



## Neutrosophic Hyper $KU$ -Ideals

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**Abstract.** We define neutrosophic hyper  $KU$ -ideal (strong, weak,  $s$ -weak) and reflexive neutrosophic hyper  $KU$ -ideal. A few key properties and their relationships are highlighted. The study of the neutrosophic (weak) hyper  $KU$ -ideal is considered. The conditions for a neutrosophic set to be a  $NS\mathbb{H}KUI$  and a (reflexive) neutrosophic hyper  $KU$ -ideal are discussed. There are also circumstances for a  $N\mathbb{W}\mathbb{H}KUI$  to be a  $Ns\mathbb{W}\mathbb{H}KUI$ , as well as conditions for a  $NS\mathbb{H}KUI$  to be a  $RN\mathbb{H}KUI$ .

**Keywords:** Hyper  $KU$ -algebra; hyper  $KU$ -ideals;  $N\mathbb{H}KUI$ ;  $NS\mathbb{H}KUI$ ;  $N\mathbb{W}\mathbb{H}KUI$ ;  $Ns\mathbb{W}\mathbb{H}KUI$ ;  $RN\mathbb{H}KUI$ .

### 1. Introduction

Prabpayak and Leerawat created a novel algebraic structure known as  $KU$ -algebras [12, 13]. In  $KU$ -algebras, they worked at ideals and congruences. They also established the concept of  $KU$ -algebra homomorphism and looked into various related features. They also deduced some simple consequences of the quotient  $KU$ -algebras and isomorphism relationships. Marty [9] presented the hyper structure theory (also known as multialgebras) at the 8th Congress of Scandinavian Mathematicians in 1934. Several authors, primarily in France and the United States, but also in Italy, Russia, and Japan, worked on hyper groups in the 1940's. Hyperstructures offer a wide range of applications in both pure and applied sciences. Jun et al. extended hyper structures to  $BCK$ -algebras, proposed the idea of a hyper  $BCK$ -algebra, which is a generalization of a  $BCK$ -algebra, and looked into several associated characteristics in [8]. They also defined a hyper  $BCK$ -ideal and a weak hyper  $BCK$ -ideal, as well as relationships between hyper  $BCK$ -ideals and weak hyper  $BCK$ -ideals. Jun et al. [7] proposed the notions of a strong hyper  $BCK$ -ideal, a weak hyper  $BCK$ -ideal, and a reflexive hyper  $BCK$ -ideal, as well as a requirement for a hyper  $BCK$ -algebra to be a  $BCK$ -algebra. Every strong hyper

$BCK$ -ideal is a hyper sub-algebra, a weak hyper  $BCK$ -ideal, and a hyper  $BCK$ -ideal, and every reflexive hyper  $BCK$ -ideal is a strong hyper  $BCK$ -ideal, they established.

Smarandache [14–16] developed the neutrosophic set, which is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set, and interval valued (intuitionistic) fuzzy set. On  $BL$ -algebras, Borzooei et al. [4] investigated neutrosophic deductive filters. Zhang et al. [19] discussed neutrosophic regular filters and fuzzy regular filters when applying the concept of neutrosophic set to pseudo- $BCI$  algebras. Neutrosophic set theory has been applied to a variety of areas, and many studies have been conducted to develop, improve, and expand the theory ([1–3, 5, 6, 10, 17] and [18]).

The goal of this paper is to introduce the concepts of neutrosophic (strong, weak,  $s$ -weak) hyper  $KU$ -ideal, as well as  $RN\mathbb{H}KUI$ . We look at their connections and properties. Characterizations of neutrosophic (weak) hyper  $KU$ -ideal are discussed. We define exactly for a neutrosophic set to be a  $NS\mathbb{H}KUI$  and a (reflexive) neutrosophic hyper  $KU$ -ideal. We're looking for some provisions that will allow a  $NS\mathbb{H}KUI$  to become a  $RN\mathbb{H}KUI$ . We go over the conditions for a  $N\mathbb{W}\mathbb{H}KUI$  to be a  $Ns\mathbb{W}\mathbb{H}KUI$ .

## 2. Preliminaries

The basic definitions of hyper  $KU$ -ideals and neutrosophic set are given in this section.

Let  $H$  be a non-empty set and let “ $\circ$ ” be a mapping

$$\circ : H \times H \rightarrow P(H) \setminus \{\emptyset\}$$

which is said to be hyperoperation. For any two subsets  $A$  and  $B$ , denote by  $A \circ B$ , the set  $\cup \{l_{01} \circ l_{02} | l_{01} \in A, l_{02} \in B\}$ . We shall use  $l_{01} \circ l_{02}$  instead of  $\{l_{01}\} \circ l_{02}, l_{01} \circ \{l_{02}\}$  or  $\{l_{01}\} \circ \{l_{02}\}$ .

By a hyper  $KU$ -algebra  $H$  ([11]), we mean a non-empty set  $H$  with a special element  $0$  and a hyperoperation  $\circ$ , for all  $l_{01}, l_{02}, l_{03} \in H$ , that satisfies the following axioms:

$$(HKU1) \quad (l_{02} \circ l_{03}) \circ (l_{01} \circ l_{03}) \ll l_{01} \circ l_{02},$$

$$(HKU2) \quad l_{01} \circ 0 = \{0\},$$

$$(HKU3) \quad 0 \circ l_{01} = \{l_{01}\},$$

(HKU4) if  $l_{01} \ll l_{02}$  and  $l_{02} \ll l_{01}$  imply  $l_{01} = l_{02}$ , for all  $l_{01}, l_{02}, l_{03} \in H$ , where  $l_{01} \ll l_{02}$  is defined by  $0 \in l_{02} \circ l_{01}$  and for any  $A, B \subseteq H$ ,  $A \ll B$  is defined by  $\forall r \in A, \exists t \in B$  such that  $r \ll t$ .

**Proposition 2.1.** [11] Let  $H$  be a hyper  $KU$ -algebra. Then for all  $l_{01}, l_{02}, l_{03} \in H$ , the following statements hold:

(i)  $A \subseteq B$  implies  $A \ll B$ , for all nonempty subsets  $A, B$  of  $H$ .

(ii)  $0 \circ 0 = \{0\}$ .

- (iii)  $0 \ll l_{01}$ .
- (iv)  $l_{03} \ll l_{03}$ .
- (v)  $l_{01} \circ l_{03} \ll l_{03}$ .
- (vi)  $A \circ 0 = \{0\}$ .
- (vii)  $0 \circ A = A$ .
- (viii)  $(0 \circ 0) \circ l_{01} = \{l_{01}\}$  and  $(l_{01} \circ (0 \circ l_{01})) = \{0\}$ .
- (ix)  $l_{01} \circ l_{01} = \{l_{01}\} \Leftrightarrow l_{01} = 0$ .
- (x)  $l_{03} \circ (l_{02} \circ l_{01}) = l_{02} \circ (l_{03} \circ l_{01})$  for all  $l_{01}, l_{02}, l_{03} \in H$ .

**Definition 2.2.** [11] Let  $(H, \circ)$  be a hyper  $KU$ -algebra. A subset  $A$  of  $H$  is called:

- A hyper  $KU$ -ideal (briefly,  $\mathbb{H}KU$ I) of  $H$  if
  - (1)  $0 \in A$ ,
  - (2)  $l_{02} \circ l_{01} \ll A$  and  $l_{02} \in A$  imply  $l_{01} \in A$ , for all  $l_{01}, l_{02} \in H$ .
- A weak hyper  $KU$ -ideal (briefly,  $\mathbb{W}HKU$ I) of  $H$  if it satisfies (1) and
  - (3)  $l_{02} \circ l_{01} \subseteq A$ ,  $l_{02} \in A \Rightarrow l_{01} \in A$ ,  $\forall l_{01}, l_{02} \in H$ ,
- A strong hyper  $KU$ -ideal (briefly,  $\mathbb{S}HKU$ I) of  $H$  if it satisfies (1) and
  - (4)  $(l_{02} \circ l_{01}) \cap A \neq \emptyset$ ,  $l_{02} \in A \Rightarrow l_{01} \in A$ ,  $\forall l_{01}, l_{02} \in H$ ,

A subset  $I$  of a hyper  $KU$ -algebra  $H$  is said to be reflexive if  $(l \circ l) \subseteq I$  for all  $l \in H$ .

Let  $H$  be a non-empty set. A neutrosophic set ( $NS$ ) in  $H$  (See [16]) is a structure of the form:

$$L := \{\langle l; L_T(l), L_I(l), L_F(l) \rangle \mid l \in H\}$$

where  $L_T : H \rightarrow [0, 1]$  is a truth membership function,  $L_I : H \rightarrow [0, 1]$  is an indeterminate membership function, and  $L_F : H \rightarrow [0, 1]$  is a false membership function. For abbreviation, we continue to write  $L = (L_T, L_I, L_F)$  for the  $NS$

$$L := \{\langle l; L_T(l), L_I(l), L_F(l) \rangle \mid l \in H\}.$$

Given a  $NS$   $L = (L_T, L_I, L_F)$  in a hyper  $KU$ -algebra  $H$  and a subset  $V$  of  $H$ , by  $*L_T, *L_T, *L_I, *L_I, *L_F$  and  $*L_F$  we mean

$$\begin{aligned} *L_T(V) &= \inf_{v \in V} L_T(v) \text{ and } *L_T(V) = \sup_{v \in V} L_T(v), \\ *L_I(V) &= \inf_{v \in V} L_I(v) \text{ and } *L_I(V) = \sup_{v \in V} L_I(v), \\ *L_F(V) &= \inf_{v \in V} L_F(v) \text{ and } *L_F(V) = \sup_{v \in V} L_F(v), \end{aligned}$$

respectively.

Notation. From now on, in this paper, we assume that  $H$  is a hyper  $KU$ -algebra.

### 3. Neutrosophic hyper $KU$ -ideals

We discussed the features of neutrosophic (strong, weak,  $s$ -weak) hyper  $KU$ -ideal and reflexive neutrosophic hyper  $KU$ -ideal in this part.

**Definition 3.1.** Let  $L = (L_T, L_I, L_F)$  be a  $NS$  in  $H$ . Then  $L$  is said to be a neutrosophic hyper  $KU$ -ideal (briefly,  $N\mathbb{H}KUI$ ) of  $H$  if it satisfies the following assertions for all  $l_{01}, l_{02} \in H$ ,

$$\left( l_{01} \ll l_{02} \Rightarrow \begin{cases} L_T(l_{01}) \geq L_T(l_{02}) \\ L_I(l_{01}) \geq L_I(l_{02}) \\ L_F(l_{01}) \leq L_F(l_{02}) \end{cases} \right), \tag{1}$$

$$\left( \begin{cases} L_T(l_{01}) \geq \min\{*_L L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \\ L_I(l_{01}) \geq \min\{*_L L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \\ L_F(l_{01}) \leq \max\{*_L L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \end{cases} \right) \tag{2}$$

**Example 3.2.** Let  $H = \{l_0, l_a, l_b\}$  be a hyper  $KU$ -algebra. The hyper operation “ $\circ$ ” on  $H$  described by Table 1.

**Table 1 :** Cayley table for the binary operation “ $\circ$ ”

$\circ$	$l_0$	$l_a$	$l_b$
$l_0$	$\{l_0\}$	$\{l_a\}$	$\{l_b\}$
$l_a$	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_a, l_b\}$
$l_b$	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_0, l_a, l_b\}$

We define a  $NS$   $L = (L_T, L_I, L_F)$  on  $H$  by Table 2.

**Table 2 :** Tabular representation of  $L = (L_T, L_I, L_F)$

$H$	$L_T(l)$	$L_I(l)$	$L_F(l)$
$l_0$	0.77	0.65	0.08
$l_a$	0.55	0.47	0.57
$l_b$	0.11	0.27	0.69

It is easy to check that  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$ .

**Proposition 3.3.** For any  $N\mathbb{H}KUI$   $L = (L_T, L_I, L_F)$  of  $H$ , the following assertions are valid.

(i)  $L = (L_T, L_I, L_F)$  satisfies

$$(\forall l_{01} \in H) \left( \begin{cases} L_T(0) \geq L_T(l_{01}) \\ L_I(0) \geq L_I(l_{01}) \\ L_F(0) \leq L_F(l_{01}) \end{cases} \right). \tag{3}$$

(ii) If  $L = (L_T, L_I, L_F)$  satisfies

$$(\forall V \subseteq H)(\exists u, v, w \in V) \left( \begin{cases} L_T(u) = *_L L_T(V) \\ L_I(v) = *_L L_I(V) \\ L_F(w) = *_L L_F(V) \end{cases} \right), \tag{4}$$

then the following assertion is valid.

$$(\forall l_{01}, l_{02} \in H)(\exists u, v, w \in l_{02} \circ l_{01}) \left( \begin{cases} L_T(l_{01}) \geq \min\{L_T(u), L_T(l_{02})\} \\ L_I(l_{01}) \geq \min\{L_I(v), L_I(l_{02})\} \\ L_F(l_{01}) \leq \max\{L_F(w), L_F(l_{02})\} \end{cases} \right). \tag{5}$$

**Proof.** By Proposition 2.1 (ii) and (1) we have

$$L_T(0) \geq L_T(l_{01}), L_I(0) \geq L_I(l_{01}) \text{ and } L_F(0) \leq L_F(l_{01}).$$

Assume that  $L = (L_T, L_I, L_F)$  satisfies the condition (4). For all  $l_{01}, l_{02} \in H$ , there exists  $u_0, v_0, w_0 \in l_{02} \circ l_{01}$  such that

$$L_T(u_0) = {}^*L_T(l_{02} \circ l_{01}), L_I(v_0) = {}^*L_I(l_{02} \circ l_{01}) \text{ and } L_F(w_0) = {}^*L_F(l_{02} \circ l_{01}).$$

Now condition (2) implies that

$$\begin{aligned} L_T(l_{01}) &\geq \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \min\{L_T(u_0), L_T(l_{02})\} \\ L_I(l_{01}) &\geq \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \min\{L_I(v_0), L_I(l_{02})\}. \\ L_F(l_{01}) &\leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \max\{L_F(w_0), L_F(l_{02})\} \end{aligned}$$

This completes the proof. ≡

We define the following sets:

$$\begin{aligned} U(L_T, \xi_T) &:= \{l_{01} \in H \mid L_T(l_{01}) \geq \xi_T\}, \\ U(L_I, \xi_I) &:= \{l_{01} \in H \mid L_I(l_{01}) \geq \xi_I\}, \\ L(L_F, \xi_F) &:= \{l_{01} \in H \mid L_F(l_{01}) \leq \xi_F\}, \end{aligned}$$

where  $L = (L_T, L_I, L_F)$  is a *NS* in  $H$  and  $\xi_T, \xi_I, \xi_F \in [0, 1]$ .

**Lemma 3.4.** Let  $L$  be a subset of  $H$ . If  $I$  is a  $\mathbb{H}KUI$  of  $H$  such that  $L \ll I$ , then  $L$  is contained in  $I$ .

**Proof.** Assume that  $L \ll H$  and let  $l \in L$ . Then  $0 \circ l = \{l\} \ll H$  and so  $l \in H$  by using Definition 2.2 (2). Therefore  $L \subseteq H$ . ≡

**Theorem 3.5.** A *NS*  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$  iff the nonempty sets  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $\mathbb{H}KUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ .

**Proof.** Assume that  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$  and suppose that  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are nonempty for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ . It is easy to see that  $0 \in U(L_T, \xi_T), 0 \in U(L_I, \xi_I)$  and  $0 \in L(L_F, \xi_F)$ . Let  $l_{01}, l_{02} \in H$  be such that  $l_{02} \circ l_{01} \ll U(L_T, \xi_T)$  and  $l_{02} \in U(L_T, \xi_T)$ . Then  $L_T(l_{02}) \geq \xi_T$  and for any  $l \in l_{02} \circ l_{01}$  there exists  $l_0 \in U(L_T, \xi_T)$  such that  $l \ll l_0$ . We conclude from (1) that  $L_T(l) \geq L_T(l_0) \geq \xi_T$  for all  $l \in l_{02} \circ l_{01}$ . Hence  ${}^*L_T(l_{02} \circ l_{01}) \geq \xi_T$ , and so

$$L_T(l_{01}) \geq \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \geq \xi_T,$$

that is,  $l_{01} \in U(L_T, \xi_T)$ . Similarly, we show that if  $l_{02} \circ l_{01} \ll U(L_I, \xi_I)$  and  $l_{02} \in U(L_I, \xi_I)$ , then  $l_{01} \in U(L_I, \xi_I)$ . Hence  $U(L_T, \xi_T)$  and  $U(L_I, \xi_I)$  are  $\mathbb{H}KUI$ 's of  $H$ . Let  $l_{01}, l_{02} \in H$  be such that

$l_{02} \circ l_{01} \ll L(L_F, \xi_F)$  and  $l_{02} \in L(L_F, \xi_F)$ . Then  $L_F(l_{02}) \leq \xi_F$ . Let  $m \in l_{02} \circ l_{01}$ . Then there exists  $m_0 \in L(L_F, \xi_F)$  such that  $m \ll m_0$ , which implies from (1) that  $L_F(m) \leq L_F(m_0) \leq \xi_F$ . Thus  $*L_F(l_{02} \circ l_{01}) \leq \xi_F$ , and so

$$L_F(l_{01}) \leq \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \leq \xi_F.$$

Hence  $l_{01} \in L(L_F, \xi_F)$  and therefore  $L(L_F, \xi_F)$  is a  $\mathbb{H}KUI$  of  $H$ .

Conversely, suppose that the nonempty sets  $U(L_T, \xi_T)$ ,  $U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $\mathbb{H}KUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ . Let  $l_{01}, l_{02} \in H$  be such that  $l_{01} \ll l_{02}$ . Then

$$l_{02} \in U(L_T, L_T(l_{02})) \cap U(L_I, L_I(l_{02})) \cap L(L_F, L_F(l_{02})),$$

and thus  $l_{01} \ll U(L_T, L_T(l_{02}))$ ,  $l_{01} \ll U(L_I, L_I(l_{02}))$  and  $l_{01} \ll L(L_F, L_F(l_{02}))$ . According to Lemma 3.4, we have  $l_{01} \in U(L_T, L_T(l_{02}))$ ,  $l_{01} \in U(L_I, L_I(l_{02}))$  and  $l_{01} \in L(L_F, L_F(l_{02}))$  which imply that  $L_T(l_{01}) \geq L_T(l_{02})$ ,  $L_I(l_{01}) \geq L_I(l_{02})$  and  $L_F(l_{01}) \leq L_F(l_{02})$ . For any  $l_{01}, l_{02} \in H$ , let  $\xi_T := \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\}$ ,  $\xi_I := \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$  and  $\xi_F := \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}$ . Then

$$l_{02} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F),$$

and for each  $u_T, v_I, w_F \in l_{02} \circ l_{01}$  we have

$$L_T(u_T) \geq *L_T(l_{02} \circ l_{01}) \geq \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \xi_T,$$

$$L_I(v_I) \geq *L_I(l_{02} \circ l_{01}) \geq \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \xi_I$$

and

$$L_F(w_F) \leq *L_F(l_{02} \circ l_{01}) \leq \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \xi_F.$$

Hence  $u_T \in U(L_T, \xi_T)$ ,  $v_I \in U(L_I, \xi_I)$  and  $w_F \in L(L_F, \xi_F)$  and so  $l_{02} \circ l_{01} \subseteq U(L_T, \xi_T)$ ,  $l_{02} \circ l_{01} \subseteq U(L_I, \xi_I)$  and  $l_{02} \circ l_{01} \subseteq L(L_F, \xi_F)$ . By Proposition 2.1, we have  $l_{02} \circ l_{01} \ll U(L_T, \xi_T)$ ,  $l_{02} \circ l_{01} \ll U(L_I, \xi_I)$  and  $l_{02} \circ l_{01} \ll L(L_F, \xi_F)$ . It follows from Definition 2.2 (2) that

$$l_{01} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F).$$

Hence

$$L_T(l_{01}) \geq \xi_T = \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \xi_I = \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$$

and

$$L_F(l_{01}) \leq \xi_F = \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Therefore  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$ . \(\Xi\)

**Theorem 3.6.** If  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$ , then the set

$$J := \{l_{01} \in H \mid L_T(l_{01}) = L_T(0), L_I(l_{01}) = L_I(0), L_F(l_{01}) = L_F(0)\} \tag{6}$$

is a  $\mathbb{H}KUI$  of  $H$ .

**Proof.** It is easy to check that  $0 \in J$ . Let  $l_{01}, l_{02} \in H$  be such that  $l_{02} \circ l_{01} \ll J$  and  $l_{02} \in J$ . Then  $L_T(l_{02}) = L_T(0), L_I(l_{02}) = L_I(0)$  and  $L_F(l_{02}) = L_F(0)$ . Let  $l \in l_{02} \circ l_{01}$ . Then there exists  $l_0 \in J$  such that  $l \ll l_0$ , and thus by (1),  $L_T(l) \geq L_T(l_0) = L_T(0), L_I(l) \geq L_I(l_0) = L_I(0)$  and  $L_F(l) \leq L_F(l_0) = L_F(0)$ . It follows from (2) that

$$L_T(l_{01}) \geq \min\{*_L L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \geq L_T(0),$$

$$L_I(l_{01}) \geq \min\{*_L L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \geq L_I(0)$$

and

$$L_F(l_{01}) \leq \max\{*_L L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \leq L_F(0).$$

Hence  $L_T(l_{01}) = L_T(0), L_I(l_{01}) = L_I(0)$  and  $L_F(l_{01}) = L_F(0)$ , that is,  $l_{01} \in J$ . Therefore  $J$  is a  $\mathbb{H}KUI$  of  $H$ . □

We define the situation under which a  $NS L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$ .

**Theorem 3.7.** Let  $H$  satisfy  $|l_{02} \circ l_{01}| < \infty$  for all  $l_{01}, l_{02} \in H$ , and let  $\{J_\beta \mid \beta \in \Lambda \subseteq [0, 0.5]\}$  be a collection of  $\mathbb{H}KUI$ 's of  $H$  such that

$$H = \bigcup_{\beta \in \Lambda} J_\beta, \tag{7}$$

$$(\forall \alpha, \beta \in \Lambda)(\alpha > \beta \Leftrightarrow J_\alpha \subset J_\beta). \tag{8}$$

Then a  $NS L = (L_T, L_I, L_F)$  in  $H$  defined by

$$L_T : H \rightarrow [0, 1], l_{01} \mapsto \sup\{\beta \in \Lambda \mid l_{01} \in J_\beta\},$$

$$L_I : H \rightarrow [0, 1], l_{01} \mapsto \sup\{\beta \in \Lambda \mid l_{01} \in J_\beta\},$$

$$L_F : H \rightarrow [0, 1], l_{01} \mapsto \inf\{\beta \in \Lambda \mid l_{01} \in J_\beta\}$$

is a  $N\mathbb{H}KUI$  of  $H$ .

**Proof.** We first shows that

$$\rho \in [0, 1] \Rightarrow \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta \text{ isa } \mathbb{H}KUI \text{ of } H. \tag{9}$$

It is clear that  $0 \in \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$  for all  $\rho \in [0, 1]$ . Let  $l_{01}, l_{02} \in H$  be such that  $l_{02} \circ l_{01} = \{l_1, l_2, \dots, l_n\}, l_{02} \circ l_{01} \ll \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$  and  $l_{02} \in \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$ . Then  $l_{02} \in J_\gamma$  for some  $\gamma \in \Lambda$  with  $\rho \leq \gamma$ , and for any  $l_i \in l_{02} \circ l_{01}$  there exists  $m_i \in \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$ , and so  $m_i \in J_{\beta_i}$  for some  $\beta_i \in \Lambda$  with  $\rho \leq \beta_i$ , such that  $l_i \ll m_i$ . If we let  $\beta := \min\{\beta_i \mid i \in \{1, 2, \dots, n\}\}$  then  $J_{\beta_i} \subset J_\beta$  for

all  $i \in \{1, 2, \dots, n\}$  and so  $l_{02} \circ l_{01} \ll J_\beta$  with  $\rho \leq \beta$ . We may assume that  $\gamma > \beta$  without loss of generality, and so  $J_\gamma \subset J_\beta$ . By Definition 2.2 (2), we have  $l_{01} \in J_\beta \subset \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$ . Hence

$\bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$  is a  $\mathbb{H}KUI$  of  $H$ . Next, we consider the following two cases:

$$(i) \beta = \sup\{\rho \in \Lambda \mid \rho < \beta\}, (ii) \beta \neq \sup\{\rho \in \Lambda \mid \rho < \beta\}. \tag{10}$$

If the first case is valid, then

$$l_{01} \in U(L_T, \beta) \Leftrightarrow l_{01} \in J_\rho \text{ for all } \rho < \beta \Leftrightarrow l_{01} \in \bigcap_{\rho < \beta} J_\rho,$$

and so  $U(L_T, \beta) = \bigcap_{\rho < \beta} J_\rho$  which is a  $\mathbb{H}KUI$  of  $H$ . Similarly, we know that  $U(L_I, \beta)$  is a  $\mathbb{H}KUI$  of  $H$ . For the second case, we will show that  $U(L_T, \beta) = \bigcup_{\rho \geq \beta} J_\rho$ . If  $l_{01} \in \bigcup_{\rho \geq \beta} J_\rho$ , then  $l_{01} \in J_\rho$  for some  $\rho \geq \beta$ . Thus  $L_T(l_{01}) \geq \rho \geq \beta$ , and so  $l_{01} \in U(L_T, \beta)$  which shows that  $\bigcup_{\rho \geq \beta} J_\rho \subseteq U(L_T, \beta)$ . Assume that  $l_{01} \notin \bigcup_{\rho \geq \beta} J_\rho$ . Then  $l_{01} \notin J_\rho$  for all  $\rho \geq \beta$ , and so there exist  $\delta > 0$  such that  $(\beta - \delta, \beta) \cap \Lambda = \emptyset$ . Thus  $l_{01} \notin J_\rho$  for all  $\rho > \beta - \delta$ , that is, if  $l_{01} \in J_\rho$  then  $\rho \leq \beta - \delta < \beta$ . Hence  $l_{01} \notin U(L_T, \beta)$ . This shows that  $U(L_T, \beta) = \bigcup_{\rho \geq \beta} J_\rho$  which is a  $\mathbb{H}KUI$  of  $H$  by (9). Similarly we can prove that  $U(L_I, \beta)$  is a  $\mathbb{H}KUI$  of  $H$ . Now we consider the following two cases:

$$\alpha = \inf\{\gamma \in \Lambda \mid \alpha < \gamma\} \text{ and } \alpha \neq \inf\{\gamma \in \Lambda \mid \alpha < \gamma\}. \tag{11}$$

The first case implies that

$$l_{01} \in L(L_F, \alpha) \Leftrightarrow l_{01} \in J_\gamma \text{ for all } \alpha < \gamma \Leftrightarrow l_{01} \in \bigcap_{\alpha < \gamma} J_\gamma,$$

and so  $L(L_F, \alpha) = \bigcap_{\alpha < \gamma} J_\gamma$  which is a  $\mathbb{H}KUI$  of  $H$ . For the second case, there exists  $\delta > 0$  such that  $(\alpha, \alpha + \delta) \cap \Lambda = \emptyset$ . If  $l_{01} \in \bigcup_{\alpha \geq \gamma} J_\gamma$ , then  $l_{01} \in J_\gamma$  for some  $\alpha \geq \gamma$ . Thus  $L_F(l_{01}) \leq \gamma \leq \alpha$ , that is,  $l_{01} \in L(L_F, \alpha)$ . Hence  $\bigcup_{\alpha \geq \gamma} J_\gamma \subseteq L(L_F, \alpha)$ . If  $l_{01} \notin \bigcup_{\alpha \geq \gamma} J_\gamma$ , then  $l_{01} \notin J_\gamma$  for all  $\gamma \leq \alpha$  and thus  $l_{01} \notin J_\gamma$  for all  $\alpha\gamma < \alpha + \delta$ . This shows that if  $l_{01} \in J_\gamma$  then  $\gamma \geq \alpha + \delta$ . Hence  $L_F(l_{01}) \geq \alpha + \delta > \alpha$ , i.e.,  $l_{01} \notin L(L_F, \alpha)$ .

Therefore  $L(L_F, \alpha) \subseteq \bigcup_{\alpha \geq \gamma} J_\gamma$ . Consequently,  $L(L_F, \alpha) = \bigcup_{\alpha \geq \gamma} J_\gamma$  which is a  $\mathbb{H}KUI$  of  $H$  by (9). It follows from Theorem 3.5 that  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{H}KUI$  of  $H$ . \(\Xi\)

**Definition 3.8.** A  $NS L = (L_T, L_I, L_F)$  in  $H$  is called a neutrosophic strong hyper  $KU$ -ideal (briefly,  $NS\mathbb{H}KUI$ ) of  $H$  if it satisfies the following assertions.

$$\begin{aligned} *L_T(l_{01} \circ l_{01}) &\geq L_T(l_{01}) \geq \min\left\{ \sup_{u_0 \in l_{02} \circ l_{01}} L_T(u_0), L_T(l_{02}) \right\}, \\ *L_I(l_{01} \circ l_{01}) &\geq L_I(l_{01}) \geq \min\left\{ \sup_{v_0 \in l_{02} \circ l_{01}} L_I(v_0), L_I(l_{02}) \right\} \\ *L_F(l_{01} \circ l_{01}) &\leq L_F(l_{01}) \leq \max\left\{ \inf_{w_0 \in l_{02} \circ l_{01}} L_F(w_0), L_F(l_{02}) \right\} \end{aligned} \tag{12}$$

for all  $l_{01}, l_{02} \in H$ .

**Example 3.9.** Consider a hyper  $KU$ -algebra  $H = \{l_0, l_a, l_b\}$  with the hyper operation “ $\circ$ ” which is given by Table 3.

**Table 3 :** Cayley table for the binary operation “ $\circ$ ”

$\circ$	$l_0$	$l_a$	$l_b$
$l_0$	$\{l_0\}$	$\{l_a\}$	$\{l_b\}$
$l_a$	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_b\}$
$l_b$	$\{l_0\}$	$\{l_b\}$	$\{l_0, l_b\}$

Let  $L = (L_T, L_I, L_F)$  be a  $NS$  in  $H$  which is described in Table 4.

**Table 4 :** Tabular representation of  $L = (L_T, L_I, L_F)$

$H$	$L_T(l)$	$L_I(l)$	$L_F(l)$
$l_0$	0.77	0.65	0.08
$l_a$	0.55	0.47	0.57
$l_b$	0.11	0.27	0.69

It is routine to verify that  $L = (L_T, L_I, L_F)$  is a  $NSHKUI$  of  $H$ .

**Theorem 3.10.** For any  $NSHKUI$   $L = (L_T, L_I, L_F)$  of  $H$ , the following assertions are valid.

- (i)  $L = (L_T, L_I, L_F)$  satisfies the conditions (1) and (3).
- (ii)  $L = (L_T, L_I, L_F)$  satisfies

$$(\forall l_{01}, l_{02} \in H)(\forall u, v, w \in l_{02} \circ l_{01}) \left( \begin{array}{l} L_T(l_{01}) \geq \min\{L_T(u), L_T(l_{02})\} \\ L_I(l_{01}) \geq \min\{L_I(v), L_I(l_{02})\} \\ L_F(l_{01}) \leq \max\{L_F(w), L_F(l_{02})\} \end{array} \right). \quad (13)$$

**Proof.** (i) Since  $l_{01} \ll l_{01}$ , i.e.,  $0 \in l_{01} \circ l_{01}$  for all  $l_{01} \in H$ , we get

$$\begin{aligned} L_T(0) &\geq *L_T(l_{01} \circ l_{01}) \geq L_T(l_{01}), \\ L_I(0) &\geq *L_I(l_{01} \circ l_{01}) \geq L_I(l_{01}), \\ L_F(0) &\leq *L_F(l_{01} \circ l_{01}) \leq L_F(l_{01}), \end{aligned}$$

which shows that (3) is valid. Let  $l_{01}, l_{02} \in H$  be such that  $l_{01} \ll l_{02}$ . Then  $0 \in l_{02} \circ l_{01}$ , and so

$$*L_T(l_{02} \circ l_{01}) \geq L_T(0), *L_I(l_{02} \circ l_{01}) \geq L_I(0) \text{ and } *L_F(l_{02} \circ l_{01}) \leq L_F(0).$$

It follows from (3) that

$$\begin{aligned} L_T(l_{01}) &\geq \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \geq \min\{L_T(0), L_T(l_{02})\} = L_T(l_{02}), \\ L_I(l_{01}) &\geq \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \geq \min\{L_I(0), L_I(l_{02})\} = L_I(l_{02}), \\ L_F(l_{01}) &\leq \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \leq \max\{L_F(0), L_F(l_{02})\} = L_F(l_{02}). \end{aligned}$$

Hence  $L = (L_T, L_I, L_F)$  satisfies the condition (1).

(ii) Let  $l_{01}, l_{02}, u, v, w \in H$  be such that  $u, v, w \in l_{02} \circ l_{01}$ . Then

$$L_T(l_{01}) \geq \min\left\{ \sup_{u_0 \in l_{02} \circ l_{01}} L_T(u_0), L_T(l_{02}) \right\} \geq \min\{L_T(u), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \min\left\{ \sup_{v_0 \in l_{02} \circ l_{01}} L_I(v_0), L_I(l_{02}) \right\} \geq \min\{L_I(v), L_I(l_{02})\},$$

$$L_F(l_{01}) \leq \max\left\{ \inf_{c_0 \in l_{02} \circ l_{01}} L_F(w_0), L_F(l_{02}) \right\} \leq \max\{L_F(w), L_F(l_{02})\}.$$

This completes the proof. □

**Theorem 3.11.** If a NS  $L = (L_T, L_I, L_F)$  is a NSHKUI of  $H$ , then the nonempty sets  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are SHKUI's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ .

**Proof.** Let  $L = (L_T, L_I, L_F)$  be a NSHKUI of  $H$ . Then  $L = (L_T, L_I, L_F)$  is a NHKUI of  $H$ . Assume that  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are nonempty for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ . Then there exist  $a \in U(L_T, \xi_T), b \in U(L_I, \xi_I)$  and  $c \in L(L_F, \xi_F)$ , that is,  $L_T(a) \geq \xi_T, L_I(b) \geq \xi_I$  and  $L_F(c) \leq \xi_F$ . It follows from (3) that  $L_T(0) \geq L_T(a) \geq \xi_T, L_I(0) \geq L_I(b) \geq \xi_I$  and  $L_F(0) \leq L_F(c) \leq \xi_F$ . Hence

$$0 \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F).$$

Let  $l_{01}, l_{02}, a, b, u, v \in H$  be such that  $(l_{02} \circ l_{01}) \cap U(L_T, \xi_T) \neq \emptyset, l_{02} \in U(L_T, \xi_T), (b \circ a) \cap U(L_I, \xi_I) \neq \emptyset, b \in U(L_I, \xi_I), (v \circ u) \cap L(L_F, \xi_F) \neq \emptyset$  and  $v \in L(L_F, \xi_F)$ . Then there exist  $x_0 \in (l_{02} \circ l_{01}) \cap U(L_T, \xi_T), a_0 \in (b \circ a) \cap U(L_I, \xi_I)$  and  $u_0 \in (v \circ u) \cap L(L_F, \xi_F)$ . It follows that

$$L_T(l_{01}) \geq \min\left\{ \sup_{c \in l_{02} \circ l_{01}} L_T(c), L_T(l_{02}) \right\} \geq \min\{L_T(x_0), L_T(l_{02})\} \geq \xi_T,$$

$$L_I(a) \geq \min\left\{ \sup_{d \in b \circ a} L_I(d), L_I(b) \right\} \geq \min\{L_I(a_0), L_I(b)\} \geq \xi_I$$

and

$$L_F(u) \leq \max\left\{ \inf_{e \in v \circ u} L_F(e), L_F(v) \right\} \leq \max\{L_F(u_0), L_F(v)\} \leq \xi_F.$$

Hence  $l_{01} \in U(L_T, \xi_T), a \in U(L_I, \xi_I)$  and  $u \in L(L_F, \xi_F)$ . Therefore  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are SHKUI of  $H$ . □

**Theorem 3.12.** For any NS  $L = (L_T, L_I, L_F)$  in  $H$  satisfying the condition

$$(\forall S \subseteq H)(\exists u, v, w \in S) \begin{pmatrix} L_T(u) = {}^*L_T(V) \\ L_I(v) = {}^*L_I(V) \\ L_F(w) = {}^*L_F(V) \end{pmatrix}, \tag{14}$$

if the nonempty sets  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are SHKUI's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ , then  $L = (L_T, L_I, L_F)$  is a NSHKUI of  $H$ .

**Proof.** Assume that  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are nonempty and  $SHKUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ . For any  $l_{01}, l_{02}, l_{03} \in H$ , such that  $l_{01} \in U(L_T, L_T(l_{01})), l_{02} \in U(L_I, L_I(l_{02}))$  and  $l_{03} \in L(L_F, L_F(l_{03}))$ , since  $l_{01} \circ l_{01} \ll l_{01}, l_{02} \circ l_{02} \ll l_{02}$  and  $l_{03} \circ l_{03} \ll l_{03}$  by Proposition 2.1 (v), we have  $l_{01} \circ l_{01} \ll U(L_T, L_T(l_{01})), l_{02} \circ l_{02} \ll U(L_I, L_I(l_{02}))$  and  $l_{03} \circ l_{03} \ll L(L_F, L_F(l_{03}))$ . By Lemma 3.4,  $l_{01} \circ l_{01} \subseteq U(L_T, L_T(l_{01})), l_{02} \circ l_{02} \subseteq U(L_I, L_I(l_{02}))$  and  $l_{03} \circ l_{03} \subseteq L(L_F, L_F(l_{03}))$ . Hence  $a \in U(L_T, L_T(l_{01})), b \in U(L_I, L_I(l_{02}))$  and  $c \in L(L_F, L_F(l_{03}))$  for all  $a \in l_{01} \circ l_{01}, b \in l_{02} \circ l_{02}$  and  $c \in l_{03} \circ l_{03}$ . Therefore  $*L_T(l_{01} \circ l_{01}) \geq L_T(l_{01}), *L_I(l_{02} \circ l_{02}) \geq L_I(l_{02})$  and  $*L_F(l_{03} \circ l_{03}) \leq L_F(l_{03})$ . Now, let  $\xi_T := \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\}, \xi_I := \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$  and  $\xi_F := \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}$ . By (14), we have

$$L_T(a_0) = *L_T(l_{02} \circ l_{01}) \geq \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \xi_T,$$

$$L_I(b_0) = *L_I(l_{02} \circ l_{01}) \geq \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \xi_I$$

and

$$L_F(c_0) = *L_F(l_{02} \circ l_{01}) \leq \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \xi_F$$

for some  $a_0, b_0, c_0 \in l_{02} \circ l_{01}$ . Hence  $a_0 \in U(L_T, \xi_T), b_0 \in U(L_I, \xi_I)$  and  $c_0 \in L(L_F, \xi_F)$  which imply that

$$(l_{02} \circ l_{01}) \cap U(L_T, \xi_T), (l_{02} \circ l_{01}) \cap U(L_I, \xi_I) \text{ and } (l_{02} \circ l_{01}) \cap L(L_F, \xi_F)$$

are nonempty. Since  $l_{02} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F)$ , it follows from Definition 2.2 (4) that  $l_{01} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F)$ . Thus

$$L_T(l_{01}) \geq \xi_T = \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \xi_I = \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$$

and

$$L_F(l_{01}) \leq \xi_F = \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Consequently,  $L = (L_T, L_I, L_F)$  is a  $NSH KUI$  of  $H$ . Ξ

We have the following corollary in a finite hyper  $KU$ -algebra since any  $NS L = (L_T, L_I, L_F)$  satisfies the condition (14).

**Corollary 3.13.** Let  $L = (L_T, L_I, L_F)$  be a  $NS$  in a finite hyper  $KU$ -algebra  $H$ . Then  $L = (L_T, L_I, L_F)$  is a  $NSH KUI$  of  $H$  iff the nonempty sets  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $SHKUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ .

**Definition 3.14.** A *NS*  $L = (L_T, L_I, L_F)$  in  $H$  is called a neutrosophic weak hyper *KU*-ideal (briefly, *NWHKUI*) of  $H$  if it satisfies the following assertions.

$$\begin{aligned}
 L_T(0) &\geq L_T(l_{01}) \geq \min\{*_L L_T(l_{02} \circ l_{01}), L_T(l_{02})\}, \\
 L_I(0) &\geq L_I(l_{01}) \geq \min\{*_L L_I(l_{02} \circ l_{01}), L_I(l_{02})\}, \\
 L_F(0) &\leq L_F(l_{01}) \leq \max\{*_L L_F(l_{02} \circ l_{01}), L_F(l_{02})\}
 \end{aligned}
 \tag{15}$$

for all  $l_{01}, l_{02} \in H$ .

**Definition 3.15.** A *NS*  $L = (L_T, L_I, L_F)$  in  $H$  is called a neutrosophic *s*-weak hyper *KU*-ideal (briefly, *NsWCHKUI*) of  $H$  if it satisfies the conditions (3) and (5).

**Example 3.16.** Consider a hyper *KU*-algebra  $H = \{l_0, l_a, l_b\}$  with the hyper operation “ $\circ$ ” which is given by Table 5.

**Table 5 :** Cayley table for the binary operation “ $\circ$ ”

$\circ$	$l_0$	$l_a$	$l_b$
$l_0$	$\{l_0\}$	$\{l_a\}$	$\{l_b\}$
$l_a$	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_a, l_b\}$
$l_b$	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_0, l_a, l_b\}$

Let  $L = (L_T, L_I, L_F)$  be a *NS* in  $H$  which is described in Table 6.

**Table 6 :** Tabular representation of  $L = (L_T, L_I, L_F)$

$H$	$L_T(l)$	$L_I(l)$	$L_F(l)$
$l_0$	0.77	0.65	0.08
$l_a$	0.55	0.47	0.57
$l_b$	0.11	0.27	0.69

It is routine to verify that  $L = (L_T, L_I, L_F)$  is a *NWHKUI* of  $H$ .

**Theorem 3.17.** Every *NsWCHKUI* is a *NWHKUI*.

**Proof.** Let  $L = (L_T, L_I, L_F)$  be a *NsWCHKUI* of  $H$  and let  $l_{01}, l_{02} \in H$ . Then there exist  $u, v, w \in l_{02} \circ l_{01}$  such that

$$\begin{aligned}
 L_T(l_{01}) &\geq \min\{L_T(u), L_T(l_{02})\} \geq \min\left\{\inf_{u_0 \in l_{02} \circ l_{01}} L_T(u_0), L_T(l_{02})\right\}, \\
 L_I(l_{01}) &\geq \min\{L_I(v), L_I(l_{02})\} \geq \min\left\{\inf_{v_0 \in l_{02} \circ l_{01}} L_I(v_0), L_I(l_{02})\right\}, \\
 L_F(l_{01}) &\leq \max\{L_F(w), L_F(l_{02})\} \leq \max\left\{\sup_{w_0 \in l_{02} \circ l_{01}} L_F(w_0), L_F(l_{02})\right\}.
 \end{aligned}$$

Hence  $L = (L_T, L_I, L_F)$  is a *NWHKUI* of  $H$ . □

The corollary of Theorem 3.17, we can assume, is not true. However, finding an example of a *NWHKUI* that is not a *NsWCHKUI* is difficult. Now we specify that a *NWHKUI* must also be a *NsWCHKUI*.

**Theorem 3.18.** If  $L = (L_T, L_I, L_F)$  is a  $N\mathbb{W}\mathbb{H}KUI$  of  $H$  which satisfies the condition (4), then  $L = (L_T, L_I, L_F)$  is a  $Ns\mathbb{W}\mathbb{H}KUI$  of  $H$ .

**Proof.** Let  $L = (L_T, L_I, L_F)$  be a  $N\mathbb{W}\mathbb{H}KUI$  of  $H$  in which the condition (4) is true. Then there exist  $u_0, v_0, c_0 \in l_{02} \circ l_{01}$  such that  $L_T(u_0) = *L_T(l_{02} \circ l_{01}), L_I(v_0) = *L_I(l_{02} \circ l_{01})$  and  $L_F(w_0) = *L_F(l_{02} \circ l_{01})$ . Hence

$$\begin{aligned} L_T(l_{01}) &\geq \min\{ *L_T(l_{02} \circ l_{01}), L_T(l_{02}) \} = \min\{ L_T(u_0), L_T(l_{02}) \}, \\ L_I(l_{01}) &\geq \min\{ *L_I(l_{02} \circ l_{01}), L_I(l_{02}) \} = \min\{ L_I(v_0), L_I(l_{02}) \}, \\ L_F(l_{01}) &\leq \max\{ *L_F(l_{02} \circ l_{01}), L_F(l_{02}) \} = \max\{ L_F(w_0), L_F(l_{02}) \}. \end{aligned}$$

Therefore  $L = (L_T, L_I, L_F)$  is a  $Ns\mathbb{W}\mathbb{H}KUI$  of  $H$ . ≡

**Remark 3.19.** In a finite hyper  $KU$ -algebra, every  $NS$  satisfies the condition (4). Hence the concept of  $Ns\mathbb{W}\mathbb{H}KUI$  and  $N\mathbb{W}\mathbb{H}KUI$  coincide in a finite hyper  $KU$ -algebra.

**Theorem 3.20.** A  $NS$   $L = (L_T, L_I, L_F)$  is a  $N\mathbb{W}\mathbb{H}KUI$  of  $H$  iff the nonempty sets  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $\mathbb{W}\mathbb{H}KUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ .

**Proof.** The proof is comparable to Theorem 3.5's proof. ≡

**Definition 3.21.** A  $NS$  in  $H$  is called a reflexive neutrosophic hyper  $KU$ -ideal (briefly,  $RN\mathbb{H}KUI$ ) of  $H$  if it satisfies

$$(\forall l_{01}, l_{02} \in H) \left( \begin{array}{l} *L_T(l_{01} \circ l_{01}) \geq L_T(l_{02}) \\ *L_I(l_{01} \circ l_{01}) \geq L_I(l_{02}) \\ *L_F(l_{01} \circ l_{01}) \leq L_F(l_{02}) \end{array} \right), \tag{16}$$

and

$$(\forall l_{01}, l_{02} \in H) \left( \begin{array}{l} L_T(l_{01}) \geq \min\{ *L_I(l_{02} \circ l_{01}), L_I(l_{02}) \} \\ L_I(l_{01}) \geq \min\{ *L_T(l_{02} \circ l_{01}), L_T(l_{02}) \} \\ L_F(l_{01}) \leq \max\{ *L_F(l_{02} \circ l_{01}), L_F(l_{02}) \} \end{array} \right). \tag{17}$$

**Theorem 3.22.** Every  $RN\mathbb{H}KUI$  is a  $NS\mathbb{H}KUI$ .

**Proof.** Straightforward. ≡

**Theorem 3.23.** If  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$ , then the nonempty sets  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $R\mathbb{H}KUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ .

**Proof.** Assume that  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are nonempty for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ . Let  $a \in U(L_T, \xi_T), b \in U(L_I, \xi_I)$  and  $c \in L(L_F, \xi_F)$ . If  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$ , then by Theorem 3.22,  $L = (L_T, L_I, L_F)$  is a  $NS\mathbb{H}KUI$  of  $H$ , and so it is a  $N\mathbb{H}KUI$  of

$H$ . It follows from Theorem 3.5 that  $U(L_T, \xi_T)$ ,  $U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $\mathbb{H}KUI$ 's of  $H$ . For each  $l_{01} \in H$ , let  $a_0, b_0, c_0 \in l_{01} \circ l_{01}$ . Then

$$\begin{aligned} L_T(a_0) &\geq \inf_{u \in l_{01} \circ l_{01}} L_T(u) \geq L_T(a) \geq \xi_T, \\ L_I(b_0) &\geq \inf_{v \in l_{01} \circ l_{01}} L_I(v) \geq L_I(b) \geq \xi_I, \\ L_F(c_0) &\leq \sup_{w \in l_{01} \circ l_{01}} L_F(w) \leq L_F(c) \leq \xi_F, \end{aligned}$$

and so  $a_0 \in U(L_T, \xi_T)$ ,  $b_0 \in U(L_I, \xi_I)$  and  $c_0 \in L(L_F, \xi_F)$ . Hence  $l_{01} \circ l_{01} \subseteq U(L_T, \xi_T)$ ,  $l_{01} \circ l_{01} \subseteq U(L_I, \xi_I)$  and  $l_{01} \circ l_{01} \subseteq L(L_F, \xi_F)$ . Therefore  $U(L_T, \xi_T)$ ,  $U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $R\mathbb{H}KUI$ 's of  $H$ . □

**Lemma 3.24.** Every  $R\mathbb{H}KUI$  is a  $S\mathbb{H}KUI$ .

By adding a condition, we explore the converse of Theorem 3.23.

**Theorem 3.25.** Let  $L = (L_T, L_I, L_F)$  be a  $NS$  in  $H$  satisfying the condition (14). If the nonempty sets  $U(L_T, \xi_T)$ ,  $U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $R\mathbb{H}KUI$ 's of  $H$  for all  $\xi_T, \xi_I, \xi_F \in [0, 1]$ , then  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$ .

**Proof.** If the nonempty sets  $U(L_T, \xi_T)$ ,  $U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $R\mathbb{H}KUI$ 's of  $H$ , then by Lemma 3.24 they are  $S\mathbb{H}KUI$ 's of  $H$ . By Theorem 3.12 that  $L = (L_T, L_I, L_F)$  is a  $NS\mathbb{H}KUI$  of  $H$ . Hence the condition (17) is valid. Let  $l_{01}, l_{02} \in H$ . Then the sets  $U(L_T, L_T(l_{02}))$ ,  $U(L_I, L_I(l_{02}))$  and  $L(L_F, L_F(l_{02}))$  are  $R\mathbb{H}KUI$ 's of  $H$ , and so  $l_{01} \circ l_{01} \subseteq U(L_T, L_T(l_{02}))$ ,  $l_{01} \circ l_{01} \subseteq U(L_I, L_I(l_{02}))$  and  $l_{01} \circ l_{01} \subseteq L(L_F, L_F(l_{02}))$ . Hence  $L_T(u) \geq L_T(l_{02})$ ,  $L_I(v) \geq L_I(l_{02})$  and  $L_F(w) \leq L_F(l_{02})$  for all  $u, v, w \in l_{01} \circ l_{01}$  and so  $*L_T(l_{01} \circ l_{01}) \geq L_T(l_{02})$ ,  $*L_I(l_{01} \circ l_{01}) \geq L_I(l_{02})$  and  $*L_F(l_{01} \circ l_{01}) \leq L_F(l_{02})$ . Therefore  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$ . □

We provide conditions for a  $NS\mathbb{H}KUI$  to be a  $RN\mathbb{H}KUI$ .

**Theorem 3.26.** Let  $L = (L_T, L_I, L_F)$  be a  $NS\mathbb{H}KUI$  of  $H$  which satisfies the condition (14). Then  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$  iff the following assertion is valid.

$$(\forall l_{01} \in H) \left( \begin{array}{l} *L_T(l_{01} \circ l_{01}) \geq L_T(0) \\ *L_I(l_{01} \circ l_{01}) \geq L_I(0) \\ *L_F(l_{01} \circ l_{01}) \leq L_F(0) \end{array} \right). \tag{18}$$

**Proof.** It is clear that if  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$ , then the condition (18) is valid.

Conversely, assume that  $L = (L_T, L_I, L_F)$  is a  $NS\mathbb{H}KUI$  of  $H$  which satisfies the conditions (14) and (18). Then  $L_T(0) \geq L_T(l_{02})$ ,  $L_I(0) \geq L_I(l_{02})$  and  $L_F(0) \leq L_F(l_{02})$  for all  $l_{02} \in H$ . Hence

$$*L_T(l_{01} \circ l_{01}) \geq L_T(l_{02}), *L_I(l_{01} \circ l_{01}) \geq L_I(l_{02}) \text{ and } *L_F(l_{01} \circ l_{01}) \leq L_F(l_{02}).$$

For any  $l_{01}, l_{02} \in H$ , let

$$\xi_T := \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$\xi_I := \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\},$$

$$\xi_F := \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Then  $U(L_T, \xi_T), U(L_I, \xi_I)$  and  $L(L_F, \xi_F)$  are  $S\mathbb{H}KUI$ 's of  $H$  by Theorem 3.11. Since  $L = (L_T, L_I, L_F)$  satisfies the condition (14), there exist  $u_0, v_0, w_0 \in l_{02} \circ l_{01}$  such that

$$L_T(u_0) = *L_T(l_{02} \circ l_{01}), L_I(v_0) = *L_I(l_{02} \circ l_{01}), L_F(w_0) = *L_F(l_{02} \circ l_{01}).$$

Hence  $L_T(u_0) \geq \xi_T, L_I(v_0) \geq \xi_I$  and  $L_F(w_0) \leq \xi_F$ , that is,  $u_0 \in U(L_T, \xi_T), v_0 \in U(L_I, \xi_I)$  and  $w_0 \in L(L_F, \xi_F)$ . Hence  $(l_{02} \circ l_{01}) \cap U(L_T, \xi_T) \neq \emptyset, (l_{02} \circ l_{01}) \cap U(L_I, \xi_I) \neq \emptyset$  and  $(l_{02} \circ l_{01}) \cap L(L_F, \xi_F) \neq \emptyset$ . Since  $l_{02} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F)$ , by Definition 2.2 (4),  $l_{01} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F)$ . Thus

$$L_T(l_{01}) \geq \xi_T = \min\{*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \xi_I = \min\{*L_I(l_{02} \circ l_{01}), L_I(l_{02})\},$$

$$L_F(l_{01}) \leq \xi_F = \max\{*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Therefore  $L = (L_T, L_I, L_F)$  is a  $RN\mathbb{H}KUI$  of  $H$ . □

## Conclusions

We have introduced the notions of  $NS\mathbb{H}KUI$ ,  $NW\mathbb{H}KUI$ ,  $NsW\mathbb{H}KUI$  and  $RN\mathbb{H}KUI$ . We have considered their relations and related properties. We have discussed characterizations of  $N\mathbb{H}KUI$  and  $NW\mathbb{H}KUI$  have given conditions for a  $NS$  to be a  $RN\mathbb{H}KUI$  and  $NS\mathbb{H}KUI$ . We have provided conditions for a  $NW\mathbb{H}KUI$  to be a  $NsW\mathbb{H}KUI$  and have provided conditions for a  $NS\mathbb{H}KUI$  to be a  $RN\mathbb{H}KUI$ .

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