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Neutrosophic Quadruple Metric Spaces and Neutrosophic Quadruple Normed Spaces

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Abstract: This study establishes fundamental results in the emerging domains of neutrosophic quadruple metric spaces and neutrosophic quadruple normed spaces. Building upon the recent definition of neutrosophic quadruple metric spaces, the first primary contribution is the development of fixed-point theory within this generalized framework. Specifically, we formulate and rigorously prove an analogue of the Banach contraction principle tailored for neutrosophic quadruple metric spaces. Furthermore, we establish additional fixed-point theorems applicable in this context, extending foundational results from classical metric spaces and simpler neutrosophic structures to handle the increased complexity and uncertainty modeled by quadruple-valued neutrosophic sets. The second major contribution involves the algebraic generalization of normed spaces. We introduce the novel concept of neutrosophic quadruple normed spaces. Within these spaces, we define an appropriate norm structure capable of measuring the "magnitude" of vectors characterized by quadruple-valued neutrosophic components. We systematically investigate and establish various fundamental properties of this newly defined norm, exploring concepts such as statistical convergence, Cauchy statistical convergence, boundedness, and continuity within the neutrosophic quadruple normed spaces. Additionally, we address the specific case of neutrosophic quadruple vector spaces, defining and analyzing the corresponding norm structure in this specialized context. The results generalize and extend prior work in fuzzy, intuitionistic fuzzy, and standard neutrosophic metric and normed spaces.

Keywords: fixed points for neutrosophic quadruple metric space; neutrosophic quadruple normed space; statistical convergence to the neutrosophic quadruple norm.

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1. Introduction

Fixed point theory stands as a cornerstone of nonlinear functional analysis, witnessing remarkable growth in both theoretical depth and practical application. Its significance extends far beyond the boundaries of pure mathematics, permeating diverse fields such as differential equations, optimization theory, game theory, economics, computer science, and engineering. This widespread utility has naturally spurred intense research activity, leading to the development of numerous fixed-point theorems tailored to increasingly sophisticated mathematical structures.

The foundation of much of this work lies in the theory of metric spaces. Seminal contributions by T. Zamfirescu [1] established crucial fixed-point results within this classical setting. As mathematical frameworks evolved to model greater complexity and uncertainty, researchers explored generalizations of metric spaces. This led to the investigation of fixed-point theorems in fuzzy metric (shortly FM) spaces [2], capturing vagueness through membership degrees, and subsequently in intuitionistic FM spaces [3], which incorporate both membership and non-membership. More recently, the advent of neutrosophic metric spaces [8], built upon the neutrosophic logic principle of characterizing truth, falsehood, and indeterminacy simultaneously, provided a richer framework for handling incomplete, inconsistent, and uncertain information, yielding novel fixed-point results.

The ongoing quest for structures capable of representing higher-dimensional uncertainty culminated in the definition of neutrosophic quadruple metric (shortly NQM) spaces [6]. This innovative framework, extending the representational capacity of neutrosophic sets to quadruples, immediately presented the compelling challenge of establishing fixed point theorems within this new context. Addressing this gap forms a primary objective of the present study. Consequently, the first part of this paper is dedicated to exploring contraction principles and fixed-point theorems specifically within NQM spaces. We establish and prove an analogue of the fundamental Banach contraction principle in this setting and investigate other relevant fixed-point results.

Parallel to the development of metric structures in uncertain environments, the concept of normed vector spaces has also been generalized. The norm, fundamentally quantifying the "length" or magnitude of a vector, is essential for analysis in vector spaces. Significant efforts have been made to define appropriate norms for vectors residing in fuzzy [11], intuitionistic fuzzy [12], and neutrosophic vector spaces, enabling the study of functional analysis in these generalized settings. In studies [14] and [15], previous studies have examined statistical convergence in standard normed spaces and intuitionistic fuzzy normed spaces. In this study, we generalize this concept to neutrosophic quadruple normed spaces. [16] Deli et al. defined n-valued neutrosophic trapezoidal numbers with similarity measures and its properties. [17] Sahin et al. investigated the extension principles of neutrosophic multi-sets and cut sets and algebraic operators. [18] Ulucay defined similarity function of trapezoidal fuzzy multi-numbers. [19] Bakbak and Ulucay analyzed Q-neutrosophic soft expert multiset and its set operations (like union, intersection,complement,subset). [20] Baser and Ulucay studied an application of neutrosophic soft sets and its properties. [21] Baser and Ulucay investigated effective Q-neutrosophic soft expert sets on an application. No prior studies address properties such as metric, continuity, and completeness in NQS. Various applications of neutrosophic sets are discussed in this essay [22-37].

Building upon this progression, the second part of our work introduces the concept of neutrosophic quadruple normed (shortly NQN) spaces. We define a suitable norm structure for vector spaces where the vectors themselves possess neutrosophic quadruple characteristics. Furthermore, we extend this concept specifically to neutrosophic quadruple vector spaces, defining and examining the properties of the norm within this specialized context.

This study thus contributes to two expanding frontiers: establishing fundamental fixed-point results in the novel setting of NQM spaces and initiating the development of norm theory for neutrosophic quadruple sets.

2. Preliminaries

Definition 2.1 [4] Let $\rhd : [0,1] \times [0,1] \to [0,1]$ be binary operation. Then $\rhd$ is called a t-norm (or TN) if it satisfies the following properties for all elements $m, n, r, s \in [0,1]$:

I. Identity: $m \rhd 1 = \mathrm{m}$.

II. Monotonicity: If $m \leq n$ and $r \leq s$, then $m \rhd r \leq n \rhd s$.

III. Symmetry: $m \rhd n = n \rhd m$.

IV. Associativity: $m \rhd (n \rhd r) = (m \rhd n) \rhd r$.

V. Continuity: The operation $\rhd$ is continous.

Definition 2.2 [5] Let $\blacktriangleright: [0,1] \times [0,1] \rightarrow [0,1]$ be binary operation. Then $\blacktriangleright$ is called a t-conorm (or TC) if it satisfies the following proporties for all elements $m, n, r, s \in [0,1]$:

I. Identity: $m \blacktriangleright 1 = m$.

II. Monotonicity: If $m \leq n$ and $r \leq s$, then $m \blacktriangleright r \leq n \blacktriangleright s$.

III. Symmetry: $m \blacktriangleright n = n \blacktriangleright m$.

IV. Associativity: $m \blacktriangleright (n \blacktriangleright r) = (m \blacktriangleright n) \blacktriangleright r$.

V. Continuity: The operation $\blacktriangleright$ is continous.

Definition 2.3 [6] $x, y, z, k \in \mathbb{R}\ or\ \mathbb{C}$, and let T represent a truth-membership degree, I an indeterminacy-membership degree, and F a falsity-membership degree. A neutrosophic quadruple set element D is defined as:

$$D = x + yT + zI + kF$$

Here, x is called the known segment of D, while the expression $yT + zI + kF$ constitutes its unknown segment. Alternatively, the quadruple number D can be represented as an ordered tuple:

$$D = (x, yT, zI, kF)$$

$NQ = \{(x, yT, zI, kF) | x, y, z, k \in \mathbb{R}(alternatively\ \mathbb{C})\}$ is neutrosophic quadruple set.

Definition 2.4 [7] Let X be a non-empty set, $x, y \in X$, and (H, J, K) be a neutrosophic metric on $X \times X \times (0, \infty)$ where $H, J, K: X \times X \times (0, \infty) \rightarrow [0,1]$ represent truth, indeterminacy, and falsity degrees respectively. Let $d: X \times X \rightarrow [0, \infty)$ be a metric, $\varepsilon, a, b, c > 0$ be positive constants, and $' \rhd ', ' \blacktriangleright '$ denote a t-norm (TN) and t-conorm (TC) respectively. The neutrosophic quadruple metric (NQM) $M: X \times X \rightarrow \mathbb{R} \times \mathbb{R}^3$ is defined as:

$$M(x, y) = \{(d(x, y), aH(x, y; \varepsilon), bJ(x, y; \varepsilon), cK(x, y; \varepsilon): x, y \in X\}.$$

Definition 2.5 [8] Let (H,J,K) be a NM on X. The mapping $T:X \to X$ is defined neutrosophic contraction (NC) if there exists $c \in (0,1)$ such that

$$H(T(x),T(y),\varepsilon) \geq cH(x,y,\varepsilon), J(T(x),T(y),\varepsilon) \leq cJ(x,y,\varepsilon),$$

$$K(T(x),T(y),\varepsilon) \leq cK(x,y,\varepsilon)$$

for each $x,y \in X$ and $\varepsilon > 0$.

Definition 2.6 [8] Let (H,J,K) be a NM on X and let $T:X \to X$ be a NC mapping. Then there exists $u \in U$ such that $k = T(k)$. That is, k is called neutrosophic fixed point (NFP) of T.

Theorem 2.7 [10] Let *K* denote one of the following fields: $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ (where *p* is a prime). Define the neutrosophic quadruple group as the set

$$NQ = \{(a, bT, cI, dF) | a, b, c, d \in K\}$$

equipped with addition (+). Then $V = (NQ, +, \bullet)$ forms a neutrosophic quadruple vector space over *K*, where the operation ·· (scalar multiplication) is a specially defined bilinear map from $K \times NQ$ to *NQ*.

Definition 2.8 [13] Let *Q* be a vector space over a field *K* (typically $\mathbb{R}$ or $\mathbb{C}$). Let $N: Q \times \mathbb{R}^{+} \to [0,1]^{3}$ Be a mapping, denoted for each $p \in Q$ and $t > 0$ by

$N = \{(p, V(p,t), Y(p,t), Z(p,t))\}: p \in Q\}$, where $V(p,t)$, $Y(p,t)$, $Z(p,t)$ represent the degrees of truth, indeterminacy, and falsity of the assertion "the norm of *p* is less than or equal to *t*", respectively. Let $\triangleright$ be a t-norm and $\blacktriangleright$ be a t-conorm. The 4-tuple $(Q, N, \triangleright, \blacktriangleright)$ is called a neutrosophic normed space if for all $p, q \in Q$ and all positive real numbers k, s, t the following conditions hold:

I. $0 \leq V(p,t) \leq 1, 0 \leq Y(p,t) \leq 1, 0 \leq Z(p,) \leq 1,$

II. $V(p,t) + Y(p,t) + Z(p,t) \leq 3,$

III. $V(p,t) = 1$ if and only if $p = 0$,

IV. $V(sp,t) = H\left(p,\frac{t}{|s|}\right)$,

V. $V(p,t) \triangleright H(r,s) \leq H(p+q,t+s)$,

VI. $V(p,t)$ is continuous non-decreasing function,

VII. $\lim\limits_{\alpha\to\infty} V(p,t) = 1$,

VIII. $Y(p,t) = 0$ if and only if $p = 0$,

IX. $Y(sp,t) = Y\left(p,\frac{t}{|s|}\right)$,

X. $Y(p,t) \blacktriangleright Y(r,s) \geq Y(p+r,t+s)$,

XI. $Y(p,t)$ is continuous non-decreasing function,

XII. $\lim\limits_{\alpha\to\infty} Y(p,t) = 0$,

XIII. $Z(p,t) = 1$ if and only if $p = 0$,

XIV. $Z(sp,t) = Z\left(p,\frac{t}{|s|}\right)$,

XV. $Z(p,t) \blacktriangleright Z(r,s) \geq Z(p+r,t+s)$,

XVI. $Z(p,t)$ is continuous non-decreasing function,

XVII. $\lim\limits_{\alpha\to\infty} Z(p,t) = 1$,

XVIII. If $t \leq 0$, then $V(p,t) = 0$, $Y(p,t) = 1$ and $Z(p,t) = 1$.

Then $N = (V,Y,Z)$ is called neutrosophic norm (shortly NM).

Definition 2.9 [14] Let A be a subset of the natural numbers $\mathbb{N}$. The asymptotic density of A, denoted by $\delta(A)$, is defined as

$$\delta(A) = \lim_{k\to\infty} \frac{|\{a \leq k : a \in A\}|}{k},$$

where $|\cdot|$ denotes the cardinality of the set.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set and let $A(x,y)$ be number pair of (a,b) in A such that $a \leq x$ and $b \leq y$. Then the two dimensional asymptotic density can be define as $\underline{\delta}_2(A) = \lim_{m,n} \inf \frac{A(x,y)}{x.y}$.

A sequence (p_n) is statistically convergent to q if for every $\varepsilon > 0$, the asymptotic density of the set $A(\varepsilon) = \{n \leq q : |p_n - q| > \varepsilon\}$ is zero. That is,

$$\lim_q \frac{1}{q} |\{n \leq q : |p_n - q| > \varepsilon\}| = 0.$$

A double sequence (p_{xy}) is statistically convergent to q if for every $\varepsilon > 0$, the double asymptotic density of the set

$$B(\varepsilon) = \{(x,y), x \leq a \ and \ y \leq b : |p_{xy} - q| \geq \varepsilon\}$$

is zero. That is,

$$\lim_{a,b\to\infty} \frac{1}{ab} \left|\{x \leq a, y \leq b : \left|p_{xy} - q\right| \geq \varepsilon\}\right| = 0.$$

Definition 2.10 [14] A sequence (p_n) is called a statistically Cauchy sequence if for every $\varepsilon > 0$,

$$\lim_q \frac{1}{q} \left|\{m \leq q : \left|p_m - p_q\right| \geq \varepsilon\}\right| = 0.$$

Definition 2.11 [15] Let $(A, \mu, \vartheta, \triangleright, \blacktriangleright)$ be an intuitionistic fuzzy normed space. A double sequence (p_{xy}) is statistically convergent to $q \in A$ with respect to the intuitionistic fuzzy norm (μ, ϑ) if for every $\varepsilon > 0$ and every $\epsilon > 0$,

$$\delta_2\{(x,y) \in \mathbb{N} \times \mathbb{N} : \mu\left(p_{xy} - q, \epsilon\right) \leq 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - q, \epsilon\right) \geq \varepsilon\} = 0.$$

That is

$$\lim_{a,b\to\infty} \frac{1}{ab} \left|\{a \leq x, b \leq y : \mu\left(p_{xy} - q, \epsilon\right) \leq 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - q, \epsilon\right) \geq \varepsilon\}\right| = 0.$$

Definition 2.12 [15] Let $(A, \mu, \vartheta, \triangleright, \blacktriangleright)$ be an intuitionistic fuzzy normed space. A double sequence (p_{xy}) is statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ϑ) if for every $\varepsilon > 0$ and every $\epsilon > 0$,

$$\delta_2\{(x,y) \in \mathbb{N} \times \mathbb{N} : \mu\left(p_{xy} - p_{ab}, \epsilon\right) \leq 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - p_{ab}, \epsilon\right) \geq \varepsilon\} = 0.$$

That is

$$\lim_{a,b\to\infty}\frac{1}{ab}\left|\left\{a\le x, b\le y: \mu\left(p_{xy}-q_{xy},\epsilon\right)\le 1-\varepsilon \text{ and } \vartheta\left(p_{xy}-q_{xy},\epsilon\right)\ge\varepsilon\right\}\right|=0.$$

3. Fixed-Point Theorems In Neutrosophic Quadruple Metric Spaces

Definition 3.1 Let $(Q, N, \triangleright, \blacktriangleright)$ be a NQM space. The mapping $T: Q \to Q$ is called neutrosophic quadruple contraction if there exist $0 < c < 1$ such that

$$d\left(T(p), T(q)\right) \le cd(p, q),$$

$$H(T(p), T(q), c\varepsilon) \ge H(p, q, \varepsilon),$$

$$J(T(p), T(q), c\varepsilon) \le J(p, q, \varepsilon),$$

$$K(T(p), T(q), c\varepsilon) \le K(p, q, \varepsilon),$$

for each $p, q \in Q$ and $\varepsilon > 0$.

We will define neutrosophic quadruple banach contraction theorem.

Theorem 3.1 Let $(Q, N, \triangleright, \blacktriangleright)$ be a neutrosophic quadruple metric space (NQM). If $T: Q \to Q$ is a neutrosophic quadruple contraction mapping, then there exists an element $c \in Q$ such that $c = T(c)$. This element *c* is called a neutrosophic quadruple fixed point of *T*.

Proof Let $p \in Q$ and $p_n = T^n p$ $(p \in \mathbb{N})$. By a induction we get

$$d(p_n, p_{n+1}) \le d(p, p_1),$$

$$H(p_n, p_{n+1}, \varepsilon) \ge H\left(p, p_1, \frac{\varepsilon}{t^n}\right),$$

$$J(p_n, p_{n+1}, \varepsilon) \le J\left(p, p_1, \frac{\varepsilon}{t^n}\right),$$

$$K(p_n, p_{n+1}, \varepsilon) \le K\left(p, p_1, \frac{\varepsilon}{t^n}\right),$$

for each $n \in \mathbb{N}$ and $\varepsilon > 0$. For any $m \in \mathbb{Z}$,we get

$$\begin{aligned}|d(p_n,p_{n+m})| &\leq |d(p_n,p_{n+1})| + \cdots + |d(p_{n+m-1},p_{n+m})|\\ &\leq |d(p_n,p_{n+1})| + \cdots + |d(p_n,p_{n+m})|,\end{aligned}$$

$$\begin{aligned}H(p_n,p_{n+k},\varepsilon) &\geq H\left(p_n,p_{n+1},\frac{\varepsilon}{m}\right) \triangleright \ldots \triangleright H\left(p_{n+m-1},p_{n+m},\frac{\varepsilon}{m}\right)\\ &\geq H\left(p,p_1,\frac{\varepsilon}{mt^n}\right) \triangleright \ldots \triangleright H\left(p,p_1,\frac{\varepsilon}{mt^{n+k-1}}\right),\end{aligned}$$

$$\begin{aligned}J(p_n,p_{n+m},\varepsilon) &\leq J\left(p_n,p_{n+1},\frac{\varepsilon}{m}\right) \blacktriangleright \ldots \blacktriangleright J\left(p_{n+m-1},p_{n+m},\frac{\varepsilon}{m}\right)\\ &\leq J\left(p,p_1,\frac{\varepsilon}{mt^n}\right) \blacktriangleright \ldots \blacktriangleright J\left(p,p_1,\frac{\varepsilon}{mt^{n+m-1}}\right),\end{aligned}$$

$$\begin{aligned}K(p_n,p_{n+m},\varepsilon) &\leq K\left(p_n,p_{n+1},\frac{\varepsilon}{m}\right) \blacktriangleright \ldots \blacktriangleright K\left(p_{n+m-1},p_{n+m},\frac{\varepsilon}{m}\right)\\ &\leq K\left(p,p_1,\frac{\varepsilon}{mt^n}\right) \blacktriangleright \ldots \blacktriangleright K\left(p,p_1,\frac{\varepsilon}{mt^{n+m-1}}\right).\end{aligned}$$

If we apply the limit conditions for the functions H, J, K in the neutrosophic metric space definition, we get

$$\lim_{n\to\infty} d(p_{n+m},p_n) = 0 + \cdots + 0 = 0,$$

$$\operatorname*{Lim}_{n\to\infty} H(p_{n+m},p_n,\varepsilon) \geq 1 \triangleright \ldots \triangleright 1 = 1,$$

$$\lim_{n\to\infty} J(p_{n+m},p_n,\varepsilon) \leq 0 \blacktriangleright \ldots \blacktriangleright 0 = 0$$

and

$$\lim_{n\to\infty} K(p_{n+m},p_n,\varepsilon) \leq 0 \blacktriangleright \ldots \blacktriangleright 0 = 0.$$

That is, $\{p_n\}$ cauchy sequence. Hence $\{p_n\}$ is convergent. We define a limit point q for $\{p_n\}$ sequence. We get

$$d(Tq,q) \leq d(Tq,Tp_n) \to 0$$

$$H(Tq,q,\varepsilon) \geq H\left(Tq,Tp_n,\frac{\varepsilon}{2}\right) \triangleright H\left(p_{n+1},q,\frac{\varepsilon}{2}\right)$$

$$\geq H\left(q,p_n,\frac{\varepsilon}{2m}\right) \triangleright H\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \to 1 \triangleright 1 = 1$$

$$J(Tq,q,\varepsilon) \leq J\left(Tq,Tp_n,\frac{\varepsilon}{2}\right) \blacktriangleright J\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \leq J\left(q,p_n,\frac{\varepsilon}{2m}\right) \blacktriangleright J\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \to 0 \blacktriangleright 0$$
$$= 0,$$

$$K(Tq,q,\varepsilon) \leq K\left(Tq,Tp_n,\frac{\varepsilon}{2}\right) \blacktriangleright K\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \leq K\left(q,p_n,\frac{\varepsilon}{2m}\right) \blacktriangleright K\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \to 0 \blacktriangleright 0$$
$$= 0.$$

Thus we get $Tq = q$, a fixed point. Now to get uniqueness, we assume $Tr = r$ for any $r \in Q$.

Then t

$$0 \leq d(r,q) = d(Tr,Tq) \to 0,$$

$$1 \geq H(r,q,\varepsilon) = H(Tr,Tq,\varepsilon) \geq H\left(r,q,\frac{\varepsilon}{m}\right) = H\left(Tr,Tq,\frac{\varepsilon}{m}\right) \geq H\left(r,q,\frac{\varepsilon}{m^2}\right) \geq \cdots$$
$$\geq H\left(r,q,\frac{\varepsilon}{m^n}\right) \to 1,$$

$$0 \leq J(r,q,\varepsilon) = J(Tr,Tq,\varepsilon) \leq J\left(r,q,\frac{\varepsilon}{m}\right) = J\left(Tr,Tq,\frac{\varepsilon}{m}\right) \leq J\left(r,q,\frac{\varepsilon}{m^2}\right) \leq \cdots$$
$$\leq J\left(r,q,\frac{\varepsilon}{m^n}\right) \to 0,$$

$$0 \leq J(r,q,\varepsilon) = J(Tr,Tq,\varepsilon) \leq J\left(r,q,\frac{\varepsilon}{m}\right) = J\left(Tr,Tq,\frac{\varepsilon}{m}\right) \leq J\left(r,q,\frac{\varepsilon}{m^2}\right) \leq \cdots$$
$$\leq J\left(r,q,\frac{\varepsilon}{m^n}\right) \to 0$$

when $n \to \infty$. So $r = q$.

Corollary 3.1 Let $(Q,N,\triangleright,\blacktriangleright)$ be a complete NQM space and let $T: Q \to Q$ be a neutrosophic quadruple contraction mapping. Then T possesses a unique fixed point in Q.

Now we will show edelstein contraction theorem for NQM spaces.

Theorem 3.2 Let $(Q,N,\triangleright,\blacktriangleright)$ be a NQM space and let $T: Q \to Q$ be a neutrosophic quadruple contraction mapping. if for all $p \neq q$

$$d\big(T(p),T(q)\big) < cd(p,q),$$

$$H(T(p),T(q),.) > H(p,q,.),$$

$$J(T(p),T(q),.) < J(p,q,.),$$

$$K(T(p),T(q),.) < K(p,q,.),$$

then T has a unique fixed point.

Proof Let $p \in Q$ ve $p_n = T^n p$ $(n \in \mathbb{N})$. $p_n \neq p_{n+1}$ for each n. Consequently for $n \neq m$, $p_n \neq p_m$. İf not, we get

$$d(p_n, p_{n+1}) = d(p_m, p_{m+1}) < d(p_{m-1}, p_m) < \cdots < d(p_n, p_{n+1}),$$

$$H(p_n, p_{n+1}, .) = H(p_m, p_{m+1}, .) > H(p_{m-1}, p_m, .) > \cdots > H(p_n, p_{n+1}, .),$$

$$J(p_n, p_{n+1}, .) = J(p_m, p_{m+1}, .) < J(p_{m-1}, p_m, .) < \cdots < J(p_n, p_{n+1}, .)$$

and

$$K(p_n, p_{n+1}, .) = K(p_m, p_{m+1}, .) < K(p_{m-1}, p_m, .) < \cdots < K(p_n, p_{n+1}, .)$$

where $m > n$, a contradiction. $\{p_n\}$ has a convergent subsequence $\{p_{n_i}\}$, because of Q is compact. Let $q = \lim_i p_{n_i}$. Moreover, we call that $q, Tq \notin \{p_{n_i} : i \in \mathbb{N}\}$. From our assumptions, we can write

$$d(Tp_n, Tq) < d(p_{n_i}, q),$$

$$H(Tp_n, Tq, .) > H(p_n, q, .),$$

$$J(Tp_n, Tq, .) < J(p_n, q, .)$$

and

$$K(Tp_n, Tq, .) < K(p_n, q, .)$$

for all $i \in \mathbb{N}$. So, we get

$$\lim_{i\to\infty} d(Tp_{n_i}, Tq) \leq \lim_{i\to\infty} d(p_{n_i}, q) = d(q,q) = 0,$$

$$\lim_{i\to\infty} H(Tp_{n_i}, Tq, \varepsilon) \geq \lim_{i\to\infty} H(p_{n_i}, q, \varepsilon) = H(q, q, \varepsilon) = 1,$$

$$\lim_{i\to\infty} J\left(Tp_{n_i}, Tq, \varepsilon\right) \leq \lim_{i\to\infty} J\left(p_{n_i}, q, \varepsilon\right) = J(q, q, \varepsilon) = 0$$

and

$$\lim_{i\to\infty} K\left(Tp_{n_i}, Tq, \varepsilon\right) \leq \lim_{i\to\infty} K\left(p_{n_i}, q, \varepsilon\right) = K(q, q, \varepsilon) = 0$$

for each $\varepsilon > 0$. Hence

$$\lim_{i\to\infty} Tp_{n_i} = Tq \ and \lim_{i\to\infty} T^2 p_{n_i} = T^2 q.$$

We get

$$d\left(p_{n_i}, Tp_{n_i}\right) \geq d\left(Tp_{n_i}, T^2 p_{n_i}\right) \geq \cdots \geq d\left(Tp_{n_{i+1}}, Tp_{n_{i+1}}\right) \geq d\left(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}\right) \geq \cdots \geq 0,$$

$$H\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \leq H\left(Tp_{n_i}, T^2 p_{n_i}, \varepsilon\right) \leq \cdots \leq H\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \leq H\left(Tp_{n_i}, T^2 p_{n_i}, \varepsilon\right) \leq \cdots$$
$$\leq H\left(p_{n_{i+1}}, Tp_{n_{i+1}}, \varepsilon\right) \leq H\left(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}, \varepsilon\right) \leq \cdots \leq 1,$$

$$J\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \geq J\left(Tp_{n_i}, T^2 p_{n_i}, \varepsilon\right) \geq \cdots \geq J\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \geq J\left(Tp_{n_i}, T^2 p_{n_i}, \varepsilon\right) \geq \cdots$$
$$\geq J\left(p_{n_{i+1}}, Tp_{n_{i+1}}, \varepsilon\right) \geq J\left(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}, \varepsilon\right) \geq \cdots \geq 0$$

and

$$K\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \geq K\left(Tp_{n_i}, T^2 p_{n_i}, \varepsilon\right) \geq \cdots \geq K\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \geq K\left(Tp_{n_i}, T^2 p_{n_i}, \varepsilon\right) \geq \cdots$$
$$\geq K\left(p_{n_{i+1}}, Tp_{n_{i+1}}, \varepsilon\right) \geq K\left(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}, \varepsilon\right) \geq \cdots \geq 0$$

for every $\varepsilon > 0$. Thus

$$d\left(p_{n_i}, Tp_{n_i}\right), H\left(p_{n_i}, Tp_{n_i}, \varepsilon\right), J\left(p_{n_i}, Tp_{n_i}, \varepsilon\right) \text{ and } K\left(p_{n_i}, Tp_{n_i}, \varepsilon\right)$$

are convert to same limit point. We get

$$d(q, Tq) \leq \lim_{i\to\infty} sum\ d\left(p_{n_i}, Tp_{n_i}\right) = \lim_{i\to\infty} sum\ d\left(Tp_{n_i}, T^2 p_{n_i}\right) \leq \lim_{i\to\infty} inf\ d\left(Tp_{n_i}, T^2 p_{n_i}\right)$$
$$\leq d(Tq, T^2 q),$$

$$H(q,Tq,\varepsilon) \geq \lim_{i\to\infty} sum\, H(p_{n_i},Tp_{n_i},\varepsilon) = \lim_{i\to\infty} sum\, H(Tp_{n_i},T^2p_{n_i},\varepsilon)$$

$$\geq \lim_{i\to\infty} inf\, H(Tp_{n_i},T^2p_{n_i},\varepsilon) \geq H(Tq,T^2q,\varepsilon),$$

$$J(q,Tq,\varepsilon) \leq \lim_{i\to\infty} sum\, J(p_{n_i},Tp_{n_i},\varepsilon) = \lim_{i\to\infty} sum\, J(Tp_{n_i},T^2p_{n_i},\varepsilon)$$

$$\leq \lim_{i\to\infty} inf\, J(Tp_{n_i},T^2p_{n_i},\varepsilon) \leq J(Tq,T^2q,\varepsilon)$$

and

$$J(q,Tq,\varepsilon) \leq \lim_{i\to\infty} sum\, J(p_{n_i},Tp_{n_i},\varepsilon) = \lim_{i\to\infty} sum\, J(Tp_{n_i},T^2p_{n_i},\varepsilon)$$

$$\leq \lim_{i\to\infty} inf\, J(Tp_{n_i},T^2p_{n_i},\varepsilon) \leq J(Tq,T^2q,\varepsilon)$$

for every $\varepsilon > 0$.

We assume $q \neq Tq$. Similarly we get

$$d(q,Tq) > d(Tq,T^2q),$$

$$H(q,Tq,.) < H(Tq,T^2q,.),$$

$$J(q,Tq,.) > J(Tq,T^2q,.)$$

and

$$K(q,Tq,.) > K(Tq,T^2q,.).$$

Those are a contradiction. Hence, $q = Tp,$ a fixed point.

4. Neutrosophic Quadruple Normed Spaces

Definition 4.1 Let Q be a vector space over K field, $p \in Q,$ and (V,Y,Z) be a **neutrosophic norm** on $Q \times (0,\infty)$ where $V,Y,Z: Q \times (0,\infty) \to [0,1]$ denote **truth, indeterminacy**, and **falsity** norm degrees, respectively. Let $\|.\|$ be a conventional norm on $Q,$ $\varepsilon, a, b, c > 0$ be fixed positive constants, and let '$\triangleright$' and '$\blacktriangleright$' denote a t-norm (TN) and t-conorm (TC), respectively. The **neutrosophic quadruple norm (NQN)** is the function $N: Q \to [0,\infty) \times [0,\infty)^3$ defined for each $p \in Q$, as:

$$N(p) = \big(\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon)\big).$$

$(N, Q, \triangleright, \blacktriangleright, \|.\|)$ is defined as NQN space.We calculate by $\|p\|$ norm of the known part of the vector p, how accurately the norm is measured with a, how ambiguously the norm is measured with b, how inaccurately the norm is measured with c, by V the degree to which the norm of vector p is correct, Y the degree to which the norm of vector p is imprecise, Z the degree to which the norm of vector p is incorrect

Example 4.1 Let $(N, \mathbb{R}, \triangleright, \blacktriangleright, \|.\|)$ be a NQN. it define as TN $p \triangleright q = pq$ and TC $p \blacktriangleright q = p + q - pq$. $\forall p \in \mathbb{R}$, $N = \{(|p|, a\frac{\varepsilon}{\varepsilon+|p|}, b\frac{|p|}{\varepsilon+|p|}, c\frac{|p|}{\varepsilon} : p \in Q\}$ neutrosophic quadruple set define NQN.

Example 2 Let $(N, Q, \triangleright, \blacktriangleright, \|.\|)$ be a NM space. If we take

$H(p,q,\delta) = V(p-q,\delta), J(p,q,\delta) = Y(p-q,\delta)$ and $K(p,q,\delta) = Z(p-q,\delta)$,

then $N = (H, J, K)$ a neutrosophic metric on Q, which is induced by the NQN N.

Definition 4.2 Let $(N, Q, \triangleright, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple-normed space (abbreviated NM-space).

I. A sequence (p_n) in N is a Cauchy sequence if for every $\delta > 0$ and $\tau > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$:

$$\|p_n - p_m\| < \delta,$$

$$aV(p_n - p_m, \tau) > 1 - \delta,$$

$$bY(p_n - p_m, \tau) < \delta,$$

$$cZ(p_n - p_m, \tau) < \delta.$$

II. The sequence (p_n) converges to $p \in N$ (denoted $p_n \to p$) if for every $\tau > 0$:

$$\lim_{n\to\infty} \|p_n - p\| = 0,$$

$$\lim_{n\to\infty} aV(p_n - p_m, \tau) = 1,$$

$$\lim_{n\to\infty} bY(p_n - p_m, \tau) = 0$$

and

$$\lim_{n\to\infty} cZ(p_n - p_m, \tau) = 0.$$

III. The space N is complete if every Cauchy sequence in N converges to a limit in N.

Definition 4.3 Let $(N, Q, \triangleright, \blacktriangleright)$ be a neutrosophic quadruple-normed space (NQN space) with $Q \subseteq N$.

I. The open ball $B(p, r, \tau)$ centered at $p \in Q$ with radius $r \in (0,1)$ and parameter $\tau > 0$ is defined as:

$$B(p,r,\tau) = \left\{ p \in Q: \begin{matrix} \|p-q\| < r, \\ aV(p-q,\tau) > 1-r, \\ bY(p-q,\tau) < r, \\ cZ(p-q,\tau) < r \end{matrix} \right\}.$$

II. A subset $P \subseteq Q$ is open if for every $p \in P$, there exist $r \in (0,1)$ and $\tau > 0$ such that $B(p, r, \tau) \subseteq P$.

III. The topology induced by the NQN, denoted t_N, is the family of all open subsets of Q.

Definition 4.4 A subset $P \subseteq Q$ is neutrosophic quadruple bounded in a neutrosophic quadruple-normed space $(N, Q, \triangleright, \blacktriangleright)$ if there exist $r \in (0,1)$ and $\tau > 0$ such that for every $p \in P$:

$$\|p - q\| < r,$$

$$aV(p - q, \tau) > 1 - r,$$

$$bY(p-q,\tau) < r,$$

$$cZ(p-q,\tau) < r.$$

Theorem 4.1 In any neutrosophic quadruple-normed space, all compact subsets exhibit neutrosophic quadruple boundedness.

Proof Let $(N, Q, \rhd, \blacktriangleright, \|.\|)$ be a NQN space and Let $P \subseteq Q$ compact subset. Consider the set $\{B(p,r,\tau): p \in P\}$ for $\tau > 0$ and $0 < r < 1$. Because of P compact, there are $p_1, p_2, \dots, p_n \in P$ such that $P \subseteq \bigcup_{m=1}^{n} B(p_m, r, \tau)$. For some $m, t \in \mathbb{N}$ and $p, q \in P$, $p \in B(p_m, r, \tau)$ and $q \in B(p_m, r, \tau)$. So we get

$$\|p-p_m\| < r, aV(p-p_m,\tau) > 1-r, bY(p-p_m,\tau) < r, cZ(p-p_m,\tau) < r$$

and

$$\|q-p_m\| < r, aV(q-p_m,\tau) > 1-r, bY(q-p_m,\tau) < r, cZ(q-p_m,\tau) < r.$$

Forthe

$\alpha = \max\{\|p_t - p_m\|: 1 \leq m,t \leq n\}, \beta = \min\{aV(p_t - p_m, \tau): 1 \leq m,t \leq n\}, \gamma = \max\{bY(p_t - p_m, \tau): 1 \leq m,t \leq n\}, \delta = \max\{cZ(p_t - p_m, \tau): 1 \leq m,t \leq n\}$

. Then $\alpha, \beta, \gamma, \delta > 0$. So we get

$$\|p_t - p_m\| \leq \|p_t - p_k\| + \|p_k - p_m\| \leq \tau_1,$$

$$V(p_t - p_m, \tau) \geq V(p_t - p_k, \tau) \rhd V(p_k - p_m, \tau) \geq (1-\tau_2) \rhd (1-\tau_2) \geq (1-\tau_2),$$

$$Y(p_t - p_m, \tau) \leq Y(p_t - p_k, \tau) \blacktriangleright Y(p_k - p_m, \tau) \leq (1-\tau_3) \blacktriangleright (1-\tau_3) \leq (1-\tau_3)$$

and

$$Z(p_t - p_m, \tau) \leq V(p_t - p_k, \tau) \blacktriangleright V(p_k - p_m, \tau) \leq (1-\tau_4) \blacktriangleright (1-\tau_4) \leq (1-\tau_4)$$

for the $\tau_1, \tau_2, \tau_3, \tau_4 \in (0,1)$. If we take $\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$, we have

$$\|p\| < r, aV(p,\tau) > 1-r, bY(p,\tau) < r, cZ(p,\tau) < r.$$

We get that compact subset P of a NQN space is neutrosophic quadruple bounded.

Corollary 4.1 In a NQN space, every compact set is closed and neutrosophic quadruple bounded.

The proof of this corollary can be easily obtained from the above theorems and definitions.

Definition 4.5 Let $(Q, N, \triangleright, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple normed space. Then a sequence $p = (p_{xy})$ is called to statistically convergent to $q \in A$ with to nutrosophic quadruple norm $N(p) = ((\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon))$ hold on that for all $\varepsilon > 0$ and $\epsilon > 0$,

$$\delta_N\{(x,y) \in \mathbb{N} \times \mathbb{N}: \|p_{xy} - q\| \leq 0, V(p_{xy} - q, \epsilon) \leq 1 - \varepsilon, Y(p_{xy} - q, \epsilon) \geq \varepsilon \text{ and}$$
$$Z(p_{xy} - q, \epsilon) \geq \varepsilon\} = 0.$$

That is

$$\lim_{x,y} \frac{1}{xy} \left|\{a \leq x, b \leq y: \|p_{xy} - q\| \leq 0, V(p_{xy} - q, \epsilon) \leq 1 - \varepsilon, Y(p_{xy} - q, \epsilon) \geq \varepsilon \text{ and } Z(p_{xy} - q, \epsilon) \geq \varepsilon\}\right| = 0.$$

Example 4.3 Let $(\mathbb{R}, |.|)$ be classic norm for real numbers and for the t-norm is $m \triangleright \mathrm{n} = \mathrm{mn}$ and t-conorm is $m \blacktriangleright \mathrm{n} = \min\{m + n, 1\}$ for every $m, n \in [0,1]$. For $p \in \mathbb{R}$ and every $\varepsilon > 0$, is defined as

$$N = (\|p\| = |p|, aV(p,\varepsilon) = \frac{\varepsilon}{\varepsilon + |p|}, bY(p,\varepsilon) = \frac{|p|}{\varepsilon + |p|}, cZ(p,\varepsilon) = \frac{|p|}{\varepsilon})$$

Then $(\mathbb{R}, N, \triangleright, \blacktriangleright, |.|)$ is a neutrosophic quadruple normed space. For the $p = (p_{xy})$ sequence is defined by

$$p_{xy} = \begin{cases} xy, if\ x\ and\ y\ is\ a\ cube\ roots. \\ 0, otherwise. \end{cases}$$

$$A_{x,y}(\varepsilon, \epsilon) = \{x \leq a, y \leq b: p_{xy} = \sqrt[3]{xy}\}.$$

Then, we get

$$\frac{1}{xy}|A_{x,y}(\varepsilon,\epsilon)| \leq \frac{1}{xy}|\{x \leq a \leq, y \leq b: a\ and\ b\ is\ a\ cube\ roots\}| \leq$$
$$\leq \frac{\sqrt[3]{a}\sqrt[3]{b}}{ab} \to 0\ as\ a, b \to \infty.$$

Definition 4.6 Let $(Q, N, \triangleright, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple normed space. Then a double sequence (p_{xy}) is called to statistically cauchy convergent to $q \in A$ with to neutrosophic quadruple norm $N(p) = ((\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon))$ if for all $\varepsilon > 0$ and $\epsilon > 0$, there exist $S = S(\varepsilon)$ and $T = T(\varepsilon)$ such that for every $x, a \geq S; y, b \geq T$

$$\delta_2\{(x,y) \in \mathbb{N} \times \mathbb{N}: \|p\| \leq 0, aV(p_{xy} - p_{ab}, \epsilon) \leq 1 - \varepsilon, cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon$$

and

$$cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon\} = 0.$$

Definition 4.6 Let $(Q, N, \triangleright, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple normed space. A double sequence (p_{xy}) is statistically Cauchy with respect to the neutrosophic quadruple norm

$N(p) = ((\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon))$ if for every $\varepsilon > 0$ and $\epsilon > 0$

$$\delta_2\{(x,y) \in \mathbb{N} \times \mathbb{N}: \|p\| \leq 0, aV(p_{xy} - p_{ab}, \epsilon) \leq 1 - \varepsilon, cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon$$

and

$$cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon\} = 0.$$

That is

$$\lim_{a,b\to\infty} \frac{1}{ab} \left| \begin{Bmatrix} a \leq x, b \leq y: \|p\| \leq 0, aV(p_{xy} - p_{ab}, \epsilon) \leq 1 - \varepsilon, \\ cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon \ and \ cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon \end{Bmatrix} \right| = 0.$$

5. Conclusions

In this study, we have defined the structure of neutrosophic quadruple normed spaces based on the neutrosophic quadruple set. Within this framework, we established the neutrosophic quadruple Banach contraction theorem and proved fundamental fixed-point theorems in neutrosophic quadruple metric spaces. We obtained definitions of statistical convergence and Cauchy statistical convergence for neutrosophic quadruple normed spaces.

This research successfully bridges a critical theoretical gap. It not only provides foundational fixed-point results for the newly introduced neutrosophic quadruple metric spaces but also pioneers the development of the theory of neutrosophic quadruple normed spaces. These contributions furnish essential mathematical tools and open significant avenues for advanced analysis under complex, multifaceted uncertainty. Collectively, the results presented offer a natural and substantial extension of existing knowledge in fixed point theory and functional analysis within fuzzy, intuitionistic fuzzy, and neutrosophic settings.

Abbreviations

FS Fuzzy set

FN Fuzzy norm

NN Neutrosophic Norm

NQM Neutrosophic quadruple metric

NQN Neutrosophic quadruple norm

TN Continuous t-norm

TC Continuous t-conorm

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