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Advancing Common Fixed Point Theorems in Bipolar Fuzzy-2 Metric Space and Applications in Nonlinear Analysis

Rajesh Kumar Saini[1] and Mukesh Kushwaha[2]

[1,2] Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi, INDIA

* Correspondence: prof.rksaini@bujhansi.ac.in, Tel.: (0919412322576)

Abstract: This study advances common fixed point theorems (CFPT) in bipolar fuzzy-2 metric space (BF2MS) and explores their applications in nonlinear analysis. BF2MS generalizes traditional metric and fuzzy metric spaces (FMS) by incorporating dualistic relationships, allowing the representation of both attraction and repulsion effects. We extend classical fixed point theorems (FPTs) to BF2MS, establishing conditions for the existence of common fixed points (CFPs). The study's applications span nonlinear analysis, stability analysis, control theory, and multi-criteria decision-making under uncertainty. Our findings refine existing results and provide practical real-world applications, supported by numerical examples, enhancing fuzzy mathematics and nonlinear systems modeling.

Keywords: Common Fixed Point Theorem, Bipolar Fuzzy-2 Metric Space, Fuzzy Metric Space

1. Introduction

For centuries, philosophers and scientists have sought to address uncertainty, ambiguity, and vagueness in mathematical and logical frameworks. Initially, probability theory served as the primary tool for dealing with such challenges (von Mises, 1957 [1]. However, the seminal work of Lotfi Zadeh's (1965) [2] on fuzzy set theory (FST) introduced a revolutionary approach to modeling imprecision. Since, then FST evolved into a robust mathematical framework applied across various disciplines. For instance, Klir & Yuan, 1995 [3] provided a comprehensive formalization on fuzzy systems, while Atanassov, 1986 [4] extended the concept of intuitionistic fuzzy set (IFS). Further refinements, such as those by Bustince & Burillo, 1996 [5], strengthened the theoretical underpinnings of FST. Despite these advances, continued research remains essential to refine and expand its theoretical foundations.

Over the past few decades, fuzzy mathematics has developed extensively, extending classical mathematical concepts into the fuzzy domain. Early contributions in this direction including the pioneering work of Bellman & Zadeh, 1970 [6] on decision theory and Dubois & Prade, 1980 [7] on operation research followed by the comprehensive treatment of fuzzy topology by Klement, Mesiar & Pap, 2000 [8]. The theory of FMS advanced through distinct formulations introduced by Kramosil and Michálek (1975) [9], and Kaleva and Seikkala (1984) [10], laying the foundations of fuzzy probabilistic topologies. Later, significant progress in the FPT within FMS was made by Grabiec, 1989 [11], and Park J. H., 2004 [12].

In metric space theory, an important breakthrough was made by Sharma, Sharma, and Iseki (1984) [13], who investigated contraction-type mappings in 2-metric spaces (2-MS), leading to further developments in FPT. These ideas were further extended into probabilistic 2-MS by Wenzhi (1997) [14], integrating probability theory with 2-metric structures to handle uncertainty more effectively. Traditional 2-MS, first introduced by Gähler, 1963 [15] and later refined by Dhage, 1992 [16] inspired by Euclidean area functions and find in spatial analysis.

Building upon these advancements, BF2MS presents a novel approach by incorporating dual-metric framework that simultaneously captures compatibility and opposition. This innovative framework broadens the applicability of CFPTs to nonlinear analysis. Jain et al., 2020 [17] emphasized on compatibility measures, while Dutta & Debnath, 2022 [18], explored opposition metrics. Further many researchers demonstrated the relevance of such structures in stability analysis, control theory and multi-criteria decision-making under uncertainty. Collectively these contributions paving the way for future advancements in open new frontiers in nonlinear and fuzzy systems research.

1.1 Motivation for the above study

The motivation for studying BF2MSs and their CFPTs arises from the need to develop advanced mathematical tools that address uncertainty, duality, and vagueness in real-world systems. By extending FMSs into BF2MSs, this framework captures the intricacies of systems influenced by opposing forces. For instance, Jain, Kaushik, & Singh, 2020 [19] highlighted their role in nonlinear dynamics, while Matloka, 1984 [20] provided foundation results in fuzzy convergence. Moreover, decision-making frameworks pioneered by Bellman and Zadeh (1970) [6] and extended by Dubois & Prade (1980) [7] have been refined in context of contradictory preferences through Xu, 2007 [21]. Such developments are particularly relevant to the applications in economics, risk assessment, and artificial intelligence, where competing objectives must be balanced.

Additionally, BF2MSs facilitate the precise mathematical representation of complex systems where cooperation and competition dynamically interact, such as in multi-agent systems, game theory, and social networks. In applied domains, this framework enhances modeling accuracy in physics, engineering, and economics, where dual influences shape system behavior. By developing CFPTs in BF2MS, this research contributes to both theoretical advancements and practical applications. It introduces a powerful mathematical toolset for analyzing layered, nonlinear relationships and addressing gaps in existing models. As such, BF2MSs offer new insights into stability, decision-making, and uncertainty modeling, reinforcing their importance in advancing modern fuzzy mathematics and nonlinear analysis.

1.2 Recent Developments

Recent advancements in BF2MSs have refined theoretical foundations and expanded applications, particularly in nonlinear analysis, decision-making models, and complex systems under uncertainty. As explored by Atanassov and Gargov (2020) [22], bipolar fuzzy methods are used in decision making problems of multi criteria in two choices that makes trade off evaluation also in [23] they work on interval-valued IFSs and their application in multi criteria decision making (MCDM) frameworks in uncertain environments.

In nonlinear analysis, Mohammadi and Jafari (2021) [24] utilized BF2MSs to model dynamical systems with the coexistence of attraction and repulsion forces, benefiting fields like physics (Aghababa, 2018 [25]) and engineering (Yao et al., 2022 [26]). Extensions proposed by Singh and Sharma (2021) [27] introduced additional parameters for more flexible modeling of multi-dimensional relationships, finding utility in neural networks and artificial intelligence (AI) systems. Integrations with AI by Gupta and Kumar (2022) [28], highlighted the effectiveness of BF2MSs in machine learning tasks, such as sentiment analysis and financial forecasting, where contradictory data must be managed. Generalized BF2MSs and their applications in neural networks were introduced by Singh and Sharma (2021) [29], showing their impact on deep learning architectures and cognitive computing. Applications in robotics and cyber-physical systems emphasized by Xu (2023) [30] demonstrated the relevance of BF2MSs for autonomous vehicle control. These studies underline the growing significance of BF2MSs in addressing dualistic and uncertain environments across diverse disciplines, including computational intelligence, engineering and decision sciences.

1.3 Novelty of the study

Our study introduces an advanced mathematical framework by extending traditional FMSs to BF2MSs, enabling the modeling of dual-dimensional relationships such as attraction-repulsion and compatibility-opposition. This approach enhances the representation of complex systems where opposing influences coexist, overcoming the limitations of traditional metrics. By generalizing classical FPTs, we establish new conditions applicable to systems governed by dual interactions. The framework is particularly valuable for analyzing stability and convergence in nonlinear systems and improving MCDM by reconciling conflicting objectives. Ultimately, this study advances fuzzy mathematics, offering new insights into fuzzy topology, nonlinear analysis, and decision theory, with broad interdisciplinary applications across computational intelligence, engineering, and control theory.

1.4 Practical Applicability

BF2MSs hold substantial potential in mathematics, particularly in fields such as FPT, topology, optimization, variational inequalities, MCDM, and convergence analysis. By incorporating both positive and negative memberships, BF2MSs provide a refined and adaptable framework for addressing uncertainty, conflict, and intricate relationships in complex systems. This innovative approach establishes BF2MSs as a vital tool in contemporary mathematical modeling and analysis, with significant implications for computational intelligence, control theory, and decision sciences.

2. Preliminaries

The study of CFPTs in BF2MS involves several foundational concepts from FST, metric spaces, and FPT. These preliminaries establish the groundwork for extending classical results into the BF2MS framework.

Definition 2.1 (MS): Let X be an arbitrary non-empty set. A function $d : X \times X \to [0,\infty)$ is called a metric on X, if for any elements $x, y, z \in X$, the following axioms are satisfied:

(i) non-negativity: $d(x, y) \geq 0$

(ii) identity of indiscernible: $d(x,y)=0$ iff $x=y$
(iii) symmetry: $d(x,y)=d(y,x)$
(iv) triangle inequality: $d(x,z)\le d(x,y)+d(y,z)$
The pair (X,d) is then called a MS.

Definition 2.2 (2-MS): Let X be a non-empty set. A function $\Omega: X\times X\times X\to[0,\infty)$ is called 2-metric on X if for all elements $x,y,z,w\in X,$ the following conditions holds:
(v) Non-negativity: for any distinct $x,y,z\in X,$ $\Omega(x,y,z)\ge 0$.
(vi) Identity: $\Omega(x,y,z)=0,$ iff x,y,z are not distinct (i.e. at least two of the points are equal)
(vii) Symmetry in arguments: the value remains unchanged for any permutation of the points x,y,z. i.e. $\Omega(x,y,z)=\Omega(y,z,x)=\Omega(z,x,y)$
(viii) tetrahedral inequality: $\Omega(x,y,z)\le d(x,u,w)+d(x,w,z)+d(w,y,z)$
The pair (X,d) is then called a 2-MS.

Definition 2.3: Let X be a non-empty set. A mapping $B: X\to[0,1]\times[0,1]$ is called bipolar fuzzy set B on X if for each element $x\in X$ the pair $\left(\mu_B(x),\nu_B(x)\right)$ representing its degree of membership and degree of non-membership, respectively i.e. $\mu_B(x)\in[0,1]$ (positive membership degree) and $\nu_B(x)\in[0,1]$ (negative membership degree). The pair $\left(\mu_B(x),\nu_B(x)\right)$ must satisfy the constraint $\mu_B(x)+\nu_B(x)\le 1$ $\forall x\in X.$

Definition 2.4: A BF2MS on X is a 3-tuple $(X,M_\Omega,*)$ where '$*$' is a continuous t-norm on [0,1]. $M_\Omega=\left(M_\Omega^+,M_\Omega^-\right)$ is a bipolar fuzzy set on $X^3\times[0,\infty)$, that is, for each triplet $x,y,z\in X$ and a real number $t>0,$ we have a pair $M_\Omega^+\left(x,y,z,t\right),$ $M_\Omega^-\left(x,y,z,t\right)$ representing the positive and negative memberships of the BF2M M_Ω, satisfies the following axioms for all $x,y,z,w\in X$ and all $s,t>0$:

(BF2M-1): $M_\Omega^+\left(x,y,z,t\right)=1$ and $M_\Omega^-\left(x,y,z,t\right)=0.$

(BF2M-2): $M_\Omega^+\left(x,y,z,t\right)\ne 1$ and $M_\Omega^-\left(x,y,z,t\right)\ne 0,\ \forall\ t>0,$ also $M_\Omega^+\left(x,y,z,t\right)=1$ and $M_\Omega^-\left(x,y,z,t\right)=0,\ \forall\ t\ge 0,$ if at least two of x,y,z are equal, and $\forall\ x,y\in X\ \exists\ z\in X.$

(BF2M-3): $M_\Omega^+\left(x,y,z,t\right)$ and $M_\Omega^+\left(p(x,y,z),t\right)$ are invariant under any permutation of $x,y,z.$

(BF2M-4): $M_\Omega^+(x,y,z,r+t+s)\ge M_\Omega^+(x,y,w,r)*M_\Omega^+(x,w,z,t)*M_\Omega^+(w,y,z,s),$

$$M_\Omega^-(x,y,z,r+t+s)\le M_\Omega^-(x,y,w,r)*M_\Omega^-(x,w,z,t)*M_\Omega^-(w,y,z,s)$$

(BF2M-5): $M_{\Omega}^{+}(x,y,z,.):(0,\infty)\rightarrow(0,1]$ and $M_{\Omega}^{-}(x,y,z,.):(0,\infty)\rightarrow[-1,0)$ are both continuous functions.

Some additional necessary conditions in BF2MS are outlined as follows:

Symmetry: $M_{\Omega}^{+}(x,y,z,t)=M_{\Omega}^{+}(y,z,x,t)$ and $M_{\Omega}^{-}(x,y,z,t)=M_{\Omega}^{-}(y,z,x,t)$

Boundary Conditions: $\lim_{t\rightarrow\infty} M_{\Omega}^{+}(x,y,z,t)=1$ and $\lim_{t\rightarrow\infty} M_{\Omega}^{-}(x,y,z,t)=0$ and

$$\lim_{t\rightarrow 0} M_{\Omega}^{+}(x,y,z,t)=0 \text{ and } \lim_{t\rightarrow 0} M_{\Omega}^{-}(x,y,z,t)=1$$

Monotonicity: $M_{\Omega}^{+}(x,y,z,t_1)\leq M_{\Omega}^{+}(x,y,z,t_2)$ and $M_{\Omega}^{-}(x,y,z,t_1)\geq M_{\Omega}^{-}(x,y,z,t_2)$ for $t_1<t_2$.

Contraction Condition: Let (X,Ω) be CMS. If $f:X\rightarrow X$ is a continuous mapping, then there exists a constant $k\in(0,1)$ Such That (s.t.), for all $x,y,z\in X$ and $t>0$, we have $M_{\Omega}^{+}(f(x),f(y),z,t)\geq M_{\Omega}^{+}(x,y,z,kt)$ and $M_{\Omega}^{-}(f(x),f(y),z,t)\leq M_{\Omega}^{-}(x,y,z,kt)$ implies f has a unique FP in X.

Example 2.1: Consider $X=[0,1]$, with $a*b=\min\{a,b\}$ for $a,b\in[0,1]$. Define the bipolar fuzzy 2-metric M_{Ω} by $M_{\Omega}^{+}\left(x,y,z,t\right)=\min\left\{|x-y|,|y-z|,|z-x|\right\}$

and $M_{\Omega}^{-}(x,y,z,t)=\min\left\{\left(1-|x-y|\right),\left(1-|y-z|\right),\left(1-|z-x|\right)\right\}$ for $x,y,z\in X,t>0.$

Then $(X,M_{\Omega},*)$ forms a BF2MS.

Example 2.2: Let $X=(x_1,x_2,x_3)$ be a set of three distinct point. Suppose we have 2-metric Ω defined on X, which intuitively measure the area of a triangle formed by three points. For simplicity, assume that $\Omega(x,y)$ is a function that outputs a value between 0 and 1, representing the degree of closeness between points in the space. We now define the following positive and negative membership functions. Let $\mu_{\Omega}(x,y,z,t)=\min\left(\Omega(x,y),\Omega(y,z),\Omega(z,x)\right)$ representing the positive membership or attraction degree between x,y,z at time t and $v_{\Omega}(x,y,z,t)=1-\min\left(\Omega(x,y),\Omega(y,z),\Omega(z,x)\right)$ representing the negative membership or repulsion degree between x,y,z. Thus, the BF2M is given by $M_{\Omega}^{+}\left(x,y,z,t\right)=\min\left\{\Omega(x,y),\Omega(y,z),\Omega(z,x)\right\}$ and $M_{\Omega}^{-}\left(x,y,z,t\right)=1-\min\left\{\Omega(x,y),\Omega(y,z),\Omega(z,x)\right\}$.

Let's assume the distances based on the 2-metric Ω are $\Omega(x_1,x_2)=0.8,$ $\Omega(x_2,x_3)=0.9$ and $\Omega(x_3,x_1)=0.7.$ Then, the positive membership for x_1,x_2,x_3 at a given time t is $M_{\Omega}^{+}(x_1,x_2,x_3,t)=\min(0.8,0.9.0.7)=0.7$. The negative membership for x_1,x_2,x_3 at the same

$M^{-}_{\Omega}(x_1,x_2,x_3,t) = 1-\min(0.8,0.9.0.7) = 0.3$. Thus, the BF2M for this set of points at time t is $M_{\Omega}(x_1,x_2,x_3,t) = \left\{M^{+}_{\Omega}(x_1,x_2,x_3,t), M^{-}_{\Omega}(x_1,x_2,x_3,t)\right\} = (0.7,0.3)$.

This illustrates how the BF2MS introduces both positive and negative memberships to model dual relationships between points, providing a more comprehensive framework for handling uncertainty and duality in complex systems.

Definition 2.5: Let us consider a BF2MS as $(X, M_{\Omega}, *)$. Then

(i) If $\lim_{n\to\infty} x_n = x$ and $x \in X$, then the sequence $\{x_n\}$ is said to converge to a point if

$$\lim_{n\to\infty} M^{+}_{\Omega}(x_n,x,z,t) = 1 \text{ and } \lim_{n\to\infty} M^{-}_{\Omega}(x_n,x,z,t) = 0, \ \forall z \in X \text{ and } t > 0.$$

(ii) If $\lim_{n,m\to\infty} M^{+}_{\Omega}(x_n,x_m,z,t) = 1$ and $\lim_{n,m\to\infty} M^{-}_{\Omega}(x_n,x_m,z,t) = 0$ for all $n,m \in N$, $z \in X$ and $t > 0$.

then the sequence $\{x_n\}$ is said to be a Cauchy sequence (CS).

(iii) Every CS in BF2MS, which is convergent is said to be complete.

Lemma 2.1: Let $(X, M^{+}_{\Omega}, M^{-}_{\Omega}, *)$ be a BF2MS. If $M^{+}_{\Omega}(x,y,z,kt) \geq M^{+}_{\Omega}(x,y,z,t)$ and $M^{-}_{\Omega}(x,y,z,kt) \leq M^{+}_{\Omega}(x,y,z,t)$, $\forall x,y,z \in X, t > 0,$ and some $k \in (0,1)$ hold good in BF2MS, then $x = y$.

Lemma 2.2: The functions $M^{+}_{\Omega}(x,y,z,t)$ and $M^{-}_{\Omega}(x,y,z,t)$ are non-decreasing in t for all $x,y,z \in X$.

Proof: Let $t', t > 0,\ t' > t$ then, by applying BF2M-4 for both the positive and negative memberships, since $M^{+}_{\Omega}(x,y,z,t') \geq M^{+}_{\Omega}(x,y,z,t)$ and $M^{-}_{\Omega}(x,y,z,t') \leq M^{+}_{\Omega}(x,y,z,t)$ which establishes the non-decreasing values in both positive and negative memberships.

Lemma 2.3: Let $\{x_n\}$ be a sequence in a BF2MS $(X, M_{\Omega}, *)$, then $\{x_n\}$ is a CS in X, if for some $k \in (0,1)$, $\exists k$ s.t. $M^{+}_{\Omega}(x_{n+1},x_{n+2},z,kt) \geq M^{+}_{\Omega}(x_n,x_{n+1},z,t)$ and

$M^{-}_{\Omega}(x_{n+1},x_{n+2},z,kt) \leq M^{-}_{\Omega}(x_n,x_{n+1},z,t)$, $\forall x,y,z \in X, t > 0$ and $n = 0,1,2,3,\ldots.$.

Lemma 2.4 (Contraction Mappings in BF2MS): Let $(X, M^{+}_{\Omega}, M^{-}_{\Omega}, *)$ be a BF2MS. If for all $x,y,z \in X$, $t > 0$, a mapping $T: X \to X$ satisfies a contraction condition in the BF2MS if $\exists k \in [0,1)$ s.t. $M^{+}(T(x),T(y),z,t) \geq \lambda.M^{+}(x,y,z,t)$ and $M^{-}(T(x),T(y),z,t) \leq \lambda.M^{-}(x,y,z,t)$.

Lemma 2.5 (Compatibility and Weak Compatibility in BF2MS): Let $(X, M_\Omega^+, M_\Omega^-, *)$ be a BF2MS. If all $x, y, z \in X,\ t > 0,$ two mappings $T_1,\ T_2 : X \to X$ are compatible if

$$\lim_{n\to\infty} M^+(T_1(x_n), T_2(x_n), z, t) \geq \lim_{n\to\infty} M^+(x_n, x_n, z, t),$$

and $$\lim_{n\to\infty} M^-(T_1(x_n), T_2(x_n), z, t) \leq \lim_{n\to\infty} M^-(x_n, x_n, z, t),$$ for any sequence $\{x_n\}$ X.

Two mappings T_1 and T_2 are weakly compatible if $T_1, T_2(x) = T_2T_1(x)$ implies $T_1(x) = T_2(x)$.

Lemma 2.6: In all the examples we define the function $\phi \in \Phi,$ which is upper semi-continuous, monotonic, non-decreasing, and satisfies diagonal dominance. We prove some FPT on $\phi : [0,1]^5 \to [0,1]$ and each variable satisfying $\phi(t,t,t,t,t) \geq t,\quad \forall t \in [0,1].$ For example, the function $\phi(t_1, t_2, t_3, t_4, t_5) = t_1,\ \ \forall t_i\text{'}s \in [0,1]$ is a member of this class.

Lemma 2.7: (Coincidentally Commuting Mappings): Two mappings $f, g : X \to X$ are said to be coincidentally commuting if $(f \circ g)(x) = (g \circ f)(x)$ whenever $f(x) = g(x).$

Notably, every commuting mapping is coincidentally commuting, but the reverse does not always hold.,

3. Fixed Point Theorem for BF2MS

Theorem 3.1: Let $(X, M_\Omega, *)$ be a complete BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-),$ and $t * t \geq t$ holds for all $t \in [0,1]$ and let $f : X \to X$ be a self-mapping. Then *f* has a CFP in *X* if there exists a continuous mapping $A : X \to f(X),$ which commutes with *f* and satisfies

$$M_\Omega^+\left(A(x), A(y), z, kt\right) \geq \phi \begin{Bmatrix} M_\Omega^+(f(x), f(y), z, t); M_\Omega^+(f(x), A(x), z, t); M_\Omega^+(f(y), A(y), z, t); \\ M_\Omega^+(f(y), A(x), z, \alpha t); M_\Omega^+(f(x), A(y), z, (2-\alpha)t) \end{Bmatrix} \tag{1}$$

and $$M_\Omega^-\left(A(x), A(y), z, kt\right) \leq \phi \begin{Bmatrix} M_\Omega^-(f(x), f(y), z, t); M_\Omega^-(f(x), A(x), z, t); M_\Omega^-(f(y), A(y), z, t); \\ M_\Omega^-(f(y), A(x), z, \alpha t); M_\Omega^-(f(x), A(y), z, (2-\alpha)t) \end{Bmatrix} \tag{2}$$

for $x, y, z \in X, t > 0,$ and $\alpha \in (0,2),$ and $k \in (0,1)$ and $\phi \in \Phi,$ as in lemma 2.6.

If $\lim_{t\to\infty} M_\Omega^+\left(A(x), A(y), z, t\right) = 1$ and $\lim_{t\to\infty} M_\Omega^-(A(x), A(y), z, t) = 0,\quad \forall x, y, z \in X.$ then *f* and *A* have a UCFP.

Proof: By setting $x_{n+1} = f(x_n) = A(x_{n-1}),\ n = 0,1,2...$, we develop a sequence $\{x_n\}$. First we prove that $\{A(x_n)\}$ is a CS. Applying condition (ii) with $\alpha = 1-k$ for $k \in (0,1)$, in (1), we have

$$M^+_\Omega(A(x_{n-1}), A(x_n), z, kt) \geq \phi\left\{\begin{array}{r} M^+_\Omega(f(x_{n-1}), f(x_n), z, t); M^+_\Omega(f(x_{n-1}), A(x_{n-1}), z, t); \\ M^+_\Omega(f(x_n), A(x_n), z, t); M^+_\Omega(f(x_n), A(x_{n-1}), z, (1-k)t); \\ M^+_\Omega(f(x_{n-1}), A(x_n), z, (1+k)t) \end{array}\right\}$$

or
$$M^+_\Omega(x_{n+1}, x_{n+2}, z, kt) \geq \phi\left\{\begin{array}{r} M^+_\Omega(x_n, x_{n+1}, z, t), M^+_\Omega(x_n, x_{n+1}, z, t), M^+_\Omega(x_{n+1}, x_{n+2}, z, t), \\ M^+_\Omega(x_{n+1}, x_{n+1}, z, (1-k)t), M^+_\Omega(x_n, x_{n+2}, z, (1+k)t) \end{array}\right\}$$

$$M^+_\Omega(x_{n+1}, x_{n+2}, z, kt) \geq \phi\left\{\begin{array}{r} M^+_\Omega(x_n, x_{n+1}, z, t), M^+_\Omega(x_n, x_{n+1}, z, t), M^+_\Omega(x_{n+1}, x_{n+2}, z, t), 1, \\ M^+_\Omega(x_n, x_{n+1}, z, t) * M^+_\Omega(x_n, x_{n+2}, z, t) \end{array}\right\} \tag{3}$$

Now two cases arise,

Case I: Because ϕ is non-decreasing i.e. $a_i \leq b_i, \forall i \Rightarrow \phi(a_1,...a_n) \leq \phi(b_1,...b_n)$, and if $M^+_\Omega(x_{n+1}, x_{n+2}, z, t) \geq M^+_\Omega(x_n, x_{n+1}, z, t)$ then equation (1) yields

$$M^+_\Omega(x_{n+1}, x_{n+2}, z, kt) \geq \phi\left\{M^+_\Omega(x_n, x_{n+1}, z, t), M^+_\Omega(x_n, x_{n+1}, z, t), M^+_\Omega(x_{n+1}, x_{n+2}, z, t), 1, M^+_\Omega(x_n, x_{n+1}, z, t)\right\}$$

from lemma 2.6, $M^+_\Omega(x_{n+1}, x_{n+2}, z, kt) \geq M^+_\Omega(x_n, x_{n+1}, z, t)$ or

$M^+_\Omega(A(x_{n-1}), A(x_n), z, kt) \geq M^+_\Omega(A(x_{n-2}), A(x_{n-1}), z, t),\ \forall t > 0,$ therefore by lemma 2.3, $\{A(x_n)\}$ is a CS.

Case II: If $M^+_\Omega\left(x_n, x_{n+1}, z, t\right) > M^+_\Omega\left(x_{n+1}, x_{n+2}, z, t\right)$ for some *n*, then by using (3)

$$M^+_\Omega(x_{n+1}, x_{n+2}, z, kt) \geq M^+_\Omega(x_{n+1}, x_{n+2}, z, t).$$

Thus by lemma 2.3, $x_{n+1} = x_{n+2}$ which implies $f(x_{n+1}) = f(x_{n+2})$ i.e $x_{n+2} = x_{n+3}$ and so on. These exists some positive integer s.t. $n \geq m$ implies $x_n = x_m$, which show that $\{x_n\}$ a convergent sequence and so Cauchy sequence. Thus in either case, $\{A(x_n)\}$ is a Cauchy sequence in *X*. Since *X* is complete, therefore there exists $\lim z = \lim_{n\to\infty} A(x_n),$ for all $z \in X,$ it follows that $Az = fz.$ Now using $\alpha = 1+q,$ $q \in (0,1),$ all $t > 0,$ in (3) yields

$$M^+_\Omega(x_{n+1}, A(z), z, kt) = M^+_\Omega(A(x_{n+1}), A(z), z, kt) \geq \phi\left\{\begin{array}{r} M^+_\Omega(x_n, A(z), z, t); M^+_\Omega(x_n, x_{n+1}, z, t); \\ M^+_\Omega(f(z), A(z), z, t); M^+_\Omega(f(z), x_{n+1}, z, (1+q)t); \\ M^+_\Omega(x_n, A(z), z, (1-q)t) \end{array}\right\}$$

or letting $n \to \infty$ and after that $q \to 0,$

$$M_\Omega^+(z,A(z),z,kt) \ge \phi\left\{M_\Omega^+(z,A(z),z,t);1;1;M_\Omega^+(A(z),z,z,t);M_\Omega^+(z,A(z),z,t)\right\}$$

from lemma 2.6, $M_\Omega^+(z,A(z),z,kt) \ge M_\Omega^+(z,A(z),z,t)$, which implies that $A(z)=z$ therefore $f(z)=A(z)=z$ i.e $f(z)=z$. Thus f has a fixed point z. That is f and A have a CFP z.

Again let $w(\neq z)$ be another fixed point of f, i.e. $f(z)=z$ and $f(w)=w$, the for the sake of uniqueness using (3) for $\alpha = 1+q$, $q\in(0,1)$, we have $M_\Omega^+(z,w,z,kt) = M_\Omega^+(A(z),A(w),z,kt)$, $t\in[0,1]$.

Letting $q \to 0$, $M_\Omega^+(z,w,z,kt) \ge \phi\left\{M_\Omega^+(z,w,z,t);M_\Omega^+(z,w,z,t);1;M_\Omega^+(z,w,z,t);M_\Omega^+(z,w,z,t)\right\}$

i.e. $M_\Omega^+(z,w,z,kt) \ge M_\Omega^+(z,w,z,t)$, for all $t > 0$, from lemma 2.2. Now using lemma 2.1 which implies z = w. Hence f and A have UCFP in X. Similarly we can prove $\lim_{t\to\infty} M_\Omega^-(A(x),A(y),z,t)=0$. This completes the theorem.

Theorem 3.2: Let $(X,M_\Omega,*)$ be a complete BF2MS with $M_\Omega=(M_\Omega^+,M_\Omega^-)$, and $t*t \ge t$ for all $t\in[0,1]$. Suppose S and T are continuous mappings of X into itself. Then S and T have a CFP in X if there exists a continuous mapping A of X into $S(X)\bigcap T(X)$, commutes with S and T, which satisfy the following conditions:

(i)•• $S(X)\subseteq T(X)$

(ii)•• $T(X)$ is complete,

(iii)•• The bipolar fuzzy 2-metric M_Ω satisfies the inequality:

$$M_\Omega^+\left(A(x),A(y),a,kt\right) \ge \phi\left\{\begin{matrix}M_\Omega^+\left(S(x),T(y),a,t\right);M_\Omega^+\left(S(x),A(x),a,t\right);M_\Omega^+\left(T(y),A(y),a,t\right);\\ M_\Omega^+\left(T(y),A(x),a,\alpha t\right);M_\Omega^+\left(S(x),A(y),a,(2-\alpha)t\right)\end{matrix}\right\} \quad (4)$$

and $$M_\Omega^-(A(x),A(y),a,kt) \ge \phi\left\{\begin{matrix}M_\Omega^-(S(x),T(y),a,t);M_\Omega^-(S(x),A(x),a,t);M_\Omega^-(T(y),A(y),a,t);\\ M_\Omega^-(T(y),A(x),a,\alpha t);M_\Omega^-(S(x),A(y),a,(2-\alpha)t)\end{matrix}\right\} \quad (5)$$

$\forall(x,y,a)\in X$, $t>0$, $\alpha\in(0,2)$, $k\in(0,1)$ and $\phi\in\Phi$, as in lemma 2.6 with $\lim_{n\to\infty} M_\Omega^+\left(x,y,z,t\right)=1$ and $\lim_{n\to\infty} M_\Omega^-(x,y,z,t)=0$, implies A, S and T A have a UCFP in X.

Proof: Define a sequence $\{y_n\}$ recursively via $y_{n+1}=f(x_n)$ for $n\ge 0$, with $x_0\in X$. By setting $y_{2n}=Ax_{2n}=Sx_{2n+1}$ and $y_{2n+1}=Ax_{2n+1} =Tx_{2n+2}$, $\forall n=0,1,2....$. We will show the sequence $\{Ax_n\}$ is CS.

Case I: If $y_{2n}\neq y_{2n+1}$, for $\alpha=1-q$, $q\in(0,1)$ we introduced $x=x_{2n}$, $y=x_{2n}$, $z=x_{2n+1}$ in (4), then

$$M_\Omega^+(Ax_{2n},Ax_{2n+1},z,qt) \ge \phi\left\{\begin{array}{r} M_\Omega^+\left(Sx_{2n},Tx_{2n+1},z,t\right); M_\Omega^+\left(Sx_{2n},Ax_{2n},z,t\right); M_\Omega^+\left(Tx_{2n+1},Ax_{2n+1},z,t\right); \\ M_\Omega^+\left(Tx_{2n+1},Ax_{2n},z,(1-q)t\right); M_\Omega^+\left(Sx_{2n},Ax_{2n+1},z,(1+q)t\right) \end{array}\right\}$$

or $$M_\Omega^+\left(y_{2n}y_{2n+1},z,qt\right) \ge \phi\left\{\begin{array}{r} M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); \\ M_\Omega^+\left(y_{2n},y_{2n},z,(1-q)t\right); M_\Omega^+\left(y_{2n-1},y_{2n+1},z,(1+q)t\right) \end{array}\right\}$$

$$\ge \phi\left\{M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); 1; M_\Omega^+\left(y_{2n-1},y_{2n+1},z,(1+q)t\right)\right\}$$

letting $q \to 1$, then and (BF2M-4),

$$M_\Omega^+\left(y_{2n},y_{2n+1},z,kt\right) \ge \phi\left\{\begin{array}{r} M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); 1; \\ M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right) \end{array}\right\}$$

which we have either $M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right) \le M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right)$

or $$M_\Omega^+\left(y_{2n},y_{2\,n+1},z,t\right) > M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right).$$

Assume that the condition $M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right) \le M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right)$ hold for a certain n. Owing to the fact that ϕ is non-decreasing in every condition, we can infer from (4) that

$$M_\Omega^+\left(y_{2n},y_{2n+1},z,kt\right) \ge \phi\left\{M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right); 1; M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right)\right\}$$

i.e. $$M_\Omega^+\left(y_{2n},y_{2n+1},z,kt\right) \ge M_\Omega^+\left(y_{2n-1},y_{2n},z,t\right) \tag{6}$$

Similarly, in later cases, it follows from that

$$M_\Omega^+\left(y_{2n},y_{2n+1},z,kt\right) \ge \phi\left\{\begin{array}{r} M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); \\ M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right); M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right) \end{array}\right\}$$

i.e. $$M_\Omega^+\left(y_{2n},y_{2n+1},z,kt\right) \ge M_\Omega^+\left(y_{2n},y_{2n+1},z,t\right) \tag{7}$$

for every $t > 0$, lemma 2.3, yields $y_{2n} = y_{2n+1}$, which leads to a contradiction, because $y_{2n} \ne y_{2n+1}$ holds $\forall n = 0,1,2.....$. Hence from (6), we obtain

$$M_\Omega^+(y_{2n},y_{2n+1},z,kt) \ge M_\Omega^+(y_{2n-1},y_{2n},z,t) \tag{8}$$

for all $t > 0$, Again for $2m > 2p+1$, then from (8),

$$M_\Omega^+\left(Ax_{2m},Ax_{2p+1},z,kt\right) \ge M_\Omega^+\left(Ax_{2m-1},Ax_{2p},z,t/k\right)$$

for every m and p in N. Further if $2m>2p+1$, then

$$M_\Omega^+(Ax_{2m},Ax_{2p+1},z,kt) \ge M_\Omega^+(Ax_{2k-1},Ax_{2p},z,t/k).....M_\Omega^+(Ax_0,Ax_{2p+-2m},z,t/k^{2m}) \tag{9}$$

If $2m>2p+1$, then $M_\Omega^+(Ax_{2m},Ax_{2p+1},z,kt) \ge M_\Omega^+(Ax_{2m-(2p+1)},Ax_0,z,t/k^{2p+1})$ (10)

by simple induction with (9) & (10) we have $M_{\Omega}^{+}\left(Ax_n, Ax_{n+1}, z, kt\right) \geq M_{\Omega}^{+}\left(Ax_0, Ax_l, z, t/k^n\right)$.

If $n = 2m$, $p = 2s + 1$ *or* $n = 2m + 1$, $p = 2s + 1$ in (BF2M-4)

$$M_{\Omega}^{+}(Ax_n, Ax_{n+p}, z, kt) \geq M_{\Omega}^{+}(Ax_0, Ax_p, Ax_1, t/3k^n) * M_{\Omega}^{+}(Ax_0, Ax_1, z, t/3k^n) * M_{\Omega}^{+}(Ax_p, Ax_1, z, t/3k^n) \quad (11)$$

If $n = 2m,\ p = 2s$ *or* $n = 2m+1,\ p = 2s$, for every positive integer p and n in N, by noting that $M_{\Omega}^{+}\left(Ax_0, Ax_p, z, t/k^n\right) \to 1$ *as* $n \to \infty$. Thus $\{Ax_n\}$ is a Cauchy sequence.

Case II: If $y_{2n} = y_{2n+1}$, for some positive integer m. Hence m may be even or odd positive integer without loss of generality take $n = p$ i.e., $x_{2p} = x_{2p+1}$ and so that

$ASx_{2p} = ASx_{2p+1}, SAx_{2p} = ASx_{2p+1}, ASx_{2p} = ATx_{2p} = ATx_{2p+1}, SAx_{2p} = TAx_{2p} = TAx_{2p+1}$ and $ASx_{2p} = SAx_{2p+1}, ATx_{2p} = TAx_{2p+1}$, which implies that $x_{2p+2} = x_{2p+3}$. In the same way $x_{2p+4} = x_{2p+5}$..... Thus $x_{2p+2r} = x_{2p+2r+1}$, $\forall r = 0,1,2......$. Thus we have for case I as

$$M_{\Omega}^{+}\left(A\,x_{2p+2r+1}, Ax_{2p+2r+2}, z, kt\right) \geq M_{\Omega}^{+}\left(Ax_{2p+2r}, Ax_{2p+2r+1}, z, t\right) = 1,$$

$\Rightarrow M_{\Omega}^{+}\left(Ax_{2p+2r+1}, Ax_{2p+2r+2}, z, kt\right) = 1,\quad \forall t > 0$, yields by $Ax_{2p+2r+1} = Ax_{2p+2r+2},\quad \forall r = 0,1,2...$. Thus $Ax_{2p} = Ax_{2p+1} = Ax_{2p+2}.....$, this establishes that the sequence $\{Ax_n\}$ is convergent and consequently it must also be Cauchy. Thus $\{Ax_n\}$ is always a Cauchy sequence. Since X is complete, the sequence $\{Ax_n\}$ converges to some point $u \in X$, which further implies $u = \lim_{n\to\infty} Ax_n$ and $z = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2}$, it follows that $Au = Su = Tu$. Applying (4) for $\alpha = 1$

$$M_{\Omega}^{+}\left(Au, A^2u, z, kt\right) = M_{\Omega}^{+}\left(Au, AAu, z, kt\right) \geq \phi\begin{Bmatrix} M_{\Omega}^{+}\left(Su, TAu, z, t\right); M_{\Omega}^{+}\left(Su, Au, z, t\right); M_{\Omega}^{+}\left(TAu, AAu, z, t\right); \\ M_{\Omega}^{+}\left(TAu, Au, z, t\right); M_{\Omega}^{+}\left(Su, AAu, z, t\right) \end{Bmatrix}$$

$$= \phi\begin{Bmatrix} M_{\Omega}^{+}\left(Au, ATu, z, t\right); M_{\Omega}^{+}\left(Au, Au, z, t\right); M_{\Omega}^{+}\left(ATu, A^2u, z, t\right); \\ M_{\Omega}^{+}\left(ATu, Au, z, t\right); M_{\Omega}^{+}\left(Au, A^2u, z, t\right) \end{Bmatrix}$$

$$= \phi\left\{M_{\Omega}^{+}\left(Au, A^2u, z, t\right); 1; 1; M_{\Omega}^{+}\left(A^2u, Au, z, t\right); M_{\Omega}^{+}\left(Au, A^2u, z, t\right)\right\}$$

i.e. $\quad M_{\Omega}^{+}\left(Au, A^2u, z, kt\right) \geq M_{\Omega}^{+}\left(Au, A^2u, z, t\right) \geq M_{\Omega}^{+}\left(Au, A^2u, z, t/k\right)$, also

$$M_{\Omega}^{+}\left(Au,A^{2}u,z,kt\right)\geq M_{\Omega}^{+}\left(Au,A^{2}u,z,t/k\right)\geq M_{\Omega}^{+}\left(Au,A^{2}u,z,t/k^{2}\right)\geq ....M_{\Omega}^{+}\left(Au,A^{2}u,z,t/k^{n}\right).$$

Science $lim_{n\to\infty}M_{\Omega}^{+}\left(Au,A^{2}u,z,t/k^{n}\right)=1.$, so that $M_{\Omega}^{+}\left(Au,A^{2}u,z,kt\right)=1$, which implies, $Au = A^2u$, and hence $A^2u = Au = Su = Tu = u$. Thus u is a CFP of S, T and A.

For uniqueness let $w(\neq u)$ be another CFPof S, T and A. From (4) for $\alpha=1$, we have

$$M_{\Omega}^{+}\left(Au,Aw,z,kt\right)\geq\phi\begin{Bmatrix}M_{\Omega}^{+}\left(Su,Tw,z,t\right);M_{\Omega}^{+}\left(Su,Au,z,t\right);M_{\Omega}^{+}\left(Tw,Aw,z,t\right);\\ M_{\Omega}^{+}\left(Tw,Au,z,t\right);M_{\Omega}^{+}\left(Su,Aw,z,t\right)\end{Bmatrix}$$

$$=\phi\begin{Bmatrix}M_{\Omega}^{+}\left(Au,Aw,z,t\right);M_{\Omega}^{+}\left(Au,Au,z,t\right);M_{\Omega}^{+}\left(Aw,Aw,z,t\right);\\ M_{\Omega}^{+}\left(Aw,Au,z,t\right);M_{\Omega}^{+}\left(Au,Aw,z,t\right)\end{Bmatrix}\geq M_{\Omega}^{+}\left(Au,Aw,z,t\right)$$

ie $M_{\Omega}^{+}\left(u,w,z,kt\right)\geq M_{\Omega}^{+}\left(u,w,z,t\right)$. Therefore by lemma 2.1, we wite $u=w$. That z is a UCFP of S, T and A.

Corollary 3.1: Let $(X,M_{\Omega},*)$ be a complete intuitionistic BF2MS with $M_{\Omega}=(M_{\Omega}^{+},M_{\Omega}^{-})$, $t*t\geq t,\forall t\in[0,1]$. Assume ($S$, T) be two continuous maps. Then S and T possess a CFP in X *provided* there exists a continuous mapping A of X into $S(X) \cap T(X)$ that commutes with S and T. Moreover $(m,p) > 0$, and there exists $0 < k < 1$, s.t.

$$M_{\Omega}^{+}(Ax,Ay,z,kt)\geq\phi\begin{Bmatrix}M_{\Omega}^{+}(S^{p}x,T^{m}y,z,t);M_{\Omega}^{+}(S^{p}x,Ax,z,t);M_{\Omega}^{+}(T^{m}y,\\ Ay,z,t);M_{\Omega}^{+}(T^{m}y,Ax,z,\alpha t);M_{\Omega}^{+}(S^{p}x,Ay,z,(2-\alpha)t)\end{Bmatrix} \tag{12}$$

and

$$M_{\Omega}^{-}(Ax,Ay,z,kt)\leq\phi\begin{Bmatrix}M_{\Omega}^{-}(S^{p}x,T^{m}y,z,t);M_{\Omega}^{-}(S^{p}x,Ax,z,t);M_{\Omega}^{-}(T^{m}y,Ay,z,t);\\ M_{\Omega}^{-}(T^{m}y,Ax,z,\alpha t);M_{\Omega}^{-}(S^{p}x,Ay,z,(2-\alpha)t)\end{Bmatrix} \tag{13}$$

for $x,y,z\in X,t>0,\alpha\in(0,2)$, and $\phi\in\Phi$, as in lemma 2.6 with $\lim_{n\to\infty}M_{\Omega}^{+}(x,y,z,t)=1$ and $\lim_{n\to\infty}M_{\Omega}^{-}(x,y,z,t)=0$. Further suppose that (a) $S^{p}(X)\subseteq T^{m}(X)$ and (b) $T^{m}(X)$ is complete, then S, T, and A have a UCFP.

Proof: Using the conditions in theorem 3.2 for BF2MS, S^p and T^m have a UCFP z s.t. $S^{p}z=T^{m}z=z$. Since $S^{p}(Az)\subseteq A(S^{p}z)=Az$ and $T^{m}(Az)\subseteq A(T^{m}z)=Az$ we conclude that Az is also a CFP of, S^p and T^m. By the uniqueness of z, it implies $S(z)=z$ and $T(z)=z$, so $A(z)=S(z)=T(z)=z$. Hence, S, T, and A have a UCFP. This completes the proof.

Corollary 3.2: Let $(X, M_\Omega, *)$ be a complete intuitionistic BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-)$, $t * t \geq t$ for all $t \in [0,1]$. Let $f : X \to X$ be a self-mapping in X. If $\exists$ a continuous mapping $A : X \to f(X)$ s.t., for $x, y, z \in X$, $t > 0, k \in (0,1)$,

$$M_\Omega^+(Ax, Ay, z, kt) \geq M_\Omega^+(fx, fy, z, t) \quad \text{and} \quad M_\Omega^-(Ax, Ay, z, kt) \leq M_\Omega^+(fx, fy, z, t),$$

then f and A has a UCFP in X.

Proof: This can be obtained by setting $\phi : [0,1]^5 \to [0,1]$ as $\phi(t_1, t_2, t_3, t_4, t_5) = t_1$ in theorem 3.2 for BF2MS.

Corollary 3.3: Let $(X, M_\Omega, *)$ be a complete BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-)$, $t * t \geq t$ for all $t \in [0,1]$. If there exists a continuous mapping $A : X \to X$ s.t., for $x, y, z \in X, t > 0, k \in (0,1), \alpha \in (0,2)$,

$$M_\Omega^+(Ax, Ay, z, kt) \geq \min \left\{ \begin{matrix} M_\Omega^+(x, y, z, t); M_\Omega^+(x, Ax, z, t); M_\Omega^+(y, Ay, z, t); \\ M_\Omega^+(y, Ax, z, \alpha t); M_\Omega^+(x, Ay, z, (2-\alpha)t) \end{matrix} \right\} \tag{14}$$

and
$$M_\Omega^-(Ax, Ay, z, kt) \leq \max \left\{ \begin{matrix} M_\Omega^-(x, y, z, t); M_\Omega^-(x, Ax, z, t); M_\Omega^-(y, Ay, z, t); \\ M^-(y, Ax, z, \alpha t); M_\Omega^-(x, Ay, z, (2-\alpha)t) \end{matrix} \right\} \tag{15}$$

then A has a UCFP.

Proof: This follows by choosing $\phi : [0,1]^5 \to [0,1]$ as $\phi(t_1, t_2, t_3, t_4, t_5) = \min(t_1, t_2, t_3, t_4, t_5)$ in the positive M_Ω^+ and $\phi(t_1, t_2, t_3, t_4, t_5) = \max(t_1, t_2, t_3, t_4, t_5)$ in the negative M_Ω^-, with $f = I$ in theorem 3.2.

4. Real-Life Applications in BF2MS

Example 4.1 (BF2MS in MCDM): A company is evaluating three alternative projects (P_1, P_2, P_3) using a BF2MS approach in MCDM. The evaluation considers the positive criteria (+) (profit potential, innovation, market demand) and negative criteria (−) (risk, environmental impact, implementation complexity). The goal is to determine the most suitable project by balancing positive and negative influences using BF2MS. Let the assign normalized values to each project based on expert opinions are as in Table 1

Criteria	P_1	P_2	P_3
Profit Potential (+)	0.837	0.745	0.923
Innovation (+)	0.945	0.665	0.824
Market Demand (+)	0.749	0.856	0.965
Risk (-)	0.475	0.555	0.391
Environmental Impact (-)	0.645	0.476	0.575
Implementation Complexity (-)	0.568	0.682	0.448

Table 1: Initial Scores for Each Project

Solution: For the solution, we find the following steps

Step 4.1.1: A BF2MS is represented by (P, d^+, d^-), where

- ➢ $P = (P_1, P_2, P_3)$ (set of projects)
- ➢ $d^+(P_1, P_2, P_3)$ represents the positive fuzzy metric (higher values are better)
- ➢ $d^-(P_1, P_2, P_3)$ represents the negative fuzzy metric (lower values are better)

Step 4.1.2: Let the bipolar fuzzy membership function be given by $\mu^+(P_1, P_2, P_3) = e^{-\alpha d^+(P_1, P_2, P_3)}$ and $\mu^-(P_1, P_2, P_3) = 1 - e^{-\beta d^-(P_1, P_2, P_3)}$, where α, β are weight parameters representing the impact of positive and negative influences.

Step 4.1.3: We compute bipolar fuzzy-2 distances by using fuzzy-2 metric computations as follows:

positive fuzzy distance $d^+(P_1, P_1, P_1) = \dfrac{w_1.\text{profit} + w_2.\text{innovation} + w_3.\text{market demand}}{w_1 + w_2 + w_3}$

negative fuzzy distance $d^-(P_1, P_1, P_1) = \dfrac{w_1.\text{risk} + w_2.\text{environment} + w_3.\text{complexity}}{w_1 + w_2 + w_3}$

let us assign weights $w_1 = 0.6,\ w_2 = 0.2,$ and $w_3 = 0.2$ for positive criteria and $w_1 = 0.4,\ w_2 = 0.4,$ and $w_3 = 0.2$ for negative criteria.

Project	d^+	d^-
P_1	0.841	0.562
P_2	0.751	0.549
P_3	0.912	0.476

Table 2: Positive and negative distance for each project

Step 4.1.4: The overall bipolar fuzzy score $S(P_i) = \mu^+(P_i) - \mu^-(P_i)$ for different values of α, β as shown in Table 3

	Project	$\mu^+(P_i) = e^{-\alpha d^+(P_i)}$	$\mu^-(P_i) = 1 - e^{-\beta d^-(P_i)}$	$S(P_i) = \mu^+(P_i) - \mu^-(P_i)$
$\alpha = 1.4$ $\beta = 0.8$	P_1	0.638	0.472	0.166
	P_2	0.645	0.475	0.169

	P_3	0.683	0.495	0.188
$\alpha = 0.8$ $\beta = 1.4$	P_1	0.456	0.366	0.090
	P_2	0.464	0.371	0.093
	P_3	0.514	0.402	0.112
$\alpha = 1$ $\beta = 1$	P_1	0.570	0.435	0.136
	P_2	0.578	0.439	0.139
	P_3	0.621	0.463	0.159

Table 3: Bipolar fuzzy score function

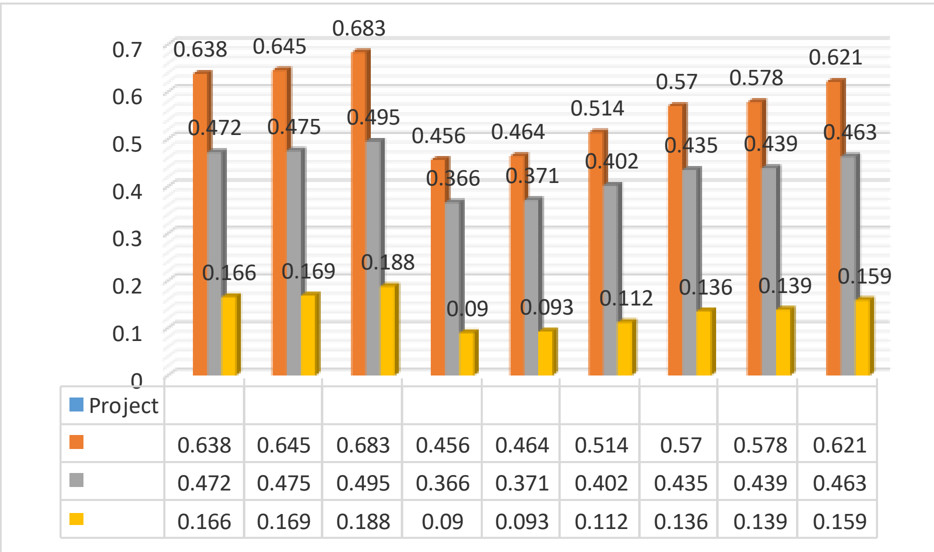

Figure 1: MCDM result

Table 3 indicates that the project P_3 is the optimal choice, the project P_2 is the next best option, and the project P_1 is the least favorable based on the lowest score. Consequently, the project P_3 is the best selection according to the BF2MS assessment methodology. This approach improves decision-making frameworks in practical contexts, including company strategy, risk evaluation, and project selection.

Example 4.2. (BF2MS in Medical Diagnosis-1): A medical diagnosis system evaluates three patients P_1, P_2 and P_3 for two symptoms such as (i) the presence of disease-related symptoms (positive membership, M^{+}_{Ω}) (ii) presence of conflicting or irrelevant symptoms (negative membership, M^{-}_{Ω}).

Calculate the BF2MS (M_Ω^+ and M_Ω^-) for each patient combination at $t=1$ and $t=2$ and determine which group of patients P_1, P_2 and P_3 has the highest positive proximity (M_Ω^+) and lowest negative proximity (M_Ω^-). Also, interpret the results and suggest how this information could be used to group patients for treatment or further analysis.

Solution: For patients P_1, P_2 and P_3 in the set *X*, $M_\Omega^+(P_1,P_2,P_3,t)=\mu_\Omega(P_1,P_2,P_3,t).e^{-t}$ is the membership function for positive symptoms (relevant to disease) and $M_\Omega^-(P_1,P_2,P_3,t)=\nu_\Omega(P_1,P_2,P_3,t).e^{-t}$, for negative symptoms (conflicting /irrelevant) in the BF2M respectively, where $t>0$ is the time decay parameter. Let the fuzzy memberships (μ_Ω and ν_Ω), based on the observed symptoms are as follows in Table 4:

Patients P_1, P_2, P_3	Positive Membership (μ_Ω)	Negative Membership (ν_Ω)
Combination 1: P_1, P_2, P_3	0.851	0.254
Combination 2: P_1, P_3, P_2	0.723	0.485
Combination 3: P_2, P_3 P_1	0.934	0.198

Table 4: Fuzzy memberships

The calculation of BF2M is shown in the table 5:

For real *t*	Combination	M_Ω^+	M_Ω^-
$t=1$	P_1, P_2, P_3	0.313	0.093
	P_1, P_3, P_2	0.266	0.178
	P_2, P_3, P_1	0.344	0.073
$t=2$	P_1, P_2, P_3	0.115	0.034
	P_1, P_3, P_2	0.098	0.066
	P_2, P_3 P_1	0.126	0.027
	P_1, P_2, P_3	0.042	0.013

$t=3$	P_1, P_3, P_2	0.036	0.024
	P_2, P_3, P_1	0.047	0.010

Table 5: Calculation of BF2M

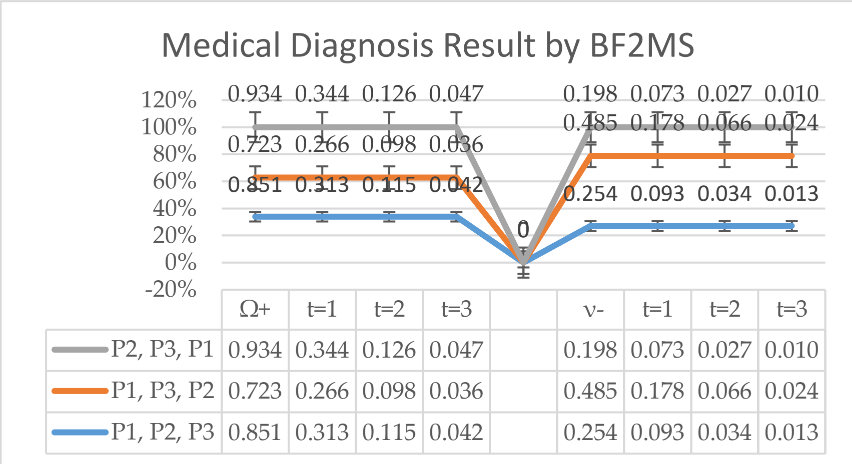

Chart 2: Medical diagnosis result

The research highlights patient clustering and decision-support findings. Patients had the highest positive closeness and the lowest negative proximity, suggesting substantial symptom similarity with few contradicting or irrelevant characteristics. By clustering such patients for comparable treatment regimens, the system optimizes diagnostic closeness for successful therapy the pattern P_2 **,** P_3 P_1 is the best medical diagnostic. The method also reveals subtle patterns in dual behaviors like overlapping illnesses or contradicting symptoms, improving medical diagnosis.

Example 4.3. (BF2MS in Medical Diagnosis-2): A medical diagnosis system analyzes symptoms and diagnostic criteria for a disease. Some symptoms may positively indicate a condition (e.g., high fever for an infection), while others may negatively correlate or suggest alternative conditions (e.g., absence of fever for non-infectious issues). A BF2MS models these dual relationships to help doctors make more precise diagnoses.

Let $X=(P_1,P_2,P_3)$ be a set of three patients. Each patient exhibits symptoms with varying levels of positive and negative memberships to a disease diagnosis. We define M^{+} a degree of positive correlation of symptoms with the disease, M^{-} a degree of negative correlation of symptoms with the disease and t is the weighting factor (e.g. severity or time duration of symptoms).

The degree of positive correlation in BF2MS is $M^{+}(x,y,z,t)=e^{-\alpha(t)\cdot[|d(x,y)+d(y,z)|]}$, where $\alpha(t)>0,$ is a decay constant controlling the rate of decrease, and $d(x,y)=(x-y)^2, \quad \forall x,y\in X$ measures the similarity/difference between patients based on symptom scores. Similarly, the degree of negative correlation $M^{-}(x,y,z,t)=e^{-\beta(t).\left[\left|d(x,y)-d(y,z)\right|\right]}$, where $\beta(t)>0$, is another decay constant controls the effect of opposing relationships. The scores for three key symptoms in relation to a disease as follows:

Symptom	Patient P_1	Patient P_2	Patient P_3
Fever	0.92	0.87	0.79
Cough	0.78	0.69	0.56
Fatigue	0.81	0.75	0.69

Symptom Difference (Euclidean Distance):

$$d(P_1,P_2)=\left(0.92-0.87\right)^2+\left(0.78-0.69\right)^2+\left(0.81-0.75\right)^2=0.0142$$

$$d(P_2,P_3)=\left(0.87-0.79\right)^2+\left(0.69-0.56\right)^2+\left(0.75-0.69\right)^2=0.0269$$

For $t=1,\alpha=0.5\Rightarrow\alpha(t)=0.5$, in a degree of positive correlation $M^{+}(x,y,z,t)=e^{-\alpha(t).\left[\left|d(x,y)+d(y,z)\right|\right]}$, i.e. $M^{+}(P_1,P_2,P_3,t)=e^{-0.5.(0.0411)}=0.97966$ and corresponding $M^{-}(P_1,P_2,P_3,t)=1-0.97966=0.02034.$ For $t=1,\beta=0.3\Rightarrow\beta(t)=0.3$, in a degree of negative correlation $M^{-}(x,y,z,t)=e^{-\beta(t).\left[\left|d(x,y)-d(y,z)\right|\right]}$, $M^{-}(P_1,P_2,P_3,t)=e^{-0.3.(0.0127)}=0.996197$ and corresponding $M^{+}(P_1,P_2,P_3,t)=1-0.996197=0.003803.$

Thus, as a result, a positive membership (M^{+}) value of 0.97966 indicates moderate compatibility among the three patients' symptoms, suggesting a shared likelihood of the disease. In contrast, a value of 0.02034 indicates relatively low opposition, meaning there is little evidence to suggest alternative diagnoses.

Similarly, a negative membership (M^{-}), which is a moderate negative membership value of 0.996197, indicates the potential conflict among the symptoms, which might indicate alternative conditions or a need for further analysis. In contrast, a value of 0.003803 indicates relatively low opposition, meaning there is little evidence to suggest alternative diagnoses. This example demonstrates how BF2MS can model dual relationships in medical diagnoses, improving the accuracy and efficiency of decision-making in healthcare systems.

Conclusion: This study introduces the concept of BF2MSs, a dual-dimensional mathematical framework that fills basic deficiencies in fuzzy metric models. BF2MSs broaden FPTs to dual-interaction systems by integrating attraction-repulsion and compatibility-opposition dynamics. These advances improve complex and nonlinear system stability, convergence, and equilibrium analysis, which benefits social networks, biological ecosystems, control systems, and optimization challenges. BF2MSs also help MCDM and medical diagnosis systems assess illness symptoms and diagnostic criteria by addressing competing aims and ambiguity, boosting decision-support models across disciplines. They are important in mathematical modeling and real-world problem-solving because they apply to topology, variational inequality, and computational intelligence. This paper lays the groundwork for future interdisciplinary research in fuzzy mathematics, nonlinear analysis, and decision sciences by connecting theoretical advances to practical applications.

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