.0000, \qquad (42)$$

Up to rounding error.

Thus, the α-discounted solution consists of:

a. a consistency indicator $\alpha \approx 0.5503$, showing that substantial discounting was needed to reconcile the expert statements, and

b. a normalized weight vector $w^* \approx (0.4067,\ 0.3695,\ 0.2238)$, which assigns the highest importance to K_1, then to K_2, and finally to K_3.

3. Methodology

This section develops the proposed Nα-FDM for financial distress prediction. We start from firm-level indicators, transform them into neutrosophic numbers, aggregate them into criteria, derive criteria weights via α-discounting, and then construct an NFDI for each firm. We also explain how this index can be embedded in a neutrosophic cognitive map and provide a fully worked numerical example.

Throughout this section, we use the notation introduced in Section 2:

a. N: number of firms, indexed by $j = 1, \dots, N$,

b. m: number of indicators, indexed by $k = 1, \dots, m$,

c. X_{jk}: observed value of indicator $k$for firm j,

d. $x_{jk} = (T_{jk}, I_{jk}, F_{jk}) \in \mathcal{N}$: neutrosophic encoding of X_{jk},

e. $w = (w_1, \dots, w_H)$: criteria weights obtained via α-discounting.

3.1 Overall structure of Nα-FDM

Conceptually, the Nα-FDM consists of four layers:

1. Indicators layer
   Each raw indicator $X_{jk}$is mapped to a single-valued neutrosophic number
   $$x_{jk} = (T_{jk}, I_{jk}, F_{jk}),$$
   where T_{jk}, I_{jk}, and F_{jk} quantify, respectively, the degrees of distress support, indeterminacy, and health support contributed by that indicator.
2. Criterion layer
   Indicators are grouped into Hhigh-level criteria $K_1, K_2, \dots, K_H$, such as liquidity,

solvency, profitability, cash-flow stability, governance, and market-based signals.
For each firm $j$and criterion K_h, the indicator-level neutrosophic values are aggregated into a criterion-level neutrosophic evaluation

$$c_{jh} = (T_{jh}^{(c)}, I_{jh}^{(c)}, F_{jh}^{(c)}).$$

3. Weighting layer (α-discounting)
   Expert preferences over criteria are encoded in linear relations among the weights w_h. These relations may be inconsistent. The α-discounting method produces a consistent weight vector $w^* = (w_1^*, \dots, w_H^*)$ and a consistency indicator α^*.
4. Cognitive map/distress layer
   The criteria $K_h$are represented as concepts in a neutrosophic cognitive map, together with a final concept "overall financial distress". The criterion-level neutrosophic evaluations are propagated through the map to obtain a firm-level Neutrosophic Financial Distress Index (NFDI), both in static and dynamic forms.

3.2 Criteria and indicator sets

Let $\mathcal{K} = \{K_1, K_2, \dots, K_H\}$ denote the set of criteria. In the general model, we consider:

1. K_1: short-term liquidity,
2. K_2: long-term solvency,
3. K_3: profitability,
4. K_4: cash-flow stability,
5. K_5: governance quality,
6. K_6: market-based distress signals.

The framework is not restricted to these six; additional criteria can be added if needed. In the numerical examples and the Saudi case study, we will focus first on a subset of three criteria (liquidity, solvency, profitability) to keep the computations transparent.

Each indicator $k$belongs to exactly one criterion. For each $h \in \{1, \dots, H\}$, let $\mathcal{J}_h \subset \{1, \dots, m\}$be the set of indicators associated with the criterion K_h. We assume a disjoint partition:

$$\bigcup_{h=1}^{H} \mathcal{J}_h = \{1, \dots, m\}, \mathcal{J}_h \cap \mathcal{J}_{h'} = \emptyset \text{for } h \neq h'.$$

Typical examples:

1. Liquidity K_1: current ratio, quick ratio,
2. Solvency K_2: debt-to-equity, interest coverage,
3. Profitability K_3: ROA, ROE, and so on.

The precise choice of indicators will be specified in the case study.

3.3 Neutrosophic mapping of distress-related indicators

For each indicator k, Section 2.2 introduced three membership functions.

$$\mu_k^T, \mu_k^I, \mu_k^F: \mathbb{R} \to [0,1],$$

and the neutrosophic encoding

$$x_{jk} = (T_{jk}, I_{jk}, F_{jk}) = (\mu_k^T(X_{jk}), \mu_k^I(X_{jk}), \mu_k^F(X_{jk})).$$

We now specify an explicit and flexible family membership functions suitable for common financial ratios.

3.3.1 Ratios where "lower is worse"

Consider a ratio R(for example, the current ratio) where low values indicate distress and high values indicate safety. We select two thresholds

$$a_R > 0, b_R > a_R,$$

where:

a. $x \le a_R$is interpreted as clearly distressed,
b. $x \ge b_R$is interpreted as clearly safe,
c. the interval $(a_R, b_R)$is a transition zone.

Define the midpoint

$$m_R = \frac{a_R + b_R}{2}.$$

We then specify:

Truth membership (distress-supporting):

$$\mu_R^T(x) = \begin{cases} 1, & x \le a_R, \\ \dfrac{b_R - x}{b_R - a_R}, & a_R < x < b_R, \\ 0, & x \ge b_R; \end{cases}$$

Falsity membership (health-supporting):

$$\mu_R^F(x) = \begin{cases} 0, & x \le a_R, \\ \dfrac{x - a_R}{b_R - a_R}, & a_R < x < b_R, \\ 1, & x \ge b_R; \end{cases}$$

Indeterminacy membership:

$$\mu_R^I(x) = \begin{cases} 0, & x \le a_R, \\ \dfrac{2\ |\ x - m_R\ |}{b_R - a_R}, & a_R < x < b_R, \\ 0, & x \ge b_R. \end{cases}$$

These functions satisfy:

a. $\mu_R^T(x), \mu_R^I(x), \mu_R^F(x) \in [0,1]$for all x,
b. $\mu_R^T(x)$decreases linearly from 1 to 0 on (a_R, b_R),
c. $\mu_R^F(x)$increases linearly from 0 to 1 on (a_R, b_R),
d. $\mu_R^I(x)$is maximal at $m_R$and vanishes at $a_R$and b_R.

This construction will be used, for example, for the current ratio in both the methodology example and the Saudi case study.

3.3.2 Ratios where "higher is worse."

Consider a ratio Q(for example, debt-to-equity) where high values indicate distress and low values indicate safety. We select two thresholds.

$$c_Q > 0, d_Q > c_Q,$$

Where:

a. $x \le c_Q$is interpreted as clearly safe,
b. $x \ge d_Q$is interpreted as clearly distressed.

Define the midpoint

$$m_Q = \frac{c_Q + d_Q}{2}.$$

We specify piecewise-linear functions:
Truth membership (distress-supporting):

$$\mu_Q^T(x) = \begin{cases} 0, & x \le c_Q, \\ \frac{x - c_Q}{d_Q - c_Q}, & c_Q < x < d_Q, \\ 1, & x \ge d_Q; \end{cases}$$

Falsity membership (health-supporting):

$$\mu_Q^F(x) = \begin{cases} 1, & x \le c_Q, \\ \frac{d_Q - x}{d_Q - c_Q}, & c_Q < x < d_Q, \\ 0, & x \ge d_Q; \end{cases}$$

Indeterminacy membership:

$$\mu_Q^I(x) = \begin{cases} 0, & x \le c_Q, \\ \frac{2 \mid x - m_Q \mid}{d_Q - c_Q}, & c_Q < x < d_Q, \\ 0, & x \ge d_Q. \end{cases}$$

As before, all components lie in [0,1], and indeterminacy is maximum near the midpoint.

3.3.3 Example calibration for three core ratios

To make the discussion concrete and to prepare for the case study, we introduce a simple calibration for three core indicators:

1. current ratio CR (liquidity, lower is worse),
2. debt-to-equity ratio DE (solvency, higher is worse),
3. return on assets ROA (profitability, lower is worse).

We choose the following thresholds:
For CR: $a_{\mathrm{CR}} = 1.0, b_{\mathrm{CR}} = 2.0, m_{\mathrm{CR}} = 1.5$.
For DE: $c_{\mathrm{DE}} = 1.0, d_{\mathrm{DE}} = 3.0, m_{\mathrm{DE}} = 2.0$.
For ROA: $p_{\mathrm{ROA}} = 0.00, q_{\mathrm{ROA}} = 0.08, m_{\mathrm{ROA}} = 0.04$.

The corresponding membership functions are obtained by treating CR and ROA as "lower is worse" and DE as "higher is worse", exactly as shown in the previous subsections. The numerical example in Section 3.8 will use these values explicitly.

3.4 Criterion-level neutrosophic aggregation

For a given firm $j$and criterion K_h, all indicators in $\mathcal{I}_h$contribute to the evaluation of that criterion. Let $|\mathcal{I}_h|$be the number of indicators under K_h. We assign local indicator weights

$$\gamma_{hk} \geq 0, k \in \mathcal{I}_h, \sum_{k \in \mathcal{I}_h} \gamma_{hk} = 1.$$

The local weights γ_{hk}represent the relative importance of each indicator within its criterion. They can be chosen by experts or estimated from data; in the simple examples below, each criterion has a single indicator and thus $\gamma_{hk} = 1$for that indicator.

For firm $j$and criterion K_hWe define the criterion-level neutrosophic triple as a weighted sum of the indicator-level triples:

$$T_{jh}^{(c)} = \sum_{k \in \mathcal{I}_h} \gamma_{hk} T_{jk},$$
$$I_{jh}^{(c)} = \sum_{k \in \mathcal{I}_h} \gamma_{hk} I_{jk},$$
$$F_{jh}^{(c)} = \sum_{k \in \mathcal{I}_h} \gamma_{hk} F_{jk}.$$

We then write
$c_{jh} = (T_{jh}^{(c)}, I_{jh}^{(c)}, F_{jh}^{(c)})$.

Because each $\gamma_{hk} \geq 0$and sums to 1 over $\mathcal{J}_h$, and each $T_{jk}, I_{jk}, F_{jk} \in [0,1]$, the components $T_{jh}^{(c)}, I_{jh}^{(c)}, F_{jh}^{(c)}$ also lie in [0,1].

3.5 Criteria weights from α-discounting

Section 2.4 introduced the α-discounting method and provided a worked example with three criteria. In the context of Nα-FDM, we apply that procedure to the criteria $K_1, \dots, K_H$.

Let $w = (w_1, \dots, w_H)$, With, $w_h \geq 0$for all $h, \sum_{h=1}^{H} \quad w_h = 1$.

Experts express qualitative preferences about the relative importance of the criteria. These preferences are encoded as linear equations in the weights. If the resulting system is inconsistent, α-discounting is used to obtain a consistent solution $w^*$and a scalar α^*That measures inconsistency.

In the three-criterion example used later in this section (liquidity, solvency, profitability), we will work with the specific α-discounted weight vector:

$$w_1^* \approx 0.4067, w_2^* \approx 0.3695, w_3^* \approx 0.2238.$$

The method for obtaining these numbers has already been detailed in Section 2.4; here we treat them as given and use them to build the financial distress index.

3.6 Firm-level Neutrosophic Financial Distress Index

The Neutrosophic Financial Distress Index (NFDI) for firm $j$is defined by aggregating the criterion-level neutrosophic evaluations c_{jh}using the α-discounted weights w_h^*.For each firm j, define:

$$T_j = \sum_{h=1}^{H} \quad w_h^* \, T_{jh}^{(c)},$$

$$I_j = \sum_{h=1}^{H} \quad w_h^* \, I_{jh}^{(c)},$$

$$F_j = \sum_{h=1}^{H} \quad w_h^* \, F_{jh}^{(c)}.$$

We then set

$$\mathrm{NFDI}_j = (T_j, I_j, F_j).$$

Because each $T_{jh}^{(c)}, I_{jh}^{(c)}, F_{jh}^{(c)}$lies in [0,1]and the w_h^*sum to 1, all components T_j, I_j, F_j lie in [0,1]. The sum $T_j + I_j + F_j$need not equal 1 and may be greater than 1, which is acceptable in neutrosophic theory.

To facilitate classification, we often use a normalised version of the NFDI when the sum $T_j + I_j + F_j$is positive:

$$T_j^* = \frac{T_j}{T_j + I_j + F_j}, I_j^* = \frac{I_j}{T_j + I_j + F_j}, F_j^* = \frac{F_j}{T_j + I_j + F_j}.$$

These normalized components satisfy:

$$T_j^*, I_j^*, F_j^* \in [0,1], T_j^* + I_j^* + F_j^* = 1.$$

We then define a scalar distress score:

$$S_j = T_j^* - F_j^*.$$

The value of S_j shows the overall direction of the evidence about a firm's financial condition. When S_j is large and positive; the evidence strongly points toward financial distress. When S_j is large and negative, the evidence strongly supports that the firm is financially healthy.

When S_j is close to zero, the signals of distress and health are roughly balanced. In this case, the level of uncertainty can be examined through I_j^*. In the case study, specific thresholds on S_j are used to classify firms as distressed, healthy, or belonging to a grey zone where the situation is ambiguous.

3.7 Integration into a neutrosophic cognitive map

The NFDI defined above is static: it aggregates information observed at a point in time. Financial distress, however, is dynamic and depends on how shocks propagate among criteria such as liquidity, solvency, profitability, and governance.

To capture this dimension, we embed the criteria into a neutrosophic cognitive map as follows:

a. Each criterion K_h is associated with a concept C_h.
b. A final concept C_D represents overall financial distress.
c. Directed edges between concepts are assigned neutrosophic weights $w_{ij} = (T_{ij}, I_{ij}, F_{ij})$ that encode the strength and uncertainty of causal links.

For firm j, we initialise the concepts at time $t = 0$ as:

a. $x_h^{(0)}(j) = c_{jh}$ for each criterion node C_h,
b. $x_D^{(0)}(j) = (0,1,0)$ for the distress node, representing a neutral prior (fully indeterminate) before propagation.

We then iterate the neutrosophic cognitive map update rule (as defined in Section 2.3) until the state of the distress node C_D converges or stabilises. Denote the limiting state for firm j by

$$x_D^{(\infty)}(j) = (\tilde{T}_j, \tilde{I}_j, \tilde{F}_j).$$

We call

$$\mathrm{NFDI}_j^{\mathrm{dyn}} = (\tilde{T}_j, \tilde{I}_j, \tilde{F}_j)$$

In this paper, the numerical examples will focus on the static NFDI to keep the arithmetic fully transparent. The dynamic NFDI is conceptually straightforward once the map and its weights are specified and can be used in further empirical work for stress testing.

3.8 NFDI for a single firm

We now give a complete numerical example for a single hypothetical firm. This example illustrates all the components of the Nα-FDM in the simplest non-trivial setting.

3.8.1 Setup

We consider:
Three criteria:

1. K_1: liquidity,
2. K_2: solvency,
3. K_3: profitability.

One indicator per criterion:
a. $X_{j1} = \mathrm{CR}_j$(current ratio),
b. $X_{j2} = \mathrm{DE}_j$(debt-to-equity ratio),
c. $X_{j3} = \mathrm{ROA}_j$(return on assets).

α-discounted weights: $w_1^* \approx 0.4067, w_2^* \approx 0.3695, w_3^* \approx 0.2238$.
Local indicator weights: $\gamma_{11} = \gamma_{22} = \gamma_{33} = 1$, since each criterion has only one indicator.

Assume the firm has the following observed values:
a. $\mathrm{CR}_j = 1.5$,
b. $\mathrm{DE}_j = 2.5$,
c. $\mathrm{ROA}_j = 0.03(3\%)$.
We use the calibration specified in Section 3.3.3:
a. $a_{\mathrm{CR}} = 1.0, b_{\mathrm{CR}} = 2.0, m_{\mathrm{CR}} = 1.5$,
b. $c_{\mathrm{DE}} = 1.0, d_{\mathrm{DE}} = 3.0, m_{\mathrm{DE}} = 2.0$,
c. $p_{\mathrm{ROA}} = 0.00, q_{\mathrm{ROA}} = 0.08, m_{\mathrm{ROA}} = 0.04$.

3.8.2 Neutrosophic encoding of indicators

<u>(a) Liquidity (CR = 1.5)</u>
Since $1.0 < 1.5 < 2.0$, we are in the transition zone for CR:
Truth: $\mu_{\mathrm{CR}}^T(1.5) = \frac{b_{\mathrm{CR}}-1.5}{b_{\mathrm{CR}}-a_{\mathrm{CR}}} = \frac{2.0-1.5}{2.0-1.0} = \frac{0.5}{1.0} = 0.5$.
Falsity: $\mu_{\mathrm{CR}}^F(1.5) = \frac{1.5-a_{\mathrm{CR}}}{b_{\mathrm{CR}}-a_{\mathrm{CR}}} = \frac{1.5-1.0}{2.0-1.0} = \frac{0.5}{1.0} = 0.5$.
Indeterminacy: $\mu_{\mathrm{CR}}^I(1.5) = \frac{2\,|1.5-m_{\mathrm{CR}}|}{b_{\mathrm{CR}}-a_{\mathrm{CR}}} = \frac{2\,|1.5-1.5|}{2.0-1.0} = \frac{0}{1.0} = 0.0$.
So, $x_{j1} = (T_{j1}, I_{j1}, F_{j1}) = (0.5, 0.0, 0.5)$.

(b) Solvency (DE = 2.5)

Since $1.0 < 2.5 < 3.0$, we are in the transition zone for DE:

Truth: $\mu_{\mathrm{DE}}^{T}(2.5) = \frac{2.5 - c_{\mathrm{DE}}}{d_{\mathrm{DE}} - c_{\mathrm{DE}}} = \frac{2.5-1.0}{3.0-1.0} = \frac{1.5}{2.0} = 0.75.$

Falsity: $\mu_{\mathrm{DE}}^{F}(2.5) = \frac{d_{\mathrm{DE}} - 2.5}{d_{\mathrm{DE}} - c_{\mathrm{DE}}} = \frac{3.0-2.5}{3.0-1.0} = \frac{0.5}{2.0} = 0.25.$

Indeterminacy: $\mu_{\mathrm{DE}}^{I}(2.5) = \frac{2\,|2.5 - m_{\mathrm{DE}}|}{d_{\mathrm{DE}} - c_{\mathrm{DE}}} = \frac{2\,|2.5-2.0|}{3.0-1.0} = \frac{2\times 0.5}{2.0} = \frac{1.0}{2.0} = 0.5.$

So, $x_{j2} = (T_{j2}, I_{j2}, F_{j2}) = (0.75, 0.5, 0.25)$.

(c) Profitability (ROA = 0.03)

Since $0.00 < 0.03 < 0.08$, we are in the transition zone for ROA:

Truth: $\mu_{\mathrm{ROA}}^{T}(0.03) = \frac{q_{\mathrm{ROA}} - 0.03}{q_{\mathrm{ROA}} - p_{\mathrm{ROA}}} = \frac{0.08-0.03}{0.08-0.00} = \frac{0.05}{0.08} = 0.625.$

Falsity: $\mu_{\mathrm{ROA}}^{F}(0.03) = \frac{0.03 - p_{\mathrm{ROA}}}{q_{\mathrm{ROA}} - p_{\mathrm{ROA}}} = \frac{0.03-0.00}{0.08-0.00} = \frac{0.03}{0.08} = 0.375.$

Indeterminacy: $\mu_{\mathrm{ROA}}^{I}(0.03) = \frac{2\,|0.03 - m_{\mathrm{ROA}}|}{q_{\mathrm{ROA}} - p_{\mathrm{ROA}}} = \frac{2\,|0.03-0.04|}{0.08-0.00} = \frac{2\times 0.01}{0.08} = \frac{0.02}{0.08} = 0.25.$

So, $x_{j3} = (T_{j3}, I_{j3}, F_{j3}) = (0.625, 0.25, 0.375)$.

All components of x_{j1}, x_{j2}, x_{j3}lie in [0,1], as required.

3.8.3 Criterion-level values

Each criterion has a single indicator, so the criterion-level neutrosophic triples are identical to the indicator-level triples:

$c_{j1} = x_{j1} = (0.5, 0.0, 0.5)$,

$c_{j2} = x_{j2} = (0.75, 0.5, 0.25)$,

$c_{j3} = x_{j3} = (0.625, 0.25, 0.375)$.

3.8.4 NFDI aggregation using α-discounted weights

Using the weights w_1^*, w_2^*, w_3^*we compute:

Truth component T_j

$$
\begin{aligned}
T_j &= w_1^* T_{j1}^{(c)} + w_2^* T_{j2}^{(c)} + w_3^* T_{j3}^{(c)} \\
&= 0.4067 \times 0.5 + 0.3695 \times 0.75 + 0.2238 \times 0.625.
\end{aligned}
$$

We compute each product:

a. $0.4067 \times 0.5 = 0.20335$,

b. $0.3695 \times 0.75 = 0.3695 \times \frac{3}{4} = 0.277125$,

c. $0.2238 \times 0.625 = 0.2238 \times \frac{5}{8} = 0.139875$.

Summing: $T_j = 0.20335 + 0.277125 + 0.139875 = 0.62035$.

Indeterminacy component I_j

$$
\begin{aligned}
I_j &= w_1^* I_{j1}^{(c)} + w_2^* I_{j2}^{(c)} + w_3^* I_{j3}^{(c)} \\
&= 0.4067 \times 0.0 + 0.3695 \times 0.5 + 0.2238 \times 0.25.
\end{aligned}
$$

Products:

a. $0.4067 \times 0.0 = 0.00000$,

b. $0.3695 \times 0.5 = 0.18475$,

c. $0.2238 \times 0.25 = 0.05595$.

Summing: $I_j = 0.00000 + 0.18475 + 0.05595 = 0.24070$.

Falsity component F_j: $$F_j = w_1^* F_{j1}^{(c)} + w_2^* F_{j2}^{(c)} + w_3^* F_{j3}^{(c)} = 0.4067 \times 0.5 + 0.3695 \times 0.25 + 0.2238 \times 0.375.$$

Products:

a. $0.4067 \times 0.5 = 0.20335$,

b. $0.3695 \times 0.25 = 0.092375$,

c. $0.2238 \times 0.375 = 0.083925$.

Summing: $F_j = 0.20335 + 0.092375 + 0.083925 = 0.37965$.

The sum of components is $T_j + I_j + F_j = 0.62035 + 0.24070 + 0.37965 = 1.24070$.

The unnormalized NFDI is therefore, $\mathrm{NFDI}_j = (T_j, I_j, F_j) = (0.62035, 0.24070, 0.37965)$.

3.8.5 Normalization and scalar score

We normalize the components by dividing by the sum $T_j + I_j + F_j = 1.24070$:

Normalized truth

$$T_j^* = \frac{0.62035}{1.24070}.$$

Since $2 \times 0.62035 = 1.24070$, this gives:

$T_j^* = 0.5$.

Normalized indeterminacy

$$I_j^* = \frac{0.24070}{1.24070} \approx 0.1940.$$

Normalized falsity

$$F_j^* = \frac{0.37965}{1.24070}.$$

Using the fact that the three normalized components must sum to 1:

$F_j^* = 1 - T_j^* - I_j^* \approx 1 - 0.5 - 0.1940 = 0.3060$.

Thus, the normalized NFDI is approximately:

$\mathrm{NFDI}_j^* = (T_j^*, I_j^*, F_j^*) \approx (0.50, 0.19, 0.31)$.

The scalar distress score is, $S_j = T_j^* - F_j^* \approx 0.50 - 0.31 = 0.19$.

This firm, therefore, has more evidence supporting distress than supporting health ($S_j > 0$), but falsity remains significant, and indeterminacy is non-negligible. In other words, it is a borderline firm with an explicit quantification of ambiguity.

4. Case Study: Evidence from Saudi Emerging Markets

This section applies the Nα-FDM framework to a concrete, fully worked case study. All data here are stylized but realistic and are meant to represent non-financial firms operating in Saudi emerging markets, with a focus on firms located in the Eastern Province. We use the same three core criteria and indicators introduced in Section 3:

a. K_1: Liquidity, measured by the current ratio (CR).
b. K_2: Solvency, measured by the debt-to-equity ratio (DE).
c. K_3: Profitability, measured by return on assets (ROA).

The α-discounted weights are those already derived:
$w_1^* \approx 0.4067, w_2^* \approx 0.3695, w_3^* \approx 0.2238$, with $w_1^* + w_2^* + w_3^* \approx 1.0000$.

4.1 Study context and sample

We consider a small, illustrative cross-section of $N = 4$
non-financial firms operating in Saudi emerging markets. To keep the focus on the method and avoid disclosure issues, we label the firms generically:

a. Firm A
b. Firm B
c. Firm C
d. Firm D

All firms are assumed to be located in the Eastern Province and to belong to sectors typical of the region (for example, petrochemical manufacturing, industrial services, logistics, and utilities). The actual sector names are not needed for the numerical calculations; what matters is the financial ratios.

4.2 Indicators and raw financial data

For each firm j, we observe:

1. Current ratio (CR) — short-term liquidity:

$$\mathrm{CR}_j = \frac{\text{current assets of firm } j}{\text{current liabilities of firm } j}.$$

2. Debt-to-equity ratio (DE) — long-term solvency:

$$\mathrm{DE}_j = \frac{\text{total debt of firm } j}{\text{total equity of firm } j}.$$

3. Return on assets (ROA) — profitability:

$$\mathrm{ROA}_j = \frac{\text{net income of firm } j}{\text{total assets of firm } j}.$$

Table 1 reports current ratio (CR), debt-to-equity ratio (DE), and return on assets (ROA) for four non-financial firms operating in Saudi emerging markets (Eastern Province).

Table 1. Raw financial indicators for four Saudi firms (stylized data)

Firm	Sector (generic description)	CR_j	DE_j	ROA_j
A	Capital-intensive petro-industrial	2.50	1.00	0.09 (9%)
B	Mixed industrial services	1.50	2.50	0.03 (3%)
C	Distressed manufacturing	0.80	3.50	−0.02 (−2%)
D	Moderately leveraged logistics	1.10	1.80	0.06 (6%)

These values will now be transformed into neutrosophic numbers using the membership functions specified in Section 3.3. For completeness, we restate the parameter values in the next subsection.

4.3 Membership function parameters

We treat each ratio as follows:

1. CR: lower values indicate worse (more distressed) liquidity.
2. DE: higher values indicate worse (more distressed) solvency.
3. ROA: lower values indicate worse profitability.

4.3.1 Current ratio (CR): "lower is worse."

Parameters: $a_{\mathrm{CR}} = 1.0, b_{\mathrm{CR}} = 2.0, m_{\mathrm{CR}} = \frac{1.0+2.0}{2} = 1.5.$

The current ratio (CR) is used to interpret a firm's liquidity position. When $CR \leq 1.0$, the firm's liquidity is clearly distressed, meaning it may struggle to meet its short-term obligations. When $CR \geq 2.0$, the liquidity position is clearly safe, indicating a comfortable cushion of current assets over current liabilities.

For values between 1.0 and 2.0, the firm is in a transition zone, where liquidity is neither clearly safe nor clearly distressed. In this range, some degree of indeterminacy or uncertainty about the true liquidity condition may be present.

<u>The membership functions are:</u>

Truth (distress-supporting):

$$\mu_{\mathrm{CR}}^{T}(x) = \begin{cases} 1, & x \leq 1.0, \\ \dfrac{2.0 - x}{2.0 - 1.0}, & 1.0 < x < 2.0, \\ 0, & x \geq 2.0; \end{cases}$$

Falsity (health-supporting):

$$\mu_{\mathrm{CR}}^{F}(x) = \begin{cases} 0, & x \leq 1.0, \\ \dfrac{x - 1.0}{2.0 - 1.0}, & 1.0 < x < 2.0, \\ 1, & x \geq 2.0; \end{cases}$$

Indeterminacy:

$$\mu^{I}_{\mathrm{CR}}(x)=\begin{cases}0, & x\le 1.0,\\ \frac{2\ |\ x-1.5\ |}{2.0-1.0}, & 1.0<x<2.0,\\ 0, & x\ge 2.0.\end{cases}$$

4.3.2 Debt-to-equity (DE): "higher is worse."

Parameters: $c_{DE}=1.0, d_{DE}=3.0, m_{DE}=\frac{1.0+3.0}{2}=2.0.$

The DE is used to describe a firm's solvency position. When $DE\le 1.0$, solvency is clearly safe, meaning the firm is not heavily dependent on debt and its capital structure is relatively conservative. When $DE\ge 3.0$, solvency is clearly distressed, indicating that the firm relies strongly on debt financing and may face higher financial risk.

For values between 1.0 and 3.0, the firm is in a transition zone. In this range, solvency is neither clearly safe nor clearly distressed, and there may be some indeterminacy or uncertainty about the true level of financial risk.

<u>Membership functions:</u>

Truth (distress-supporting):

$$\mu^{T}_{\mathrm{DE}}(x)=\begin{cases}0, & x\le 1.0,\\ \frac{x-1.0}{3.0-1.0}, & 1.0<x<3.0,\\ 1, & x\ge 3.0;\end{cases}$$

Falsity (health-supporting):

$$\mu^{F}_{\mathrm{DE}}(x)=\begin{cases}1, & x\le 1.0,\\ \frac{3.0-x}{3.0-1.0}, & 1.0<x<3.0,\\ 0, & x\ge 3.0;\end{cases}$$

Indeterminacy:

$$\mu^{I}_{\mathrm{DE}}(x)=\begin{cases}0, & x\le 1.0,\\ \frac{2\ |\ x-2.0\ |}{3.0-1.0}, & 1.0<x<3.0,\\ 0, & x\ge 3.0.\end{cases}$$

4.3.3 Return on assets (ROA): "lower is worse."

Parameters: $p_{\mathrm{ROA}}=0.00, q_{\mathrm{ROA}}=0.08, m_{\mathrm{ROA}}=\frac{0.00+0.08}{2}=0.04.$

ROA is used to evaluate a firm's profitability. When $\mathrm{ROA}\le 0.00$, profitability is clearly distressed, meaning the firm is generating zero or negative profit from its assets. This signals weak performance and potential financial problems.

When $\mathrm{ROA} \geq 0.08$, profitability is clearly healthy, showing that the firm is earning a strong return on its assets. For values between 0.00 and 0.08, the firm is in a transition zone. In this range, profitability is neither clearly distressed nor clearly healthy, and there may be some uncertainty about the firm's true financial performance.

<u>Membership functions:</u>
Truth (distress-supporting):

$$\mu_{\mathrm{ROA}}^{T}(x) = \begin{cases} 1, & x \leq 0.00, \\ \dfrac{0.08 - x}{0.08 - 0.00}, & 0.00 < x < 0.08, \\ 0, & x \geq 0.08; \end{cases}$$

Falsity (health-supporting):

$$\mu_{\mathrm{ROA}}^{F}(x) = \begin{cases} 0, & x \leq 0.00, \\ \dfrac{x - 0.00}{0.08 - 0.00}, & 0.00 < x < 0.08, \\ 1, & x \geq 0.08; \end{cases}$$

Indeterminacy:

$$\mu_{\mathrm{ROA}}^{I}(x) = \begin{cases} 0, & x \leq 0.00, \\ \dfrac{2\ |\ x - 0.04\ |}{0.08 - 0.00}, & 0.00 < x < 0.08, \\ 0, & x \geq 0.08. \end{cases}$$

All three ratios now have fully specified membership functions with no missing parameters.

4.4 Neutrosophic encoding of the four firms

For each firm $j$and each indicator, we compute the neutrosophic triple
$x_{jk} = (T_{jk}, I_{jk}, F_{jk}) = (\mu_k^T(X_{jk}), \mu_k^I(X_{jk}), \mu_k^F(X_{jk}))$.
We show the computations explicitly.
4.4.1 Liquidity (CR)
<u>Using the CR membership functions:</u>
<u>Firm A: $\mathrm{CR}_A = 2.50 \geq 2.0$</u>
$\mu_{\mathrm{CR}}^T(2.5) = 0, \mu_{\mathrm{CR}}^I(2.5) = 0, \mu_{\mathrm{CR}}^F(2.5) = 1.$
So $x_{A,1} = (0.0,0.0,1.0)$.
<u>Firm B: $\mathrm{CR}_B = 1.50$, with $1.0 < 1.5 < 2.0$</u>

$$\mu_{\mathrm{CR}}^T(1.5) = \frac{2.0 - 1.5}{2.0 - 1.0} = \frac{0.5}{1.0} = 0.5,$$

$$\mu_{\mathrm{CR}}^F(1.5) = \frac{1.5 - 1.0}{2.0 - 1.0} = \frac{0.5}{1.0} = 0.5,$$

$$\mu_{\mathrm{CR}}^I(1.5) = \frac{2\ |\ 1.5 - 1.5\ |}{2.0 - 1.0} = 0.$$

So $x_{B,1} = (0.5,0.0,0.5)$.

Firm C: $\mathrm{CR}_C = 0.80 \leq 1.0$

$\mu_{\mathrm{CR}}^T(0.8) = 1, \mu_{\mathrm{CR}}^I(0.8) = 0, \mu_{\mathrm{CR}}^F(0.8) = 0.$

So $x_{C,1} = (1.0,0.0,0.0)$.

Firm D: $\mathrm{CR}_D = 1.10$, with $1.0 < 1.1 < 2.0$

$$\mu_{\mathrm{CR}}^T(1.1) = \frac{2.0 - 1.1}{2.0 - 1.0} = 0.9,$$

$$\mu_{\mathrm{CR}}^F(1.1) = \frac{1.1 - 1.0}{2.0 - 1.0} = 0.1,$$

$$\mu_{\mathrm{CR}}^I(1.1) = \frac{2 \ | \ 1.1 - 1.5 \ |}{2.0 - 1.0} = 2 \times 0.4 = 0.8.$$

So $x_{D,1} = (0.9,0.8,0.1)$.

4.4.2 Solvency (DE)

Using the DE membership functions:

Firm A: $\mathrm{DE}_A = 1.00 \leq 1.0$

$\mu_{\mathrm{DE}}^T(1.0) = 0, \mu_{\mathrm{DE}}^I(1.0) = 0, \mu_{\mathrm{DE}}^F(1.0) = 1.$

So $x_{A,2} = (0.0,0.0,1.0)$.

Firm B: $\mathrm{DE}_B = 2.50$, with $1.0 < 2.5 < 3.0$

$$\mu_{\mathrm{DE}}^T(2.5) = \frac{2.5 - 1.0}{3.0 - 1.0} = \frac{1.5}{2.0} = 0.75,$$

$$\mu_{\mathrm{DE}}^F(2.5) = \frac{3.0 - 2.5}{3.0 - 1.0} = \frac{0.5}{2.0} = 0.25,$$

$$\mu_{\mathrm{DE}}^I(2.5) = \frac{2 \ | \ 2.5 - 2.0 \ |}{3.0 - 1.0} = \frac{2 \times 0.5}{2.0} = 0.5.$$

So $x_{B,2} = (0.75,0.5,0.25)$.

Firm C: $\mathrm{DE}_C = 3.50 \geq 3.0$

$\mu_{\mathrm{DE}}^T(3.5) = 1, \mu_{\mathrm{DE}}^I(3.5) = 0, \mu_{\mathrm{DE}}^F(3.5) = 0.$

So $x_{C,2} = (1.0,0.0,0.0)$.

Firm D: $\mathrm{DE}_D = 1.80$, with $1.0 < 1.8 < 3.0$

$$\mu_{\mathrm{DE}}^T(1.8) = \frac{1.8 - 1.0}{3.0 - 1.0} = \frac{0.8}{2.0} = 0.4,$$

$$\mu_{\mathrm{DE}}^F(1.8) = \frac{3.0 - 1.8}{3.0 - 1.0} = \frac{1.2}{2.0} = 0.6,$$

$$\mu_{\mathrm{DE}}^I(1.8) = \frac{2 \ | \ 1.8 - 2.0 \ |}{3.0 - 1.0} = \frac{2 \times 0.2}{2.0} = 0.2.$$

So $x_{D,2} = (0.4,0.2,0.6)$.

4.4.3 Profitability (ROA)

Using the ROA membership functions:

Firm A: $\mathrm{ROA}_A = 0.09 \geq 0.08$

$\mu_{\mathrm{ROA}}^{T}(0.09)=0, \mu_{\mathrm{ROA}}^{I}(0.09)=0, \mu_{\mathrm{ROA}}^{F}(0.09)=1.$
So $x_{A,3}=(0.0,0.0,1.0)$.
<u>Firm B: $\mathrm{ROA}_{B}=0.03$, with $0.00<0.03<0.08$</u>

$$\mu_{\mathrm{ROA}}^{T}(0.03)=\frac{0.08-0.03}{0.08-0.00}=\frac{0.05}{0.08}=0.625,$$
$$\mu_{\mathrm{ROA}}^{F}(0.03)=\frac{0.03-0.00}{0.08-0.00}=\frac{0.03}{0.08}=0.375,$$
$$\mu_{\mathrm{ROA}}^{I}(0.03)=\frac{2\ |\ 0.03-0.04\ |}{0.08-0.00}=\frac{2\times 0.01}{0.08}=\frac{0.02}{0.08}=0.25.$$

So $x_{B,3}=(0.625,0.25,0.375)$.
<u>Firm C: $\mathrm{ROA}_{C}=-0.02\leq 0.00$</u>

$\mu_{\mathrm{ROA}}^{T}(-0.02)=1, \mu_{\mathrm{ROA}}^{I}(-0.02)=0, \mu_{\mathrm{ROA}}^{F}(-0.02)=0.$
So $x_{C,3}=(1.0,0.0,0.0)$.
<u>Firm D: $\mathrm{ROA}_{D}=0.06$, with $0.00<0.06<0.08$</u>

$$\mu_{\mathrm{ROA}}^{T}(0.06)=\frac{0.08-0.06}{0.08-0.00}=\frac{0.02}{0.08}=0.25,$$
$$\mu_{\mathrm{ROA}}^{F}(0.06)=\frac{0.06-0.00}{0.08-0.00}=\frac{0.06}{0.08}=0.75,$$
$$\mu_{\mathrm{ROA}}^{I}(0.06)=\frac{2\ |\ 0.06-0.04\ |}{0.08-0.00}=\frac{2\times 0.02}{0.08}=\frac{0.04}{0.08}=0.50.$$

So $x_{D,3}=(0.25,0.50,0.75)$.

4.4.4 Summary table of neutrosophic indicators

Table 2. Neutrosophic encoding of financial indicators for four Saudi firms Each row shows the observed ratio and the corresponding neutrosophic triple (T_{jk}, I_{jk}, F_{jk}), computed using the membership functions in Section 4.3. All components lie in [0,1].

Table 2. Neutrosophic encoding of financial indicators for four Saudi firms

Firm	Indicator	Value	Neutrosophic triple (T_{jk}, I_{jk}, F_{jk})
A	CR	2.50	(0.00, 0.00, 1.00)
A	DE	1.00	(0.00, 0.00, 1.00)
A	ROA	0.09	(0.00, 0.00, 1.00)
B	CR	1.50	(0.50, 0.00, 0.50)
B	DE	2.50	(0.75, 0.50, 0.25)
B	ROA	0.03	(0.625, 0.25, 0.375)
C	CR	0.80	(1.00, 0.00, 0.00)
C	DE	3.50	(1.00, 0.00, 0.00)
C	ROA	−0.02	(1.00, 0.00, 0.00)
D	CR	1.10	(0.90, 0.80, 0.10)
D	DE	1.80	(0.40, 0.20, 0.60)
D	ROA	0.06	(0.25, 0.50, 0.75)

Firms A and C appear as "pure" cases: all three indicators support health for A and distress for C. Firms B and D have mixed truth–falsity patterns and non-zero indeterminacy, capturing ambiguous financial profiles.

4.5 Aggregation to firm-level NFDI

Since each criterion has one indicator, the criterion-level neutrosophic triples are simply the indicator triples:

a. For Liquidity K_1: $c_{j1} = x_{j1}$,
b. For Solvency K_2: $c_{j2} = x_{j2}$,
c. For Profitability K_3: $c_{j3} = x_{j3}$.

The firm-level NFDI for firm $j$is:
$T_j = w_1^* T_{j1}^{(c)} + w_2^* T_{j2}^{(c)} + w_3^* T_{j3}^{(c)}$,
$I_j = w_1^* I_{j1}^{(c)} + w_2^* I_{j2}^{(c)} + w_3^* I_{j3}^{(c)}$,
$F_j = w_1^* F_{j1}^{(c)} + w_2^* F_{j2}^{(c)} + w_3^* F_{j3}^{(c)}$.
We now compute T_j, I_j, F_jfirm by firm.

4.5.1 Firm A

From Table 2:
$c_{A1} = (0,0,1), c_{A2} = (0,0,1), c_{A3} = (0,0,1)$.
Thus:
Truth: $T_A = w_1^* \cdot 0 + w_2^* \cdot 0 + w_3^* \cdot 0 = 0$.
Indeterminacy: $I_A = w_1^* \cdot 0 + w_2^* \cdot 0 + w_3^* \cdot 0 = 0$.
Falsity: $F_A = w_1^* \cdot 1 + w_2^* \cdot 1 + w_3^* \cdot 1 = w_1^* + w_2^* + w_3^* = 1.0$.
So, $\text{NFDI}_A = (T_A, I_A, F_A) = (0.0, 0.0, 1.0)$.
The sum is 1, so normalization leaves the triple unchanged:
$T_A^* = 0, I_A^* = 0, F_A^* = 1$.
The scalar distress score is: $S_A = T_A^* - F_A^* = 0 - 1 = -1.0$.
Firm A is therefore clearly financially healthy in the model.

4.5.2 Firm C

From Table 2:
$c_{C1} = (1,0,0), c_{C2} = (1,0,0), c_{C3} = (1,0,0)$.
Thus:
Truth: $T_C = w_1^* \cdot 1 + w_2^* \cdot 1 + w_3^* \cdot 1 = 1.0$.
Indeterminacy: $I_C = 0$.
Falsity: $F_C = 0$.
So, $\text{NFDI}_C = (1.0, 0.0, 0.0)$.
Normalization again leaves it unchanged:
$T_C^* = 1, I_C^* = 0, F_C^* = 0$.

The scalar distress score is: $S_C = T_C^* - F_C^* = 1 - 0 = 1.0$.
Firm C is clearly financially distressed.

4.5.3 Firm B

From Table 2:
$c_{B1} = (0.5,0.0,0.5), c_{B2} = (0.75,0.5,0.25), c_{B3} = (0.625,0.25,0.375)$.
Compute each component.

Truth T_B

$$T_B = w_1^* \cdot 0.5 + w_2^* \cdot 0.75 + w_3^* \cdot 0.625$$
$$= 0.4067 \times 0.5 + 0.3695 \times 0.75 + 0.2238 \times 0.625.$$

Products:

a. $0.4067 \times 0.5 = 0.20335,$
b. $0.3695 \times 0.75 = 0.277125,$
c. $0.2238 \times 0.625 = 0.139875.$

Sum: $T_B = 0.20335 + 0.277125 + 0.139875 = 0.62035.$

Indeterminacy I_B

$$I_B = w_1^* \cdot 0.0 + w_2^* \cdot 0.5 + w_3^* \cdot 0.25$$
$$= 0.4067 \times 0.0 + 0.3695 \times 0.5 + 0.2238 \times 0.25.$$

Products:

a. $0.4067 \times 0.0 = 0.00000,$
b. $0.3695 \times 0.5 = 0.18475,$
c. $0.2238 \times 0.25 = 0.05595.$

Sum: $I_B = 0.18475 + 0.05595 = 0.24070.$

Falsity F_B

$$F_B = w_1^* \cdot 0.5 + w_2^* \cdot 0.25 + w_3^* \cdot 0.375$$
$$= 0.4067 \times 0.5 + 0.3695 \times 0.25 + 0.2238 \times 0.375.$$

Products:

a. $0.4067 \times 0.5 = 0.20335,$
b. $0.3695 \times 0.25 = 0.092375,$
c. $0.2238 \times 0.375 = 0.083925.$

Sum: $F_B = 0.20335 + 0.092375 + 0.083925 = 0.37965.$

Sum of components

$T_B + I_B + F_B = 0.62035 + 0.24070 + 0.37965 = 1.24070.$
So, $\mathrm{NFDI}_B = (0.62035,0.24070,0.37965)$.

Normalization and score

$$T_B^* = \frac{0.62035}{1.24070} = 0.5$$

(since $2 \times 0.62035 = 1.24070$),

$$I_B^* = \frac{0.24070}{1.24070} \approx 0.1940,$$

$F_B^* = 1 - T_B^* - I_B^* \approx 1 - 0.5 - 0.1940 = 0.3060.$

The scalar distress score is: $S_B = T_B^* - F_B^* \approx 0.5 - 0.3060 = 0.1940$. Firm B, therefore, has more evidence for distress than for health, but falsity and indeterminacy remain significant. It is a borderline firm.

4.5.4 Firm D

From Table 2:

$$c_{D1} = (0.9,0.8,0.1), c_{D2} = (0.4,0.2,0.6), c_{D3} = (0.25,0.5,0.75).$$

Compute each component.

Truth T_D

$T_D = w_1^* \cdot 0.9 + w_2^* \cdot 0.4 + w_3^* \cdot 0.25$

$= 0.4067 \times 0.9 + 0.3695 \times 0.4 + 0.2238 \times 0.25.$

Products:

a. $0.4067 \times 0.9 = 0.36603,$
b. $0.3695 \times 0.4 = 0.14780,$
c. $0.2238 \times 0.25 = 0.05595.$

Sum:

$T_D = 0.36603 + 0.14780 + 0.05595 = 0.56978.$

Indeterminacy I_D

$I_D = w_1^* \cdot 0.8 + w_2^* \cdot 0.2 + w_3^* \cdot 0.5$

$= 0.4067 \times 0.8 + 0.3695 \times 0.2 + 0.2238 \times 0.5.$

Products:

a. $0.4067 \times 0.8 = 0.32536,$
b. $0.3695 \times 0.2 = 0.07390,$
c. $0.2238 \times 0.5 = 0.11190.$

Sum:

$I_D = 0.32536 + 0.07390 + 0.11190 = 0.51116.$

Falsity F_D

$F_D = w_1^* \cdot 0.1 + w_2^* \cdot 0.6 + w_3^* \cdot 0.75$

$= 0.4067 \times 0.1 + 0.3695 \times 0.6 + 0.2238 \times 0.75.$

Products:

a. $0.4067 \times 0.1 = 0.04067,$
b. $0.3695 \times 0.6 = 0.22170,$
c. $0.2238 \times 0.75 = 0.16785.$

Sum:

$F_D = 0.04067 + 0.22170 + 0.16785 = 0.43022.$

Sum of components

$T_D + I_D + F_D = 0.56978 + 0.51116 + 0.43022 = 1.51116.$

So, $\mathrm{NFDI}_D = (0.56978,0.51116,0.43022)$.

Normalisation and score:

$$T_D^* = \frac{0.56978}{1.51116} \approx 0.3770,$$

$I_D^* = \dfrac{0.51116}{1.51116} \approx 0.3383,$

$F_D^* = \dfrac{0.43022}{1.51116} \approx 0.2847.$

They are satisfied,

$T_D^* + I_D^* + F_D^* \approx 0.3770 + 0.3383 + 0.2847 \approx 1.0000.$

The scalar distress score: $S_D = T_D^* - F_D^* \approx 0.3770 - 0.2847 = 0.0923.$

Firm D leans slightly towards distress ($S_D > 0$), but indeterminacy is high, and falsity is not negligible. It is more ambiguous than Firm B.

4.6 Classification and interpretation in the Saudi context

To classify firms into "distressed", "healthy", and "grey zone", we introduce symmetric thresholds $\tau_D$and $\tau_H$with $0 < \tau_H < \tau_D \leq 1$. For illustration, we set $\tau_D = 0.40, \tau_H = 0.40$. We classify:

a. Distressed if $S_j \geq \tau_D$,
b. Healthy if $S_j \leq -\tau_H$,
c. Grey zone otherwise.

Using the computed scores:

1. Firm A: $S_A = -1.0000 \leq -0.40$ ⇒Healthy.
2. Firm B: $S_B \approx 0.1940$(between −0.40 and 0.40) ⇒Grey zone.
3. Firm C: $S_C = 1.0000 \geq 0.40$ ⇒Distressed.
4. Firm D: $S_D \approx 0.0923$(between −0.40 and 0.40) ⇒Grey zone.

Table 3. Static NFDI components, scores, and classification

Firm	(T_j, I_j, F_j)	(T_j^*, I_j^*, F_j^*)	Score S_j	Classification
A	(0.00, 0.00, 1.00)	(0.00, 0.00, 1.00)	−1.0000	Healthy
B	(0.62035, 0.24070, 0.37965)	(0.50, ≈0.1940, ≈0.3060)	≈0.1940	Grey zone
C	(1.00, 0.00, 0.00)	(1.00, 0.00, 0.00)	1.0000	Distressed
D	(0.56978, 0.51116, 0.43022)	(≈0.3770, ≈0.3383, ≈0.2847)	≈0.0923	Grey zone

In the Saudi context, Firm A appears financially strong. It has comfortable liquidity, reasonable debt levels, and solid profitability, so it looks like a robust non-financial company with almost no uncertainty in its status.

Firm C is at the opposite extreme. Its low liquidity, heavy reliance on debt, and negative profitability together point to a clear distress case, which the model captures by assigning it full distress and no financial health.

Firms B and D sit in the middle. Their financial ratios do not clearly place them in either the safe or distressed group. In Saudi emerging markets, where oil price changes and frequent regulatory shifts are common, this kind of borderline profile is often observed.

The indeterminacy term I_j^* is especially useful here. Firm D has the highest level of indeterminacy, which means its future is highly uncertain and very sensitive to new shocks and information. From a practical point of view, this calls for closer follow-up—such as tighter monitoring, better disclosure, or scenario analysis—rather than a quick decision that it is simply safe or distressed.

Overall, the case study shows that the Nα-FDM model can both match common sense for very strong and very weak firms and give a richer picture of borderline firms by explicitly separating evidence for distress, evidence for health, and the uncertainty between them.

5. Discussion and Implications

This section interprets the results of the Nα-FDM model from Section 4, connects them to the theoretical structure from Sections 2–3, and highlights their meaning for financial distress assessment in Saudi emerging markets.

5.1 Relation to classical financial distress models

Classical models such as the Altman Z-score [1] and the Ohlson O-score [2] compress a set of financial ratios into a single real-valued index or probability of default. Conceptually, the scalar output can be viewed as an implicit mapping, ratios $\longrightarrow \mathbb{R}$, with no explicit separation between "evidence for distress", "health evidence", and "uncertainty about both". By contrast, the Nα-FDM approach maps each firm j to a triple $\mathrm{NFDI}_j = (T_j, I_j, F_j)$, and, when normalized, $\mathrm{NFDI}_j^* = (T_j^*, I_j^*, F_j^*)$, together with a scalar score $S_j = T_j^* - F_j^*$.

If one ignores I_j^* and looks only at S_j, the model plays a role similar to a traditional distress score: high positive S_j suggests distress, high negative S_j suggests health, and values near zero indicate intermediate risk. However, the neutrosophic representation is strictly richer, because it distinguishes:

a. How much of the information pushes towards distress (T_j^*),
b. how much pushes towards health (F_j^*),
c. How much reflects unresolved ambiguity (I_j^*).

<u>In the case study:</u>

1. Firm A and Firm C mimic the "extreme" cases of a very high Z-score or very low O-score (for A), and the opposite for C. They have $\mathrm{NFDI}_A^* = (0,0,1)$ and $\mathrm{NFDI}_C^* = (1,0,0)$.

2. Firms B and D resemble companies that would fall in a classical grey band, but the neutrosophic model further clarifies how they are ambiguous: B has balanced truth and falsity with moderate I_B^*, while D exhibits substantial indeterminacy I_D^*alongside a smaller truth–falsity imbalance.

Thus, Nα-FDM can be viewed as a generalization of classical ratio-based models, where a scalar distress measure is retained through S_j, but is systematically accompanied by an explicit decomposition into truth, falsity, and indeterminacy components.

5.2 Reading the neutrosophic outputs: four firm profiles

The four firms in the Saudi case study illustrate four qualitatively distinct configurations:

1. Firm A — robust health
   $\mathrm{NFDI}_A^* = (0,0,1), S_A = -1.0.$
   Liquidity, solvency, and profitability all lie in clearly safe regions. The model attributes full weight to the falsity of distress and zero to both truth and indeterminacy. This corresponds to a firm that is far into the "safe" region of any classical distress scale and for which the available information is internally coherent.
2. Firm C — clear distress
   $\mathrm{NFDI}_C^* = (1,0,0), S_C = 1.0.$
   All three ratios strongly support distress: liquidity is very weak, leverage is high, and profitability is negative. The model reflects this unanimity by assigning full mass to truth and none to falsity or indeterminacy. There is no ambiguity: both the level and direction of the signal are clear.
3. Firm B — symmetric but leaning towards distress
   $\mathrm{NFDI}_B^* \approx (0.50,0.19,0.31), S_B \approx 0.19.$
   This firm has moderately weak liquidity, high leverage, and low but positive profitability. The triple shows that evidence in favour of distress is stronger than evidence in favour of health, yet indeterminacy is not negligible. In practical terms, this is a firm that should be treated as elevated risk but not as unequivocally distressed. The model makes this explicit instead of forcing a sharp binary classification.
4. Firm D — high indeterminacy and mild distress tilt
   $\mathrm{NFDI}_D^* \approx (0.38,0.34,0.28), S_D \approx 0.09.$

Here, all three components are substantial. Liquidity is weak, but solvency and profitability are more acceptable. The model signals a slight leaning towards distress, but, more importantly, a large indeterminate component. From a decision-making perspective, D is not merely average; it is structurally ambiguous. This is an important distinction: two firms with similar $S_j$can have very different I_j^*, leading to different policy responses.

These four patterns illustrate the key strength of the neutrosophic representation: the ability to distinguish "clearly safe", "clearly distressed", "borderline but relatively well understood" (B), and "borderline and fundamentally uncertain" (D).

5.3 Role of α-discounting and expert inconsistency

The α-discounting method connects expert knowledge to the quantitative model in a way that is both systematic and transparent.
In the three-criterion example (liquidity, solvency, profitability), experts provide a set of relations among the weights, such as "liquidity is more important than solvency" and "solvency is more important than profitability". These relations, taken together, can be mutually incompatible. Rather than discarding statements or solving an approximate least-squares system without interpretation, the α-discounting method:

a. introduces a single scalar α(under the fairness principle),
b. adjusts each relation multiplicatively by α,
c. finds a weight vector w^*That satisfies the adjusted relations exactly.

The outcome is twofold:

1. A weight vector w^*That respects the overall pattern of preferences and is guaranteed to be non-negative and normalised.
2. A consistency indicator α^*Whose deviation from 1 reflects how strongly the original system had to be altered.

In the present setting, the resulting weights :$w_1^* \approx 0.4067, w_2^* \approx 0.3695, w_3^* \approx 0.2238$

assign slightly more importance to liquidity than to solvency, and clearly more to both than to profitability. This aligns with the intuition that in Saudi emerging markets, short-term liquidity and leverage constraints can trigger distress quickly, while profitability is important but somewhat less immediate as a default driver.

Importantly, if future expert assessments or regulatory priorities change (for example, giving more weight to profitability in highly competitive sectors), the same α-discounting machinery can be applied to generate a new weight vector, without altering the neutrosophic encoding of indicators or the structure of the cognitive map.

5.4 Managerial and regulatory implications for Saudi emerging markets

In the context of Saudi emerging markets, and particularly in regions like the Eastern Province where firms are exposed to sectoral concentration and macro-financial shocks, the Nα-FDM outputs suggest several practical uses:

1. Screening and prioritization of supervisory effort

   a. Firms like C, with $S_C$close to +1 and I_C^*near zero, are clear candidates for immediate supervisory attention, restructuring plans, or stricter credit conditions.
   b. Firms like A, with $S_A$close to −1 and zero indeterminacy, require only routine monitoring.
   c. Firms like B and D fall in the grey zone; here, the distinction between moderate and high indeterminacy is crucial. Firm D, with larger I_D^*, would merit closer qualitative analysis, additional disclosures, or scenario-based stress tests, even though its scalar score $S_D$is lower than S_B.
2. The neutrosophic framework is designed to handle indicators that may be qualitative or subjectively scored (such as governance quality or exposure to specific regional risks). In the Saudi setting, this is very relevant: data on governance practices, ownership structures, or project-specific guarantees can be incorporated as additional criteria without breaking the structure of the model. Their uncertain nature is naturally captured through non-zero indeterminacy components.
3. Regulators and lenders frequently need to justify decisions to stakeholders. Expressing firm-level risk as a triple (T_j^*, I_j^*, F_j^*)rather than only as a single probability allows for a more nuanced narrative:
   a. "The firm appears risky, but information is incomplete" (high $T_j^*$and high I_j^*), versus
   b. "The firm appears risky, and the evidence is consistent" (high T_j^*, low I_j^*). In emerging markets, where data gaps are common, this explicit separation can improve transparency and trust.
4. By extending the static NFDI to a dynamic version via the neutrosophic cognitive map, one can simulate how shocks (for example, a drop in oil prices or a tightening of credit conditions) propagate through liquidity, solvency, and profitability to affect overall distress. This opens the door to stress-testing frameworks tailored to Saudi market conditions, while preserving the explicit representation of indeterminacy.

5.5 Limitations and methodological reflections

Although the model is internally consistent and the case study is numerically complete, several limitations must be acknowledged:

1. The case study uses stylised data for four firms and three criteria. A full empirical implementation would require a larger sample, more criteria (e.g., governance, cash-flow volatility, market-based indicators), and actual historical distress events.
2. The membership function parameters are chosen to be plausible and economically sensible, but they have not been statistically calibrated. Calibration using historical Saudi firm data would refine the boundaries between "distressed", "ambiguous", and "safe" regions in indicator space.

3. The neutrosophic cognitive map has been described conceptually. The static NFDI already provides useful insight, but a dynamic implementation—where edge weights are estimated from data or expert elicitation and the map is iterated to convergence—would increase realism at the cost of additional modelling effort.
4. Finally, like any model, Nα-FDM does not remove uncertainty; instead, it makes uncertainty visible. Its value lies in exposing where the information set is strong, where it is weak, and where experts disagree.

Despite these limitations, the case study demonstrates that the Nα-FDM approach is capable of capturing both the level of distress risk and the structure of ambiguity surrounding it. For emerging markets such as Saudi Arabia, where incomplete and conflicting information is routine rather than exceptional, this explicit handling of indeterminacy is a key advantage.

6. Conclusion and Future Research

This paper proposed a Nα-FDM framework for financial distress prediction in emerging markets, with an application to Saudi non-financial firms. The model combines:

1. neutrosophic encoding of indicators into triples (T, I, F),
2. α-discounted multi-criteria weighting of high-level factors,
3. construction of a firm-level Neutrosophic Financial Distress Index $\mathrm{NFDI}_j$and scalar score S_j.

Compared to classical ratio-based models, Nα-FDM does not compress all uncertainty into a single scalar. Instead, it separates supporting evidence, contradicting evidence, and indeterminacy, and directly incorporates expert inconsistency through α-discounting. The Saudi case study shows that the model behaves intuitively in extreme cases and provides meaningful nuance for borderline firms by highlighting where ambiguity dominates.

Future research directions include:

I. empirical calibration on larger Saudi datasets and comparison with machine-learning models [3],
II. extension of the criteria set to include governance, market-based signals, and macroeconomic stress,
III. Implementation of a dynamic neutrosophic cognitive map for stress testing,
IV. hybrid models where neutrosophic outputs serve as inputs to advanced classifiers.

The framework developed here is mathematically rigorous, conceptually transparent, and tailored to the reality of emerging markets where indeterminacy is not an exception, but a central feature of the information environment.

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