Neutrosophic Sets and Systems, Vol. 98, 2026
University of New Mexico

Integral Type Contractive Condition in ε-Chainable Neutrosophic Metric Space and Common Fixed Point Theorem

Rajesh Kumar Saini[1], Mukesh Kushwaha[2] and Moh. Kasim[3]

[1,2]Department of Mathematical Sciences and Computer Applications

Bundelkhand University, Jhansi, INDIA

*Correspondence: prof.rksaini@bujhansi.ac.in; Tel.: (011-9412322576)

Abstract: Nonlinear Optimization, Game theory, economics and study of differential equations are just a few of the many domains in which fixed point theory (FPT) is essential. It is possible to construct new forms of infinite products by using continuous triangular norms (TN) and continuous triangular co-norms (TC). Banach contraction principal has been established in the context of neutrosophic metric space (NMS) within the framework through the use of these newly define infinite products. We introduced integral type contractive condition in ε-chainable NMS and establishes a common fixed point theorems (CFPTs) in the current work. The result acquired in this study are intended to consolidate and expand upon numerous existing discoveries in the field of NMS.

Key Words: Integral type contractive condition, ε-chainable neutrosophic metric space, Common fixed point theorem

Abstract: Nonlinear Optimization, Game theory, economics and study of differential equations are just a few of the many domains in which fixed point theory (FPT) is essential. It is possible to construct new forms of infinite products by using continuous triangular norms (TN) and continuous triangular co-norms (TC). Banach contraction principal has been established in the context of neutrosophic metric space (NMS) within the framework through the use of these newly define infinite products. We introduced integral type contractive condition in ε-chainable NMS and establishes a common fixed point theorems (CFPTs) in the current work. The result acquired in this study are intended to consolidate and expand upon numerous existing discoveries in the field of NMS.

Key Words: Integral type contractive condition, ε-chainable neutrosophic metric space, Common fixed point theorem

1. Introduction

In 1965, Zadeh [37] introduced the fuzzy set (FS) as a set that is defined by a membership function, serving as the first mathematical formalization of the concept. Although fundamental, the single membership grade of an FS may not be adequate to capture the complete complexity of real-world uncertainty. Kramosil and Michálek [18] introduced fuzzy metric spaces (FMS) that were subsequently refined by George and

Veeramani [9] using continuous TC, building upon this concept. The concept on FMS is explored by Grabiec [8] in 1988.

Atanassov [1, 2] introduced the intuitionistic fuzzy set (IFS) in 1986 to expand the expressive capability of FS theory. Independent degrees of both belongingness and non-belongingness for each element are incorporated into this framework to more effectively model uncertainty. The application of IFS theory rapidly penetrated all domains that were impacted by FS, including metric spaces. By employing the TN and TC structure of George and Veeramani, Park [24] extended the concept of FMS to intuitionistic fuzzy metric spaces (IFMS) and subsequently investigated its fundamental topological properties. For more results on IFS (see [23], [25]).

Heilpern [11] was the first to investigate fuzzy contraction mappings in FPT. Bose and Sahani [5] expanded upon this work, while Alaca et al. [3] demonstrated FPTs in the context of IFMS. Mohamad [21] and a multitude of other researchers [10, 12, 14, 19] have since made substantial contributions to the field of fixed point results for both FMS and IFMS.

Smarandache [29] was the first to introduce the neutrosophic set (NS) in 1998, after recognizing that neither FS nor IFS could entirely resolve issues involving contradictory or indeterminate information. NSs are defined by three independent membership functions: truth (T), indeterminacy (I), and falsity (F) as a generalization of crisp, fuzzy, and intuitionistic sets. Smarandache [30] observed that an NS is reduced to an IFS when its indeterminacy membership grade I(x) is equivalent to the hesitancy degree h(x) of the IFS. Kirişci et al. [15] contributed to the development of the fixed point theory in NMS by defining neutrosophic contractive mappings and establishing corresponding FPTs.

Branciari's [4] integral version of the Banach contraction principle was a significant parallel development in metric FPT. The results of Rhoades [26], Vijayaraju [32], and Djoudi, Aliouche [7] and Saini [28] demonstrate that this seminal work inspired extensive research, resulting in the development of a variety of CFPTs and FPTs for integral-type contractive conditions in diverse spaces.

Our contribution to this ongoing research is the definition of an integral-type contractive condition within the framework of ε-chainable NMS in this paper. We establish a CFPT for four weakly compatible mappings [13]. Our findings are a significant extension and generalization of several well-known theorems in NMS, such as those of Mohamad [22], Kirisci M. [17], Kirişci et al. [15], and Kirişci and Simsek [16].

2. Preliminaries

The following section presents fundamental definitions concerning fuzziness, intuitionistic fuzziness and neutrosophic concepts.

Definition 2.1: A FS $\tilde{F}$ is defined as a mapping $\tilde{F}: X \to [0,1]$, where X is a universe of discourse.

Definition 2.2. ([37]): For a non-empty set X, a FS $\tilde{F}$ is expressed as $\tilde{F} = \{\langle a, \mu_{\tilde{F}}(a)\rangle : a \in X\}$ where $\mu_{\tilde{F}}(a)$ is the membership function that assigns to each element $a \in X$ a degree of membership in [0,1]. If the FS $\tilde{F}$ is both convex and normalized, then it is referred to as a fuzzy number (FN) on real line $\mathbb{R}$.

Definition 2.3. ([1]): An IFS $\tilde{F}^I$ in a non-empty set X is represented as $\tilde{F}^I = \left\{\left\langle a, \mu_{\tilde{F}^I}(a), \nu_{\tilde{F}^I}(a)\right\rangle : a \in X\right\}$, where $\mu_{\tilde{F}^I}: X \to [0,1]$ denotes the membership function and $\nu_{\tilde{F}^I}: X \to [0,1]$ denotes the non-membership function with condition $\mu_{\tilde{F}^I} + \nu_{\tilde{F}^I} \le 1, \forall a \in X.$ The hesitation or indeterminacy degree is given by $h(a) = 1 - \mu_{\tilde{F}^I}(a) - \nu_{\tilde{F}^I}(a).$ An IFS $\tilde{F}^I$ becomes intuitionistic fuzzy number (IFN), if

- An IFN is a special type of subset of the $\mathbb{R}$,

- An IFN is said to be normal if $\mu_{\tilde{F}^I}(a)=1$ and $\nu_{\tilde{F}^I}(a)=0$ for each $a \in R$,
- The membership function $\mu_{\tilde{F}^I}(a)$ is considered convex, if for any $a_1, a_2 \in R, \gamma \in [0,1]$, we have $\mu_{\tilde{F}^I}\left(\gamma a_1+(1-\gamma)a_2\right) \geq \min\left\{\mu_{\tilde{F}^I}(a_1), \mu_{\tilde{F}^I}(a_2)\right\}$.
- The membership function $\mu_{\tilde{F}^I}(a)$ is considered concave, if for any $a_1, a_2 \in R, \gamma \in [0,1]$, we have $\mu_{\tilde{F}^I}\left(\gamma a_1+(1-\gamma)a_2\right) \leq \max\left\{\mu_{\tilde{F}^I}(a_1), \mu_{\tilde{F}^I}(a_2)\right\}$.
- $\mu_{\tilde{F}^I}(a)$ $\mu_{\tilde{F}^I}$ is assumed to be upper semi continuous while $\nu_{\tilde{F}^I}$ is assumed as lower semi continuous,
- $supp\,\mu_{\tilde{F}^I}(a) = cl\left(\{a \in \tilde{F}^I; \mu_{\tilde{F}^I} < 1\}\right)$

An IFS $\tilde{F}^I = \left\{\left\langle x, \mu_{\tilde{F}^I}(x), \nu_{\tilde{F}^I}(x)\right\rangle : x \in X\right\}$ s.t. $\mu_{\tilde{F}^I}(a)$ and $1-\nu_{\tilde{F}^I}(a)$ are IFNs, where $(1-\nu_{\tilde{F}^I})(a) = 1-\nu_{\tilde{F}^I}(a)$ and $\mu_{\tilde{F}^I}(a)+\mu_{\tilde{F}^I}(a) \leq 1$ is called an IFN.

Definition 2.4. ([29]): Let X be non-empty set and $a \in X$. A NS $\tilde{F}_N$ is expressed as $\tilde{F}_N = \left\{\langle a, \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\rangle : a \in X\right\}$, for each number a in X and $\mu_{\tilde{F}_N}(a)$, $\nu_{\tilde{F}_N}(a)$ and $\omega_{\tilde{F}_N}(a)$ belongs $]^{-}0,1^{+}[$ where $\mu_{\tilde{F}_N}(a): X \to]0^{-},1^{+}[$ represents the truth membership (TM), $\nu_{\tilde{F}_N}(a): X \to]0^{-},1^{+}[$ represents the indeterminacy membership (IM) and $\omega_{\tilde{F}_N}(a): X \to]0^{-},1^{+}[$ represents the falsity membership (FM) in $\tilde{F}_N$ respectively with condition $^{-}0 \leq \mu_{\tilde{F}_N}(a)+\nu_{\tilde{F}_N}(a)+\omega_{\tilde{F}_N}(a) \leq 3^{+}$.

In 2010, Wang et.al [34, 35] and Deli & S¸uba [6] introduce the single valued neutrosophic numbers (SVNN) which provides a fundamental for applying neutrosophic theory in practical settings. Later Ye [36], introduced the notion of simplified NSs, characterized by three real-valued components within [0,1]. However the improved NSs' operations may be impractical at certain times.

Definition 2.5. Let X be non-empty set and $a \in \mathbb{R}$. A NS in $\mathbb{R}$ is represented as $\tilde{F}_N = \left\{\langle a, \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\rangle : a \in \mathbb{R}\right\}$, for each number $a \in \mathbb{R}$, and $\mu_{\tilde{F}_N}(a)$, $\nu_{\tilde{F}_N}(a)$, $\omega_{\tilde{F}_N}(a)$ belongs to $]^{-}0,1^{+}[$ where $\mu_{\tilde{F}_N}(a): X \to]0^{-},1^{+}[$ represents the TM, $\nu_{\tilde{F}_N}(a): X \to]0^{-},1^{+}[$ represents the IM and $\omega_{\tilde{F}_N}(a): X \to]0^{-},1^{+}[$ represents the FM in $\tilde{F}_N$ respectively with condition $0 \leq \mu_{\tilde{F}_N}(a)+\nu_{\tilde{F}_N}(a)+\omega_{\tilde{F}_N}(a) \leq 3$. For continuous SVNS, $\tilde{F}_N = \int_{\tilde{F}_N} \left\langle \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\right\rangle \big/ a : a \in \mathbb{R}$ If X is discrete then SVNS $\tilde{F}_N = \sum_{i=1}^{n} \left\langle \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\right\rangle \big/ a_i : a_i \in \mathbb{R}$.
If NS has only one element then in simplified form $\tilde{F}_N$ express as $\left\langle \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\right\rangle$ for each $a \in \mathbb{R}$.

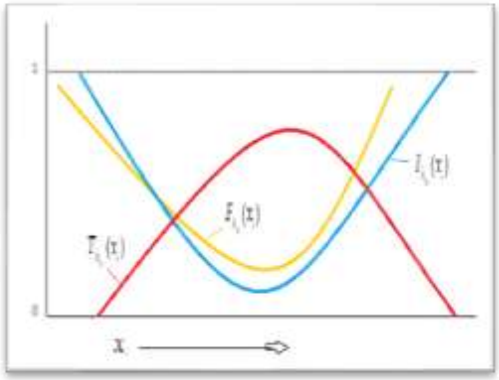

Figure 1: Neutrosophic set

Thus it is evident that NS extends the concept of IFSs within [0,1]. For all $a \in \tilde{F}_N$, a NS $\tilde{F}_N(V)$ contains the NS $\tilde{F}_N(U)$, $(U \subseteq V)$

$$\begin{pmatrix} \inf\left(\mu_{\tilde{F}_N(U)}(a)\right) \le \inf\left(\mu_{\tilde{F}_N(V)}(a)\right) \quad \text{and} \quad \sup\left(\mu_{\tilde{F}_N(U)}(a)\right) \le \sup\left(\mu_{\tilde{F}_N(V)}(a)\right) \\ \inf\left(\nu_{\tilde{F}_N(U)}(a)\right) \ge \inf\left(\nu_{\tilde{F}_N(V)}(a)\right) \quad \text{and} \quad \sup\left(\nu_{\tilde{F}_N(U)}(a)\right) \ge \sup\left(\nu_{\tilde{F}_N(V)}(a)\right) \\ \inf\left(\omega_{\tilde{F}_N(U)}(a)\right) \ge \inf\left(\omega_{\tilde{F}_N(V)}(a)\right) \quad \text{and} \quad \sup\left(\omega_{\tilde{F}_N(U)}(a)\right) \ge \sup\left(\omega_{\tilde{F}_N(V)}(a)\right) \end{pmatrix}$$

$$\tilde{F}_N(U) + \tilde{F}_N(V) = \left\langle \begin{array}{l} \mu_{\tilde{F}_N(U)}(a) + \mu_{\tilde{F}_N(V)}(a) - \mu_{\tilde{F}_N(U)}(a).\mu_{\tilde{F}_N(V)}(a), \\ \nu_{\tilde{F}_N(U)}(a) + \nu_{\tilde{F}_N(V)}(a) - \nu_{\tilde{F}_N(U)}(a).\nu_{\tilde{F}_N(V)}(a), \\ \omega_{\tilde{F}_N(U)}(a) + \omega_{\tilde{F}_N(V)}(a) - \omega_{\tilde{F}_N(U)}(a).\omega_{\tilde{F}_N(V)}(a) \end{array} \right\rangle$$

$$\tilde{F}_N(U).\tilde{F}_N(V) = \left\langle \mu_{\tilde{F}_N(U)}(a).\mu_{\tilde{F}_N(V)}(a), \nu_{\tilde{F}_N(U)}(a).\nu_{\tilde{F}_N(V)}(a), \omega_{\tilde{F}_N(U)}(a).\omega_{\tilde{F}_N(V)}(a) \right\rangle$$

$$\alpha.\tilde{F}_N(U) = \left\langle 1-\left(1-\mu_{\tilde{F}_N(U)}(a)\right)^{\alpha}, 1-\left(1-\nu_{\tilde{F}_N(U)}(a)\right)^{\alpha}, 1-\left(1-\omega_{\tilde{F}_N(U)}(a)\right)^{\alpha} \right\rangle, \quad \text{for } \alpha > 0,$$

$$\left(\tilde{F}_N(U)\right)^{\alpha} = \left\langle \mu^{\alpha}_{\tilde{F}_N(U)}(a), \nu^{\alpha}_{\tilde{F}_N(U)}(a), \omega^{\alpha}_{\tilde{F}_N(U)}(a) \right\rangle, \text{ for } \alpha > 0.$$

Definition 2.6. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is continuous TN if '$*$' is satisfying:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a*1 = a$ for all $a \in [0,1]$,

(iv) $a*b = c*d$ whenever $a \le c$ and $b \le d$, $\forall a,b,c,d \in [0,1]$.

Definition 2.7. A binary operation $\Diamond:[0,1]\times[0,1]\to[0,1]$ is continuous TC if $\Diamond$ is satisfying:

(i) $\Diamond$ is commutative and associative,

(ii) $\Diamond$ is continuous,

(iii) $a\Diamond 0 = a$ for all $a \in [0,1]$

(iv) $a\Diamond b = c\Diamond d$ whenever $a \le c = b \le d$ $\forall a,c,b,d \in [0,1]$.

From above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then $\exists$ $0 < \varepsilon_3, \varepsilon_4 < 1$ s.t. $\varepsilon_1 * \varepsilon_3 \ge \varepsilon_2$ and $\varepsilon_1 \ge \varepsilon_4 \Diamond \varepsilon_2$. Further if we choose $\varepsilon_5 \in (0,1)$, then $\exists$ $\varepsilon_6, \varepsilon_7 \in (0,1)$ s.t. $\varepsilon_6 * \varepsilon_6 \ge \varepsilon_5$ and

$\varepsilon_7 \Diamond \varepsilon_7 \geq \varepsilon_5$.

Remark 2.1 [23]. For $l,m,n,p \in [0,1]$, take $*$ and $\Diamond$ are continuous TN and TC, respectively

(i) If $l > m$, then there are n,p s.t. $l * n \geq m$ and $l \geq m \Diamond p$.

(ii) There are s,m ss.t. $m * m \geq l$ and $l \geq s \Diamond s$.

Example 2.1 [7]: Assume $X = N$. Define $x * y = \max\{0, x+y-1\}$, $\forall x,y \in [0,1]$. Again let $\tilde{F}$ be FS on $X \times X \times (0,\infty)$ is defined as $\tilde{F}(a,b,t) = \begin{cases} a/b, & \text{if } a \leq b \\ b/a, & \text{if } a \geq b \end{cases}$, $\forall a,b \in X$, $t > 0$. Then $(X, \tilde{F}, *)$ is a FMS described as $\tilde{F}(a,b,t) = \dfrac{t}{t + d(a,b)}$, where $d(a,\ b)$ is a MS in X, $\forall a,b \in X$.

Remark 2.2: Every FMS $(X, \tilde{F}, *)$ is an IFS of the form $(X, \tilde{F}, 1-\tilde{F}, *, \Diamond)$ s.t. '$*$' a TN and '$\Diamond$' a TC are associated, i.e. $a * b = 1-(1-a)\Diamond(1-b)$, $\forall\ a,b \in X$.

Definition 2.8 ([22]). Let $\tilde{F}_N = \{\langle a, \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\rangle / a \in X\}$, be a NS for an arbitrary set X s.t. $\tilde{F}_N = X \times X \times R^+ \to [0,1]$. Let $*$ and $\Diamond$ are continuous TN and continuous TC, respectively. The four tuples $\Im = (X, \tilde{F}_N, *, \Diamond)$ is said to be a NMS, when the following conditions satisfied for all $a,b,c \in X$,

(i) $0 \leq \mu_{\tilde{F}_N}(a,b,\lambda) \leq 1$, $0 \leq \nu_{\tilde{F}_N}(a,b,\lambda) \leq 1$, $0 \leq \omega_{\tilde{F}_N}(a,b,\lambda) \leq 1$, $\forall \lambda \in R^+$,

(ii) $0 < \mu_{\tilde{F}_N}(a,b,\lambda) + \nu_{\tilde{F}_N}(a,b,\lambda) + \omega_{\tilde{F}_N}(a,b,\lambda) \leq 3$, (for $\lambda \in R^+$),

(iii) $\mu_{\tilde{F}_N}(a,b,\lambda) = 1$ (for $\lambda > 0$), iff $a = b$,

(iv) $\mu_{\tilde{F}_N}(a,b,\lambda) = \mu_{\tilde{F}_N}(b,a,\lambda)$ (for $\lambda > 0$)

(v) $\mu_{\tilde{F}_N}(a,c,\lambda+\eta) \geq \mu_{\tilde{F}_N}(a,b,\lambda) * \mu_{\tilde{F}_N}(b,c,\eta)$ (for $\lambda,\eta > 0$),

(vi) $\mu_{\tilde{F}_N}(a,b,.) : [0,\infty) \to [0,1]$ is continuous,

(vii) $\lim_{\lambda\to\infty} \mu_{\tilde{F}_N}(a,b,\lambda) = 1$ ($\forall\ \lambda > 0$),

(viii) $\nu_{\tilde{F}_N}(a,b,\lambda) = 0$ (for $\lambda > 0$), iff $a = b$,

(ix) $\nu_{\tilde{F}_N}(a,b,\lambda) = \nu_{\tilde{F}_N}(b,a,\lambda)$ (for $\lambda > 0$)

(x) $\nu_{\tilde{F}_N}(a,b,\lambda) \Diamond \nu_{\tilde{F}_N}(b,c,\eta) \geq \nu_{\tilde{F}_N}(a,c,\lambda+\eta)$ (for $\lambda,\eta > 0$),

(xi) $\nu_{\tilde{F}_N}(a,b,.) : [0,\infty) \to [0,1]$ is continuous,

(xii) $\lim_{\lambda\to\infty} \nu_{\tilde{F}_N}(a,b,\lambda) = 0$ ($\forall\ \lambda > 0$),

(xiii) $\omega_{\tilde{F}_N}(a,b,\lambda) = 0$ (for $\lambda > 0$), iff $a = b$,

(xiv) $\omega_{\tilde{F}_N}(a,b,\lambda) = \omega_{\tilde{F}_N}(b,a,\lambda)$ (for $\lambda > 0$)

(xv) $\omega_{\tilde{F}_N}(a,b,\lambda) \Diamond \omega_{\tilde{F}_N}(b,c,\eta) \geq \omega_{\tilde{F}_N}(a,c,\lambda+\eta)$ (for $\lambda,\eta > 0$),

(xvi) $\omega_{\tilde{F}_N}(a,b,.) : [0,\infty) \to [0,1]$ is continuous,

(xvii) $\lim_{\lambda\to\infty} \omega_{\tilde{F}_N}(a,b,\lambda) = 0$ ($\forall\ \lambda > 0$),

(xviii) If $\lambda \leq 0$, then $\mu_{\tilde{F}_N}(a,b,\lambda) = 0$, $\nu_{\tilde{F}_N}(a,b,\lambda) = 1$, $\omega_{\tilde{F}_N}(a,b,\lambda) = 1$, $\forall \lambda \in R^+$,

Then the set $\tilde{F}_N = (\mu_{\tilde{F}_N}, \nu_{\tilde{F}_N}, \omega_{\tilde{F}_N})$ is called NM on X.

The function $\mu_{\tilde{F}_N}(a,b,\lambda)$ denotes the degree of nearness, $\nu_{\tilde{F}_N}(a,b,\lambda)$ denotes the degree of neutralness and $\omega_{\tilde{F}_N}(a,b,\lambda)$ denotes the degree of non-nearness between a,b with respect to λ.

Definition 2.9 [22]. Let $\Im$ be a NMS, $0<\varepsilon<1$, $\lambda>0$ and $a\in X$. The set $D(a,\varepsilon,\lambda)=\{b\in X:\mu_{\tilde{F}_N}(a,b,\lambda)>1-\varepsilon,\ \nu_{\tilde{F}_N}(a,b,\lambda)<\varepsilon,\ \omega_{\tilde{F}_N}(a,b,\lambda)<\varepsilon\}$ is said to be the open ball (center a and radius ε with respect to λ).

Lemma 2.1 [22]. Every open ball $D(a,\varepsilon,\lambda)$ is open set.

Definition 2.10 [22]. Let $\{a_n\}$ be a sequence in $\Im=(X,\tilde{F}_N,*,\Diamond)$, then the sequence converges to a point $a\in X$ iff for a given $\varepsilon\in(0,1)$, $\lambda>0$ there exists $n_0\in N$ s.t. for all $n\ge n_0$

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda)>1-\varepsilon, \text{ or } \lim_{n\to\infty}\mu_{\tilde{F}_N}(a_n,a_m,\lambda)=1 \\ \nu_{\tilde{F}_N}(a,b,\lambda)<\varepsilon \text{ or } \lim_{n\to\infty}\nu_{\tilde{F}_N}(a_n,a_m,\lambda)=0 \\ \omega_{\tilde{F}_N}(a,b,\lambda)<\varepsilon \text{ or } \lim_{n\to\infty}\omega_{\tilde{F}_N}(a_n,a_m,\lambda)=0 \end{pmatrix} \text{ as } \lambda\to\infty \tag{1}$$

Definition 2.11 [22]. Let $\Im=(X,\tilde{F}_N,*,\Diamond)$, be a NMS. A sequence $\{a_n\}$ in X is called a Cauchy sequence (CS) if for each $\varepsilon>0$, $\lambda>0$ there exists $n_0\in N$ s.t. $\mu_{\tilde{F}_N}(a_n,b_m,\lambda)>1-\varepsilon$, $\nu_{\tilde{F}_N}(a_n,b_m,\lambda)<\varepsilon$, and $\omega_{\tilde{F}_N}(a_n,b_m,\lambda)<\varepsilon$, for all $n,m\ge n_0$. A NMS $\Im$ is called complete if every CS is a convergent sequence.

Lemma 2.2. Let $\{a_n\}$ be a sequence in $\Im=(X,\tilde{F}_N,*,\Diamond)$, with (vii, xii, xviii). If there is a number q where

$$q\in(0,1) \text{ s.t. } \begin{pmatrix} \mu_{\tilde{F}_N}(a_{n+1},a_{n+2},q\lambda)\ge\mu_{\tilde{F}_N}(a_n,a_{n+1},\lambda), \\ \nu_{\tilde{F}_N}(a_{n+1},a_{n+2},q\lambda)\le\nu_{\tilde{F}_N}(a_n,a_{n+1},\lambda), \\ \omega_{\tilde{F}_N}(a_{n+1},a_{n+2},q\lambda)\le\omega_{\tilde{F}_N}(a_n,a_{n+1},\lambda) \end{pmatrix} \text{ for all } \lambda>0 \text{ and } n=0,1,2......., \tag{2}$$

then $\{a_n\}$ is a CS in X.

Proof: Let p be any positive integer, then by repeated application of (v, x, xv) and in view of (2), we have

$$\begin{aligned}\mu_{\tilde{F}_N}(a_n,a_{n+p},\lambda) &\ge \mu_{\tilde{F}_N}(a_n,a_{n+1},\lambda/2)*\mu_{\tilde{F}_N}(a_{n+1},a_{n+p},\lambda/2)\\ &\ge \mu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)*\mu_{\tilde{F}_N}(a_{n+1},a_{n+2},\lambda/2^2)*\mu_{\tilde{F}_N}(a_{n+2},a_{n+p},\lambda/2^2)\\ &\ge \mu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)*\mu_{\tilde{F}_N}(a_1,a_2,\lambda/2^2q^{n+1})*\mu_{\tilde{F}_N}(a_2,a_3,\lambda/2^3q^{n+2})*\mu_{\tilde{F}_N}(a_{n+3},a_{n+p},\lambda/2^3)\end{aligned}$$

Continuing this procedure, we obtain

$$\mu_{\tilde{F}_N}(a_n,a_{n+p},\lambda)\ge\mu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)*\mu_{\tilde{F}_N}(a_1,a_2,\lambda/2^2q^{n+1})*\mu_{\tilde{F}_N}(a_2,a_3,\lambda/2^3q^{n+2})*....*\mu_{\tilde{F}_N}(a_{p-1},a_{n+p},\lambda/2^pq^{n+p-1})$$

since * is the continuous TN and $\mu_{\tilde{F}_N}(a,b,.):[0,\infty)\to[0,1]$ is continuous, letting $\lim_{n\to\infty}$ we have

$$\lim_{n\to\infty}\mu_{\tilde{F}_N}(a_n,a_{n+p},\lambda)\ge 1*1*1*........*1=1. \tag{I}$$

similarly

$$\begin{aligned}\nu_{\tilde{F}_N}(a_n,a_{n+p},\lambda) &\le \nu_{\tilde{F}_N}(a_n,a_{n+1},\lambda/2)\Diamond\nu_{\tilde{F}_N}(a_{n+1},a_{n+p},\lambda/2)\\ &\le \nu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)\Diamond\nu_{\tilde{F}_N}(a_{n+1},a_{n+2},\lambda/2^2)\Diamond\nu_{\tilde{F}_N}(a_{n+2},a_{n+p},\lambda/2^2)\\ &\le \nu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)\Diamond\nu_{\tilde{F}_N}(a_1,a_2,\lambda/2^2q^{n+1})\Diamond\nu_{\tilde{F}_N}(a_2,a_3,\lambda/2^3q^{n+2})\Diamond\nu_{\tilde{F}_N}(a_{n+3},a_{n+p},\lambda/2^3)\end{aligned}$$

Continuing this procedure, we obtain

$$\nu_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le \nu_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \nu_{\tilde{F}_N}(a_1, a_2, \lambda/2^2 q^{n+1}) \diamond \nu_{\tilde{F}_N}(a_2, a_3, \lambda/2^3 q^{n+2}) \diamond \ldots\ldots \diamond \nu_{\tilde{F}_N}(a_{p-1}, a_{n+p}, \lambda/2^p q^{n+p-1})$$

Since $\diamond$ is continuous TC and $\nu_{\tilde{F}_N}(a,b,.):[0,\infty) \to [0,1]$ is continuous, letting $\lim_{n\to\infty}$ we have

$$\lim_{n\to\infty} \nu_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le 0 \diamond 0 \diamond 0 \diamond \ldots\ldots\ldots \diamond 0 = 0. \quad \text{(II)}$$

and $\omega_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le \omega_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/2) \diamond \omega_{\tilde{F}_N}(a_{n+1}, a_{n+p}, \lambda/2)$

$$\le \omega_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \omega_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/2^2) \diamond \omega_{\tilde{F}_N}(a_{n+2}, a_{n+p}, \lambda/2^2)$$

$$\le \omega_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \omega_{\tilde{F}_N}(a_1, a_2, \lambda/2^2 q^{n+1}) \diamond \omega_{\tilde{F}_N}(a_2, a_3, \lambda/2^3 q^{n+2}) \diamond \omega_{\tilde{F}_N}(a_{n+3}, a_{n+p}, \lambda/2^3)$$

Continuing this procedure, we obtain

$$\omega_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le \omega_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \omega_{\tilde{F}_N}(a_1, a_2, \lambda/2^2 q^{n+1}) \diamond \omega_{\tilde{F}_N}(a_2, a_3, \lambda/2^3 q^{n+2}) \diamond \ldots\ldots \diamond \omega_{\tilde{F}_N}(a_{p-1}, a_{n+p}, \lambda/2^p q^{n+p-1})$$

Since $\diamond$ is CTC and $\omega_{\tilde{F}_N}(a,b,.):[0,\infty) \to [0,1]$ is continuous, letting $\lim_{n\to\infty}$ we have

$$\lim_{n\to\infty} \omega_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le 0 \diamond 0 \diamond 0 \diamond \ldots\ldots\ldots \diamond 0 = 0. \quad \text{(III)}$$

From (I), (II) and (III) shows that $\{a_n\}$ is a CS and thus the lemma is proved.

Lemma 2.3: If for all $a,b \in X$, $\lambda > 0$ and for a number $q \in (0,1)$ in NMS $(X, \tilde{F}_N, *, \diamond)$, then

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,q\lambda) \ge \mu_{\tilde{F}_N}(a,b,\lambda), \\ \nu_{\tilde{F}_N}(a,b,q\lambda) \le \nu_{\tilde{F}_N}(a,b,\lambda), \\ \omega_{\tilde{F}_N}(a,b,q\lambda) \le \omega_{\tilde{F}_N}(a,b,\lambda) \end{pmatrix} \Rightarrow a = b.$$

Proof: In view of conditions (*v*, *x*, *xv*), we have $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) \ge \mu_{\tilde{F}_N}(a,b,\lambda/q) * \mu_{\tilde{F}_N}(a,b,\lambda/q^2), \\ \nu_{\tilde{F}_N}(a,b,\lambda) \le \nu_{\tilde{F}_N}(a,b,\lambda/q) * \nu_{\tilde{F}_N}(a,b,\lambda/q^2), \\ \omega_{\tilde{F}_N}(a,b,\lambda) \le \omega_{\tilde{F}_N}(a,b,\lambda/q) * \omega_{\tilde{F}_N}(a,b,\lambda/q^2) \end{pmatrix}$

Proceeding in the same way, we obtain, for $n = 1, 2, 3........$ $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) \ge \mu_{\tilde{F}_N}(a,b,\lambda/q^n), \\ \nu_{\tilde{F}_N}(a,b,\lambda) \le \nu_{\tilde{F}_N}(a,b,\lambda/q^n), \\ \omega_{\tilde{F}_N}(a,b,\lambda) \le \omega_{\tilde{F}_N}(a,b,\lambda/q^n) \end{pmatrix}.$

By noting $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda/q^n) \to 1, \\ \nu_{\tilde{F}_N}(a,b,\lambda/q^n) \to 0, \\ \omega_{\tilde{F}_N}(a,b,\lambda/q^n) \to 0 \end{pmatrix}$ as $n \to \infty$. It follows that $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) = 1, \\ \nu_{\tilde{F}_N}(a,b,\lambda) = 0, \\ \omega_{\tilde{F}_N}(a,b,\lambda) = 0 \end{pmatrix}$ for all $\lambda > 0$. Therefore by (iii), (viii) and (xiii), $a = b$.

3. Neutrosophic Contractive Mapping (NCM)

The following definitions and results are given:

Definition 3.1. Let $\Im = (X, \tilde{F}_N, *, \diamond)$ be the NMS. The mapping $f: X \to X$ is called NC if there exists $\delta \in (0,1)$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(f(a), f(b), \lambda) \geq \delta\left(\mu_{\tilde{F}_N}(a,b,\lambda)\right) \\ \nu_{\tilde{F}_N}(f(a), f(b), \lambda) \leq \delta\left(\nu_{\tilde{F}_N}(a,b,\lambda)\right) \\ \omega_{\tilde{F}_N}(f(a), f(b), \lambda) \leq \delta\left(\omega_{\tilde{F}_N}(a,b,\lambda)\right) \end{pmatrix} \text{ for each } a,b \in X \text{ and } \lambda > 0.$$

Here δ is said to be contractive constant of f and $0 < \delta < 1$.

Definition 3.2. Let $\Im = (X, \tilde{F}_N, *, \diamond)$ be the NMS and let $f : X \to X$ is a NC mapping. There exists $c \in X$ s.t. $f(c) = c$, then c is called neutrosophic fixed point (NFP) of *f*.

Proposition 3.1. Suppose f is a NC. Then f^n is also a NC. Furthermore if k is constant for f, then k^n is constant for f^n.

Proposition3.2. Suppose f is a NC and $a \in X$. Then $f[D(a,\varepsilon,\lambda)] \subset D(a,\varepsilon,\lambda)$ for large enough value of ε.

Proposition 3.3. The inclusion $f^n[D(a,\varepsilon,\lambda)] \subset D(f^n(a), \varepsilon^*, \lambda)$ is hold for all *n*, where $\varepsilon^* = \delta^n \times \varepsilon$.

Lemma 3.1: Let $(X, \tilde{F}_N, *, \diamond)$ be a NMS and $\{b_n\}$ be a sequence in *X*. There exists a number $q \in X$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(b_{n+2}, b_{n+1}, q\lambda) \geq \mu_{\tilde{F}_N}(b_{n+1}, b_n, \lambda), \\ \nu_{\tilde{F}_N}(b_{n+2}, b_{n+1}, q\lambda) \leq \mu_{\tilde{F}_N}(b_{n+1}, b_n, \lambda), \\ \omega_{\tilde{F}_N}(b_{n+2}, b_{n+1}, q\lambda) \leq \omega_{\tilde{F}_N}(b_{n+1}, b_n, \lambda) \end{pmatrix} \text{for all } \lambda > 0, \text{ and } n = 1,2,3......,$$

then $\{b_n\}$ is a Cauchy sequence in *X*.

Definition 3.3. Let us choose two NMS $(X, \tilde{F}_{N_1}, *, \diamond)$ and $(Y, \tilde{F}_{N_2}, *, \diamond)$. Let Δ_1 the uniformly generated by $\Im_i (i = 1,2)$. A mapping $f : X \to Y$ is uniformly continuous with respect to Δ_1 and Δ_2 iff for a given $\varepsilon_2 \in (0,1)$ *and* $\lambda_2 > 0$, there exists $\varepsilon_1 \in (0,1)$ *and* $\lambda_1 > 0$, s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_{N_1}}(a,b,\lambda_1) > 1-\varepsilon_1 \text{ implies } \mu_{\tilde{F}_{N_2}}(a,b,\lambda_2) > 1-\varepsilon_2, \\ \nu_{\tilde{F}_{N_1}}(a,b,\lambda_1) < \varepsilon_1 \text{ implies } \nu_{\tilde{F}_{N_2}}(a,b,\lambda_2) < \varepsilon_2, \\ \omega_{\tilde{F}_{N_1}}(a,b,\lambda_1) < \varepsilon_1 \text{ implies } \omega_{\tilde{F}_{N_2}}(a,b,\lambda_2) < \varepsilon_2 \end{pmatrix} \text{ for each } a,b \in X.$$

Definition 3.4[39]: Let $\Im = (X, \tilde{F}_N, *, \diamond)$ be a complete NMS and $\varepsilon > 0$. A finite sequence $a = a_0, a_1, a_2 .....a_n = b$ is called $\varepsilon-$chainable from *a* to *b* if

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(a,b,\lambda) < \varepsilon, \\ \omega_{\tilde{F}_N}(a,b,\lambda) < \varepsilon, \end{pmatrix} \text{ for all } \lambda > 0 \text{ and } i = 1,2,3...n.$$

A NMS $\Im = (X, \tilde{F}_N, *, \diamond)$ is called $\varepsilon-$chainable if for $a,b \in X$, there exists a $\varepsilon-$chain from *a* to b.

4. Main Result

For the proof of main result, the following definitions for compatibility [13] and weak compatibility are necessary.

Definition 4.1: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\diamond)$ are called compatible if

$$\begin{pmatrix} \lim_{n\to\infty}\mu_{\tilde{F}_N}(\mathrm{AS}a_n,\mathrm{SA}a_n,\lambda)=1, \\ \lim_{n\to\infty}\nu_{\tilde{F}_N}(\mathrm{AS}a_n,\mathrm{SA}a_n,\lambda)=0, \\ \lim_{n\to\infty}\omega_{\tilde{F}_N}(\mathrm{AS}a_n,\mathrm{SA}a_n,\lambda)=0, \end{pmatrix} \text{ whenever } \{a_n\} \text{ is a sequence in } X \text{ s.t.}$$

$\lim_{n\to\infty}\mathrm{A}a_n=\mathrm{S}a_n=a,$ for some a in X.

Definition 4.2: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\diamond)$ are called weak commuting if

$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\ge\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda), \\ \nu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda), \\ \omega_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda), \end{pmatrix} \text{ for all } a \text{ in } X \text{ and } \lambda>0.$$

Definition 4.3.[27]: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\diamond)$ are called point wise R-weakly commuting if $\exists,\mathrm{R}>0$, s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\ge\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda/\mathrm{R}), \\ \nu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda/\mathrm{R}), \\ \omega_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda/\mathrm{R}), \end{pmatrix} \text{ for all } a \text{ in } X \text{ and } \lambda>0.$$

Definition 4.4: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\diamond)$ are called reciprocal continuous on X if $\lim_{n\to\infty}\mathrm{AS}a_n=\mathrm{A}a$ and $\lim_{n\to\infty}\mathrm{SA}a_n=\mathrm{S}a$ whenever $\{a_n\}$ is a sequence in X s.t. $\lim_{n\to\infty}\mathrm{A}a_n=\lim_{n\to\infty}\mathrm{S}a_n=a$ for some a in X.

Lemma 4.1: Let $\psi:R^+\to R^+$ be a left continuous function s.t. $\psi(\lambda)>\lambda$ for every $\lambda>0$, then $\lim_{n\to\infty}\psi^n(\lambda)=1,$ where ψ^n denotes the n-times repeated composition of ψ with itself.

Theorem 4.1: Let S and T be two self-continuous mappings of a complete $\varepsilon-$chainable NMS $(X,\tilde{F}_N,*,\diamond)$ with $t*t\ge t$ and $(1-t)\diamond(1-t)\le(1-t)$, $\forall t\in[0,1]$. Let A and B be two self-mappings of X satisfying the following conditions:

(i) $A(X)\subseteq S(X)$ and $B(X)\subseteq A(X)$

(ii) for all $a,b\in X,\lambda>0$ and $k\in(0,1)$ $\exists,\psi:[0,1]\to[0,1],$ $\psi(0)=0,$ and $\psi(s)>s$ (a left continuous function)

$$\forall,s>0 \text{ s.t. } \begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{B}b,k\lambda)}\phi(\theta)d\theta\ge\psi\left(\int_0^{\mu_{\tilde{F}_N}(a,b,\lambda)}\phi(\theta)d\theta\right) \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{B}b,k\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\nu_{\tilde{F}_N}(a,b,\lambda)}\phi(\theta)d\theta\right) \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{B}b,k\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\omega_{\tilde{F}_N}(a,b,\lambda)}\phi(\theta)d\theta\right) \end{pmatrix}$$

where $\phi(\theta):R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0<\int_0^{\varepsilon}\phi(\theta)d\theta<1$, for all $\varepsilon>0$ and

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) = r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda),\mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda),\mu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{B}b,\lambda), \\ \max\{\mu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda),\mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{B}b,\lambda)\}\} \end{bmatrix} \\ \nu_{\tilde{F}_N}(a,b,\lambda) = r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda),\nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{B}b,\lambda), \\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda),\nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{B}b,\lambda)\}\} \end{bmatrix} \\ \omega_{\tilde{F}_N}(a,b,\lambda) = r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda),\omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda),\omega_{\tilde{F}_N}(\mathrm{T}b,\mathrm{B}b,\lambda), \\ \max\{\omega_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda),\omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{B}b,\lambda)\}\} \end{bmatrix} \end{pmatrix} \tag{3}$$

where $r:[0,1]\to[0,1]$, is continuous function s.t. $r(a)=\begin{cases} > a, & if\ \ a\in[0,1) \\ 1, & if\ \ a=1. \end{cases}$, then the continuity of one of the mapping in compatible pair $\{\mathrm{A},\mathrm{S}\}$ or $\{\mathrm{B},\mathrm{T}\}$ implies their reciprocal continuity, and the unique CFP of $\mathrm{A},\mathrm{S},\mathrm{B}$ and T.

Proof: Let $n_0\in X$ be an arbitrary point of *X*. From (i) we can construct a sequence $\{b_n\}$ in *X* as follows:

$$b_{2n}=\mathrm{A}a_{2n}=\mathrm{S}a_{2n+1},\quad b_{2n+1}=\mathrm{B}a_{2n+1}=\mathrm{T}a_{2n+2},\quad \text{for all } n=1,2,3......$$

We define $\begin{pmatrix} (\mu_{\tilde{F}_N})_{2n}(q\lambda)=\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda), \\ (\nu_{\tilde{F}_N})_{2n}(q\lambda)=\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda), \\ (\omega_{\tilde{F}_N})_{2n}(q\lambda)=\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda) \end{pmatrix}$ for $b_{2n}\neq b_{2n+1}$. Let us take $a=a_{2n},\ b=a_{2n+1}$ in (ii),

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a_{2n},\mathrm{B}a_{2n+1},k\lambda)}\phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)}\phi(\theta)d\theta\right) \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a_{2n},\mathrm{B}a_{2n+1},k\lambda)}\phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)}\phi(\theta)d\theta\right) \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a_{2n},\mathrm{B}a_{2n+1},k\lambda)}\phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)}\phi(\theta)d\theta\right) \end{pmatrix} \tag{4}$$

$$\mu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)=r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{A}a_{2n},\lambda),\mu_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{T}a_{2n+1},\lambda),\mu_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{B}a_{2n+1},\lambda), \\ \max\{\mu_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{A}a_{2n},\lambda),\mu_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{B}a_{2n+1},\lambda)\}\} \end{bmatrix}$$

where $\geq r\left[\min\{\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\mu_{\tilde{F}_N}(b_{2n},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n-1},b_{2n+1},\lambda)\}\}\right]$

$$\geq r\left[\min\{\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda)*\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\}\}\right]$$

$$\geq r\left[\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\right]$$

$$\nu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)=r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{A}a_{2n},\lambda),\nu_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{T}a_{2n+1},\lambda),\nu_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{B}a_{2n+1},\lambda), \\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{A}a_{2n},\lambda),\nu_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{B}a_{2n+1},\lambda)\}\} \end{bmatrix}$$

$$\leq r\left[\min\{\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\nu_{\tilde{F}_N}(b_{2n},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n-1},b_{2n+1},\lambda)\}\}\right]$$

$$\leq r\left[\min\{\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda)*\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\}\}\right]$$

$$\leq r\left[\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\right]$$

$$\omega_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)=r\begin{bmatrix}\min\{\omega_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{A}a_{2n},\lambda),\omega_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{T}a_{2n+1},\lambda),\omega_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{B}a_{2n+1},\lambda),\\ \max\{\omega_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{A}a_{2n},\lambda),\omega_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{B}a_{2n+1},\lambda)\}\}\end{bmatrix}$$
$$\le r\left[\min\{\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\omega_{\tilde{F}_N}(b_{2n},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n-1},b_{2n+1},\lambda)\}\}\right]$$
$$\le r\left[\min\{\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda)*\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\}\}\right]$$
$$\le r\left[\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\right]$$

Thus from (3), we have

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\ge\psi\left(\int_0^{\mathrm{r}\left[(\mu_{\tilde{F}_N})_{2n-1}(\lambda),(\mu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right),\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\mathrm{r}\left[(\nu_{\tilde{F}_N})_{2n-1}(\lambda),(\nu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right),\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\mathrm{r}\left[(\omega_{\tilde{F}_N})_{2n-1}(\lambda),(\omega_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)\end{pmatrix} \tag{5}$$

Now if $\begin{pmatrix}(\mu_{\tilde{F}_N})_{2n}(\lambda)>(\mu_{\tilde{F}_N})_{2n-1}(\lambda),\\ (\nu_{\tilde{F}_N})_{2n}(\lambda)<(\nu_{\tilde{F}_N})_{2n-1}(\lambda),\\ (\omega_{\tilde{F}_N})_{2n}(\lambda)<(\omega_{\tilde{F}_N})_{2n-1}(\lambda)\end{pmatrix}$ for some n, then from (5)

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\ge\psi\left(\int_0^{\mathrm{r}\left[(\mu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)>\psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\right)>\int_0^{(\mu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta,\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\mathrm{r}\left[(\nu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\right)<\int_0^{(\nu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta,\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\mathrm{r}\left[(\omega_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\right)<\int_0^{(\omega_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\end{pmatrix}$$

which implies $\begin{pmatrix}\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda)\ge\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\\ \nu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda)\le\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\\ \omega_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda)\le\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\end{pmatrix}$. Thus $b_{2n}=b_{2n+1}$, by lemma 2.3, which is contradiction, since $b_{2n}\ne b_{2n+1}$. Thus we have $\begin{pmatrix}(\mu_{\tilde{F}_N})_{2n}(q\lambda)\le r\left[(\mu_{\tilde{F}_N})_{2n-1}(\lambda)\right],\\ (\nu_{\tilde{F}_N})_{2n}(q\lambda)\ge r\left[(\nu_{\tilde{F}_N})_{2n-1}(\lambda)\right],\\ (\omega_{\tilde{F}_N})_{2n}(q\lambda)\ge r\left[(\omega_{\tilde{F}_N})_{2n-1}(\lambda)\right]\end{pmatrix}$

for which

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\ge\psi\left(\int_0^{\mathrm{r}\left[(\mu_{\tilde{F}_N})_{2n-1}(\lambda)\right]}\phi(\theta)d\theta\right)>\psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\mathrm{r}\left[(\nu_{\tilde{F}_N})_{2n-1}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\mathrm{r}\left[(\omega_{\tilde{F}_N})_{2n-1}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right)\end{pmatrix} \tag{6}$$

letting $q\to1$, then we have

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\ge\psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right)\end{pmatrix} \tag{7}$$

Similarly $\begin{pmatrix} \int_0^{(\mu_{\tilde{F}_N})_{2n-1}(\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n-2}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\nu_{\tilde{F}_N})_{2n-1}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n-2}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\omega_{\tilde{F}_N})_{2n-1}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n-2}(\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ and so on.

In general we have for all $n = 1, 2, 3......$.

$$\begin{pmatrix} \int_0^{(\mu_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(\mu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\nu_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\nu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\omega_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\omega_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \end{pmatrix} \tag{8}$$

from (8), we have

$$\begin{pmatrix} \int_0^{(\mu_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(\mu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \geq \psi^2\left(\int_0^{(\mu_{\tilde{F}_N})_{n-2}(\lambda)} \phi(\theta)d\theta\right) .... \geq \psi^n\left(\int_0^{(\mu_{\tilde{F}_N})_{0}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\nu_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\nu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \leq \psi^2\left(\int_0^{(\nu_{\tilde{F}_N})_{n-2}(\lambda)} \phi(\theta)d\theta\right) .... \leq \psi^n\left(\int_0^{(\nu_{\tilde{F}_N})_{0}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\omega_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\omega_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \leq \psi^2\left(\int_0^{(\omega_{\tilde{F}_N})_{n-2}(\lambda)} \phi(\theta)d\theta\right) .... \leq \psi^n\left(\int_0^{(\omega_{\tilde{F}_N})_{0}(\lambda)} \phi(\theta)d\theta\right) \end{pmatrix} \tag{9}$$

and taking the limit as $n \to \infty$ and using lemma 2.3, we have

$$\begin{pmatrix} \lim_{n\to\infty} \int_0^{(\mu_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \geq \lim_{n\to\infty} \psi^n\left(\int_0^{(\mu_{\tilde{F}_N})_{0}(\lambda)} \phi(\theta)d\theta\right) = 1, \\ \lim_{n\to\infty} \int_0^{(\nu_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \leq \lim_{n\to\infty} \psi^n\left(\int_0^{(\nu_{\tilde{F}_N})_{0}(\lambda)} \phi(\theta)d\theta\right) = 0, \\ \lim_{n\to\infty} \int_0^{(\omega_{\tilde{F}_N})_{n}(\lambda)} \phi(\theta)d\theta \leq \lim_{n\to\infty} \psi^n\left(\int_0^{(\omega_{\tilde{F}_N})_{0}(\lambda)} \phi(\theta)d\theta\right) = 0 \end{pmatrix} \tag{10}$$

which from (1) implies that $\begin{pmatrix} \lim_{n\to\infty} (\mu_{\tilde{F}_N})_n(\lambda) = \lim_{n\to\infty} \mu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) = 1, \\ \lim_{n\to\infty} (\nu_{\tilde{F}_N})_n(\lambda) = \lim_{n\to\infty} \nu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) = 0, \\ \lim_{n\to\infty} (\omega_{\tilde{F}_N})_n(\lambda) = \lim_{n\to\infty} \omega_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) = 0 \end{pmatrix}$ for all $n \in N$ and $\lambda > 0$. (11)

Now for each $\varepsilon > 0$ and each $\lambda > 0$, choose $n_0 \in N$ s.t. $\begin{pmatrix} \mu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) < \varepsilon, \\ \omega_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) < \varepsilon \end{pmatrix}$ for all $n > n_0$ (12)

Letting $m > n (m, n \in N)$, then

$$\begin{pmatrix} \mu_{\tilde{F}_N}(b_n, b_m, \lambda) \geq r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda/(m-n)), \mu_{\tilde{F}_N}(b_{n+1}, b_{n+2}, \lambda/(m-n)) .... \\ \max\{\mu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda/(m-n)), ...\mu_{\tilde{F}_N}(b_{m-1}, b_m, \lambda/(m-n))\}\} \end{bmatrix} \\ > r\left[\min\{(1-\varepsilon), (1-\varepsilon), ... \max\{(1-\varepsilon), (1-\varepsilon), , ...(1-\varepsilon)\}\}\right] \geq (1-\varepsilon) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(b_n, b_m, \lambda) \leq r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda/(m-n)), \nu_{\tilde{F}_N}(b_{n+1}, b_{n+2}, \lambda/(m-n)) .... \\ \max\{\nu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda/(m-n)), ...\nu_{\tilde{F}_N}(b_{m-1}, b_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, \varepsilon, ... \max\{\varepsilon, ...\varepsilon\}\}\right] \leq \varepsilon < \varepsilon, \\ \omega_{\tilde{F}_N}(b_n, b_m, \lambda) \leq r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(b_n, b_{n+1}, \lambda/(m-n)), \omega_{\tilde{F}_N}(b_{n+1}, b_{n+2}, \lambda/(m-n)) .... \\ \max\{\omega_{\tilde{F}_N}(b_{s-1}, b_s, \lambda/(m-n)), ...\omega_{\tilde{F}_N}(b_{m-1}, b_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, \varepsilon, ... \max\{\varepsilon, ...\varepsilon\}\}\right] \leq \varepsilon < \varepsilon \end{pmatrix}. \tag{13}$$

Thus from definitions 2.10, 2.11 and conditions (11), (13) and lemma 3.1 $\{b_n\}$ is a CS in X . Since X is complete so that $\{b_n\} \to z \in X$ and sub sequences $\{Aa_{2n}\}$, $\{Ba_{2n+1}\}$, $\{Sa_{2n+1}\}$ and $\{Ta_{2n+2}\}$, of $\{b_n\}$ also converges to z. Thus $Aa_{2n} \to z,\ Ba_{2n+1} \to z,\ Sa_{2n+1} \to z$ and $Ta_{2n+2} \to z.$ (14)

Again, since X is ε-chainable, $\exists$ ε-chain from a_n to a_{n+1} i.e. $\exists$ a finite sequence $a_n = b_1, b_2, ...b_l = a_{n+1}$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(b_i, b_{i-1}, \lambda) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(b_i, b_{i-1}, \lambda) < \varepsilon, \\ \omega_{\tilde{F}_N}(b_i, b_{i-1}, \lambda) < \varepsilon \end{pmatrix} \text{ for all } \lambda > 0 \text{ and } i = 1, 2, ...l.,$$

Thus, we have

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a_n, a_{n+1}, q\lambda) \geq r\left[\min\{\mu_{\tilde{F}_N}(b_1, b_2, \lambda), \mu_{\tilde{F}_N}(b_2, b_3, \lambda), ..., \max\{\mu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda), ...\mu_{\tilde{F}_N}(b_{l-1}, b_l, \lambda)\}\}\right] \\ > r\left[\min\{(1-\varepsilon), (1-\varepsilon), ...\max\{(1-\varepsilon), (1-\varepsilon), ...(1-\varepsilon)\}\}\right] \geq r\left[1-\varepsilon\right] > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(a_n, a_{n+1}, q\lambda) \leq r\left[\min\{\nu_{\tilde{F}_N}(b_1, b_2, \lambda), \nu_{\tilde{F}_N}(b_2, b_3, \lambda), ..., \max\{\nu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda), ...\nu_{\tilde{F}_N}(b_{l-1}, b_l, \lambda)\}\}\right] \\ < r\left[\min\{\varepsilon, \varepsilon, ...\max\{\varepsilon, \varepsilon, ...\varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon, \\ \omega_{\tilde{F}_N}(a_n, a_{n+1}, q\lambda) \leq r\left[\min\{\omega_{\tilde{F}_N}(b_1, b_2, \lambda), \nu_{\tilde{F}_N}(b_2, b_3, \lambda), ..., \max\{\omega_{\tilde{F}_N}(b_{s-1}, b_s, \lambda), ...\omega_{\tilde{F}_N}(b_{l-1}, b_l, \lambda)\}\}\right] \\ < r\left[\min\{\varepsilon, \varepsilon, ...\max\{\varepsilon, \varepsilon, ...\varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon. \end{pmatrix} \quad (15)$$

For $m, n \in N$, $m > n$, we have

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a_n, a_m, \lambda) \geq r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/(m-n)), \mu_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/(m-n))........ \\ \max\{\mu_{\tilde{F}_N}(a_{p-1}, a_p, \lambda/(m-n)), ...\mu_{\tilde{F}_N}(a_{m-1}, a_m, \lambda/(m-n))\}\} \end{bmatrix} \\ r\left[\min\{(1-\varepsilon), (1-\varepsilon), ...\max\{(1-\varepsilon), (1-\varepsilon), ...(1-\varepsilon)\}\}\right] \geq r\left[1-\varepsilon\right] > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(a_n, a_m, \lambda) \leq r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/(m-n)), \nu_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/(m-n))........ \\ \max\{\nu_{\tilde{F}_N}(a_{p-1}, a_p, \lambda/(m-n)), ...\nu_{\tilde{F}_N}(a_{m-1}, a_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, ...\max\{\varepsilon, \varepsilon, ...\varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon, \\ \omega_{\tilde{F}_N}(a_n, a_m, \lambda) \leq r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/(m-n)), \omega_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/(m-n))........ \\ \max\{\omega_{\tilde{F}_N}(a_{p-1}, a_p, \lambda/(m-n)), ...\omega_{\tilde{F}_N}(a_{m-1}, a_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, ...\max\{\varepsilon, \varepsilon, ...\varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon. \end{pmatrix} \quad (16)$$

i.e.

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(a_n, a_m, \lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(1-\varepsilon)} \phi(\theta)d\theta\right) > \int_0^{(1-\varepsilon)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(a_n, a_m, \lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\varepsilon} \phi(\theta)d\theta\right) < \int_0^{\varepsilon} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(a_n, a_m, \lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\varepsilon} \phi(\theta)d\theta\right) < \int_0^{\varepsilon} \phi(\theta)d\theta \end{pmatrix} \quad (17)$$

and so from definitions 2.10; 2.11 and conditions (15) and (17), $\{a_n\}$ is a CS in X . Since X is complete so that $\{a_n\} \to a \in X.$ since $\{A, S\}$ is reciprocally continuous, so that A and S are continuous. Thus

$$Aa_{2n} = Aa,\ \ Sa_{2n+1} = Sa, \quad (18)$$

since $\{A, S\}$ are compatible, so R -weakly commuting mapping. Then

$$\begin{pmatrix} \mu_{\tilde{F}_N}(ASa_{2n}, SAa_{2n}, q\lambda) \geq \mu_{\tilde{F}_N}(Aa_{2n}, Sa_{2n}, \lambda/R), \\ \nu_{\tilde{F}_N}(ASa_{2n}, SAa_{2n}, q\lambda) \leq \nu_{\tilde{F}_N}(Aa_{2n}, Sa_{2n}, \lambda/R), \\ \omega_{\tilde{F}_N}(ASa_{2n}, SAa_{2n}, q\lambda) \leq \omega_{\tilde{F}_N}(Aa_{2n}, Sa_{2n}, \lambda/R) \end{pmatrix}$$

gives $ASa_{2n} = Aa, \quad SAa_{2n+1} = Sa.$ Also $\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda) \ge 1, \\ \nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda) \le 0, \\ \omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda) \le 0 \end{pmatrix}$ which implies $\mathrm{A}a = \mathrm{S}a.$ (19)

From (14) , (18) and (19), $\mathrm{A}z = \mathrm{S}z.$ Since $\mathrm{A}(a) \subseteq \mathrm{S}(a),\ \exists, u \in X$ s.t. $\mathrm{A}z = \mathrm{S}u.$ then from (4)

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \ge \psi\left(\int_0^{\mu_{\tilde{F}_N}(z,u,q\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \le \psi\left(\int_0^{\nu_{\tilde{F}_N}(z,u,q\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \le \psi\left(\int_0^{\omega_{\tilde{F}_N}(z,u,q\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$$

where

$$\begin{aligned} \mu_{\tilde{F}_N}(z,u,\lambda) &= r\left[\min\{\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}u,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}u,\lambda)\}\}\right] \\ &= r\left[\min\{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \mu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{B}u,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\}\right] \\ &\ge r\left[\min\{1, \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda),1,\}, \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\right] \ge r\left[\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\right] \ge \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \\ \nu_{\tilde{F}_N}(z,u,\lambda) &= r\left[\min\{\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}u,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}u,\lambda)\}\}\right] \\ &= r\left[\min\{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \nu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{B}u,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\}\right] \\ &\le r\left[\min\{1, \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda),1,\}, \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\right] \le r\left[\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\right] \ge \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \\ \omega_{\tilde{F}_N}(z,u,\lambda) &= r\left[\min\{\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}u,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{T}u,\mathrm{A}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}u,\lambda)\}\}\right] \\ &= r\left[\min\{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{A}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \omega_{\tilde{F}_N}(\mathrm{B}u,\mathrm{B}u,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{B}u,\mathrm{A}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\}\right] \\ &\le r\left[\min\{1, \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda),1,\}, \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\right] \le r\left[\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\right] \ge \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \end{aligned}$$

i.e. $$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \ge \psi\left(\int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) > \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \le \psi\left(\int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \le \psi\left(\int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta \end{pmatrix} \quad (20)$$

which implies $$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda) \ge \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \\ \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda) \le \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \\ \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda) \le \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \end{pmatrix} \quad (21)$$

from lemma 2.3, $\mathrm{A}z = \mathrm{B}u$ i.e. $\mathrm{A}z = \mathrm{S}u = \mathrm{T}z.$ Again let $\mathrm{A}z = \mathrm{S}u = \mathrm{B}u = \mathrm{T}u.$

Since the pair $\{\mathrm{A},\mathrm{S}\}$ is point-wise R–weakly commuting mappings, so there exists $\mathrm{R} > 0$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AS}z,\mathrm{SA}z,\lambda) \ge \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{S}z,\lambda/\mathrm{R}) = 1, \\ \nu_{\tilde{F}_N}(\mathrm{AS}z,\mathrm{SA}z,\lambda) \le \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{S}z,\lambda/\mathrm{R}) = 0, \\ \omega_{\tilde{F}_N}(\mathrm{AS}z,\mathrm{SA}z,\lambda) \le \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{S}z,\lambda/\mathrm{R}) = 0, \end{pmatrix} \text{ i.e. } \quad \mathrm{AS}z = \mathrm{SA}z \text{ and } \mathrm{AA}z = \mathrm{AS}z = \mathrm{SA}z = \mathrm{SS}z.$$

Similarly it can be for the pair $\{\mathrm{B},\mathrm{T}\}$ which implies $\mathrm{BB}u = \mathrm{BT}u = \mathrm{TB}u = \mathrm{TT}u.$ For this in (4), we put $a = \mathrm{A}z,$ $b = u,$ we have
$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,u,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,u,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,u,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$$

where
$$\mu_{\tilde{F}_N}(\mathrm{A}z,u,\lambda) = r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}u,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \\ \max\{\mu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{AA}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}u,\lambda)\}\} \end{bmatrix}$$
$$\geq r\left[\min\{1, \mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), 1, \max\{\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), \mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)\}\}\right]$$
$$\geq r\left[\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)\right] > \mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)$$
$$\nu_{\tilde{F}_N}(\mathrm{A}z,u,\lambda) = r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}u,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{AA}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}u,\lambda)\}\} \end{bmatrix}$$
$$\leq r\left[\min\{0, \nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), 0, \max\{\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), \nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)\}\}\right]$$
$$\leq r\left[\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)\right] < \nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)$$
$$\omega_{\tilde{F}_N}(\mathrm{A}z,u,\lambda) = r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}u,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \\ \max\{\omega_{\tilde{F}_N}(\mathrm{T}u,\mathrm{AA}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}u,\lambda)\}\} \end{bmatrix}$$
$$\leq r\left[\min\{0, \omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), 0, \max\{\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), \omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)\}\}\right]$$
$$\leq r\left[\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)\right] < \omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)$$

i.e.
$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) > \int_0^{\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta \end{pmatrix} \tag{22}$$

which implies
$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda) \geq \mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), \\ \nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda) \leq \nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda), \\ \omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda) \leq \omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,\lambda) \end{pmatrix} \tag{23}$$

from lemma 2.3, we have $\mathrm{AA}z = \mathrm{B}u = \mathrm{A}z.$ Thus $\mathrm{A}z = \mathrm{AA}z$ and $\mathrm{A}z = \mathrm{AA}z = \mathrm{SA}z,$ which shows that $\mathrm{A}z$ is common fixed point of A and S. Also $\mathrm{A}z = \mathrm{B}u = \mathrm{S}u = \mathrm{T}z.$ Hence $\mathrm{A}z$ is common fixed point of A, B, S and T.

Now again suppose that $\mathrm{A}z = z$ is a common fixed point of A, B, S and T. For this from (4), we have
$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{r\left[\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}z,\lambda)\right]} \phi(\theta)d\theta\right) > \int_0^{r\left[\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}z,\lambda)\right]} \phi(\theta)d\theta > \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{r\left[\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}z,\lambda)\right]} \phi(\theta)d\theta\right) < \int_0^{r\left[\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}z,\lambda)\right]} \phi(\theta)d\theta < \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{AA}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{r\left[\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}z,\lambda)\right]} \phi(\theta)d\theta\right) < \int_0^{r\left[\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}z,\lambda)\right]} \phi(\theta)d\theta < \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)} \phi(\theta)d\theta \end{pmatrix}$$

$$\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)=r\begin{bmatrix}\min\{\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{B}z,\lambda),\\ \max\{\mu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{AA}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}z,\lambda)\}\}\end{bmatrix}$$

where

$$\geq r\left[\min\{1,1,\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\},1)\}\right]>\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),$$

$$\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)=r\begin{bmatrix}\min\{\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{B}z,\lambda),\\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}z,\lambda)\}\}\end{bmatrix}$$

$$\leq r\left[\min\{0,0,\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\},0)\}\right]<\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),$$

$$\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)=r\begin{bmatrix}\min\{\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{B}z,\lambda),\\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}z,\lambda)\}\}\end{bmatrix}$$

$$\leq r\left[\min\{0,0,\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\},0)\}\right]<\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),$$

which implies that $\begin{pmatrix}\mu_{\tilde{F}_N}(\mathrm{A}z,z,q\lambda)>\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),\\ \nu_{\tilde{F}_N}(\mathrm{A}z,z,q\lambda)<\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),\\ \omega_{\tilde{F}_N}(\mathrm{A}z,z,q\lambda)<\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\end{pmatrix}$ for all $\lambda>0$, i.e. from lemma 2.3, $\mathrm{A}z=z$. Thus z is a common fixed point of A, B, S and T. For uniqueness of z let $w\neq z$ be another common fixed point of A, B, S and T, then from (4), we have

$$\int_0^{\mu_{\tilde{F}_N}(z,w,q\lambda)}\phi(\theta)d\theta\geq\psi\left(\int_0^{r\left[\min\{\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}w,\lambda),\mu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{B}w,\lambda),\max\{\mu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{A}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}w,\lambda)\}\}\right]}\phi(\theta)d\theta\right)$$

$$>\psi\left(\int_0^{r\left[\min\{1,\mu_{\tilde{F}_N}(z,w,\lambda),1,\mu_{\tilde{F}_N}(z,w,\lambda)\}\right]}\phi(\theta)d\theta\right)>\int_0^{r\left[\mu_{\tilde{F}_N}(z,w,\lambda)\right]}\phi(\theta)d\theta>\int_0^{\mu_{\tilde{F}_N}(z,w,\lambda)}\phi(\theta)d\theta,$$

$$\int_0^{\nu_{\tilde{F}_N}(z,w,q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{r\left[\min\{\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}w,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{B}w,\lambda),\max\{\nu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{A}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}w,\lambda)\}\}\right]}\phi(\theta)d\theta\right)$$

$$<\psi\left(\int_0^{r\left[\min\{0,\nu_{\tilde{F}_N}(z,w,\lambda),0,\nu_{\tilde{F}_N}(z,w,\lambda)\}\right]}\phi(\theta)d\theta\right)<\int_0^{r\left[\nu_{\tilde{F}_N}(z,w,\lambda)\right]}\phi(\theta)d\theta<\int_0^{\mu_{\tilde{F}_N}(z,w,\lambda)}\phi(\theta)d\theta,$$

$$\int_0^{\omega_{\tilde{F}_N}(z,w,q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{r\left[\min\{\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda),\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}w,\lambda),\omega_{\tilde{F}_N}(\mathrm{T}w,\mathrm{B}w,\lambda),\max\{\omega_{\tilde{F}_N}(\mathrm{T}w,\mathrm{A}z,\lambda),\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}w,\lambda)\}\}\right]}\phi(\theta)d\theta\right)$$

$$<\psi\left(\int_0^{r\left[\min\{0,\omega_{\tilde{F}_N}(z,w,\lambda),0,\omega_{\tilde{F}_N}(z,w,\lambda)\}\right]}\phi(\theta)d\theta\right)<\int_0^{r\left[\omega_{\tilde{F}_N}(z,w,\lambda)\right]}\phi(\theta)d\theta<\int_0^{\omega_{\tilde{F}_N}(z,w,\lambda)}\phi(\theta)d\theta,$$

which from lemma 2.3, implies $\begin{pmatrix}\mu_{\tilde{F}_N}(z,w,q\lambda)\geq\mu_{\tilde{F}_N}(z,w,\lambda),\\ \nu_{\tilde{F}_N}(z,w,q\lambda)\leq\nu_{\tilde{F}_N}(z,w,\lambda),\\ \omega_{\tilde{F}_N}(z,w,q\lambda)\leq\omega_{\tilde{F}_N}(z,w,\lambda)\end{pmatrix}$ i.e. $z=w$, for all $\lambda>0$. Thus z is a unique common fixed point of A, B, S and T.

Corollary 4.1: Let $\{\mathrm{A},\mathrm{S}\}$ and $\{\mathrm{A},\mathrm{T}\}$ be point wise *R*–weakly commuting pairs of self mappings of a complete $\varepsilon-$chainable NMS $(X,\tilde{F}_N,*,\diamondsuit)$ with $t*t\geq t$ and $(1-t)\diamondsuit(1-t)\leq(1-t)$ for all $t\in[0,1]$ satisfying the following conditions:

(i)′ $\mathrm{A}(\mathrm{X})\subseteq\mathrm{S}(\mathrm{X})$ and $\mathrm{A}(\mathrm{X})\subseteq\mathrm{T}(\mathrm{X})$

(ii)′ for all $a,b \in X, \lambda > 0$ and $q \in (0,1)$ there exists a left continuous function $\psi:[0,1] \to [0,1]$, $\psi(0)=0$, and $\psi(s) > s$ for all $s > 0$ s.t. $\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ where $\phi(\theta): R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0 < \int_0^{\varepsilon} \phi(\theta)d\theta < 1$, for all $\varepsilon > 0$.

$$\mu_{\tilde{F}_N}(a,b,\lambda) = r\Big[\min\{\mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}b,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}b,\lambda)\}\}\Big]$$
$$\nu_{\tilde{F}_N}(a,b,\lambda) = r\Big[\min\{\nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}b,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}b,\lambda)\}\}\Big]$$
$$\omega_{\tilde{F}_N}(a,b,\lambda) = r\Big[\min\{\omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}b,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}b,\lambda)\}\}\Big]$$

where $r:[0,1] \to [0,1]$, is continuous function s.t. $r(a) > a$ and $r(a) = 1$ for $a = 1$, $a \in [0,1)$. Then the continuity of one of the mapping in compatible pair $\{\mathrm{A},\mathrm{S}\}$ or $\{\mathrm{A},\mathrm{T}\}$ on $\varepsilon-$chainable NMS implies the unique common fixed point of A, S and T.

Proof: If we put S = T, in theorem 4.1, then we get proof of corollary 4.1, easily.

Corollary 4.2: Let $\{\mathrm{A},\mathrm{T}\}$ be point wise *R*–weakly commuting pairs of self-mappings of a complete $\varepsilon-$chainable NMS $(X, \tilde{F}_N, *, \diamond)$ satisfying the following conditions:

(i)″ $\mathrm{A}(\mathrm{X}) \subseteq \mathrm{T}(\mathrm{X})$

(ii)″ for all $a,b \in X, \lambda > 0$ there exists a left continuous function $\psi:[0,1] \to [0,1]$, $\psi(0)=0$, and $\psi(s) > s$ for all $s > 0$ s.t. $\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ where $\phi(\theta): R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0 < \int_0^{\varepsilon} \phi(\theta)d\theta < 1$, for all $\varepsilon > 0$.

$$\mu_{\tilde{F}_N}(a,b,\lambda) = r\Big[\min\{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{T}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}a,\mathrm{T}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}a,\lambda)\}\}\Big]$$
$$\nu_{\tilde{F}_N}(a,b,\lambda) = r\Big[\min\{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{T}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}a,\mathrm{T}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}a,\lambda)\}\}\Big]$$
$$\omega_{\tilde{F}_N}(a,b,\lambda) = r\Big[\min\{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{T}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}a,\mathrm{T}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}a,\lambda)\}\}\Big]$$

where $r:[0,1] \to [0,1]$, is continuous function s.t. $r(a) > a$ and $r(a) = 1$ for $a = 1$, $a \in [0,1)$. Then the continuity of one of the mapping in compatible pair $\{\mathrm{A},\mathrm{T}\}$ on $\varepsilon-$chainable NMS implies the unique common fixed point of A and T.

Proof: If we put B = A and S = T in theorem 4.1, we get the proof of corollary 4.2.

Theorem 4.2: Let S and T be two self-continuous mappings of a complete $\varepsilon-$chainable NMS $(X, \tilde{F}_N, *, \diamond)$ with $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$. Let A and B be two self-mappings of X satisfying the following conditions:

(i)''' $A(X) \subseteq S(X)$ and $A(X) \subseteq T(X)$

(ii)''' for all $a,b \in X, \lambda > 0$ and $q \in (0,1)$ there exists a left continuous function $\psi : [0,1] \to [0,1]$, $\psi(0) = 0$, and $\psi(s) > s$ for all $s > 0$ s.t. $\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(Aa,Ab,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu'_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(Aa,Ab,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu'_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(Aa,Ab,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega'_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ where

$\phi(\theta) : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0 < \int_0^{\varepsilon} \phi(\theta)d\theta < 1$, for all $\varepsilon > 0$.

$$\mu'_{\tilde{F}_N}(a,b,\lambda) = l\mu_{\tilde{F}_N}(Sa,Aa,\lambda) + m\mu_{\tilde{F}_N}(Sa,Tb,\lambda) + n\mu_{\tilde{F}_N}(Tb,Ab,\lambda) + \max\{\mu_{\tilde{F}_N}(Tb,Aa,\lambda), \mu_{\tilde{F}_N}(Sa,Ab,\lambda)\}$$

$$\nu'_{\tilde{F}_N}(a,b,\lambda) = l\nu_{\tilde{F}_N}(Sa,Aa,\lambda) + m\nu_{\tilde{F}_N}(Sa,Tb,\lambda) + n\nu_{\tilde{F}_N}(Tb,Ab,\lambda) + \max\{\nu_{\tilde{F}_N}(Tb,Aa,\lambda), \nu_{\tilde{F}_N}(Sa,Ab,\lambda)\}$$

$$\omega'_{\tilde{F}_N}(a,b,\lambda) = l\omega_{\tilde{F}_N}(Sa,Aa,\lambda) + m\omega_{\tilde{F}_N}(Sa,Tb,\lambda) + n\omega_{\tilde{F}_N}(Tb,Ab,\lambda) + \max\{\omega_{\tilde{F}_N}(Tb,Aa,\lambda), \omega_{\tilde{F}_N}(Sa,Ab,\lambda)\}$$

for all $0 < q < l + m + n + 1$. Then A, B, S and T have a unique common fixed point.

Proof: Similar to theorem 4.1.

Conclusion

The newly defined infinite products establish the Banach contraction theorem for NMS. In this context, we introduce an integral-type contractive condition in a ε-chainable neutrosophic metric space and prove a common fixed point theorem for four weakly compatible mappings. Our findings extend and unify well-known results in neutrosophic metric spaces, such as those presented by Kirisci and Simsek [28]. Furthermore, Kirisci et al. [21] discussed fixed point results within the framework of NMS.

Funding: No funding from any agencies.

Acknowledgments: The authors sincerely acknowledge Prof. Florentin Smarandache for his invaluable guidance, constructive feedback, and encouragement during the preparation of this manuscript.

Author Contributions: The conceptualization and methodology of this study were carried out by Rajesh Kumar Saini, while the formal analysis and theoretical investigation were jointly conducted by Rajesh Kumar Saini and Mukesh Kushwaha. The original draft of the manuscript was prepared by Rajesh Kumar Saini, and the review and editing were undertaken by Mukesh Kushwaha. Supervision and project administration were led by Rajesh Kumar Saini. Both authors have read and approved the final manuscript.

Ethical Approval: This article does not involve any studies with human participants or animals performed by any of the authors. Therefore, ethical approval is not applicable.

Conflict of Interest: No potential conflict of interest was reported by the author(s).

References

1. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3

2. Atanassov, K., & Gargov, G. (1989). Interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, 31(3), 343–349. https://doi.org/10.1016/0165-0114(89)90205-4
3. Alaca, C., Turkoglu, D., & Yildiz, C. (2006). Fixed points in intuitionistic fuzzy metric spaces. Chaos, Solitons & Fractals, 29(5), 1073–1078. https://doi.org/10.1016/j.chaos.2005.08.066
4. Branciari, A. (2002). A fixed point theorem for mappings satisfying a general contractive condition of integral type. International Journal of Mathematics and Mathematical Sciences, 29(9), 531–536. https://doi.org/10.1155/S0161171202108013
5. Bose, B. K., & Sahani, D. (1987). Fuzzy mappings and fixed point theorems. Fuzzy Sets and Systems, 21(1), 53–58. https://doi.org/10.1016/0165-0114(87)90056-0
6. Deli, I., & Şuba, Y. (2014). Single valued neutrosophic numbers and their applications to multicriteria decision making problem. Neural Computing and Applications, 25(5), 1147–1154. https://doi.org/10.1007/s00521-014-1596-y
7. Djoudi, A., & Aliouche, A. (2006). Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type. Journal of Mathematical Analysis and Applications, 322(2), 796–802. https://doi.org/10.1016/j.jmaa.2005.09.059
8. Grabiec, M. (1988). Fixed points in fuzzy metric space. Fuzzy Sets and Systems, 27(3), 385–389. https://doi.org/10.1016/0165-0114(88)90123-2
9. George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. Fuzzy Sets and Systems, 64(3), 395–399. https://doi.org/10.1016/0165-0114(94)90343-3
10. Gregori, V., & Sapena, A. (2002). On fixed point theorem in fuzzy metric spaces. Fuzzy Sets and Systems, 125(2), 245–252. https://doi.org/10.1016/S0165-0114(01)00059-3
11. Heilpern, S. (1981). Fuzzy mappings and fixed point theorem. Journal of Mathematical Analysis and Applications, 83(2), 566–569. https://doi.org/10.1016/0022-247X(81)90127-1
12. Hussain, N., Khaleghizadeh, S., Salimi, P., & Abdou, A. A. N. (2014). A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces. Abstract and Applied Analysis, 2014, 1–16. https://doi.org/10.1155/2014/151679
13. Jungck, G. (1986). Compatible mappings and common fixed points. International Journal of Mathematics and Mathematical Sciences, 9(4), 771–779. https://doi.org/10.1155/S0161171286000935
14. Kaleva, O., & Seikkala, S. (1984). On fuzzy metric spaces. Fuzzy Sets and Systems, 12(3), 215–229. https://doi.org/10.1016/0165-0114(84)90075-8
15. Kirişci, M., Şimşek, N., & Akyiğit, M. (2019). Fixed point results for a new metric space. Mathematical Methods in the Applied Sciences, 42(17), 5556–5568. https://doi.org/10.1002/mma.5740
16. Kirişci, M., & Şimşek, N. (2019). Neutrosophic metric spaces. arXiv preprint arXiv:1907.00798. https://arxiv.org/abs/1907.00798
17. Kirişci, M. (2017). Integrated and differentiated spaces of triangular fuzzy numbers. Fasciculi Mathematici, 59, 75–89. https://doi.org/10.1515/fascmath-2017-0007
18. Kramosil, I., & Michálek, J. (1975). Fuzzy metric and statistical metric spaces. Kybernetika, 11(5), 326–334.
19. Kutukch, S., Sharma, S., & Tokgoz, H. (2007). A fixed point theorem in fuzzy metric spaces. International Journal of Mathematical Analysis, 1(18), 861–872.
20. Menger, K. M. (1942). Statistical metrics. Proceedings of the National Academy of Sciences, 28(12), 535–537. https://doi.org/10.1073/pnas.28.12.535
21. Mohamad, A. (2007). Fixed-point theorems in intuitionistic fuzzy metric spaces. Chaos, Solitons & Fractals, 34(5), 1689–1695. https://doi.org/10.1016/j.chaos.2006.04.040
22. Necip, Ş., & Murat, K. (2019). Fixed point theorems in neutrosophic metric spaces. Sigma Journal of Engineering and Natural Sciences, 10(2), 221–230.

23. Peng, J. J., Wang, J. Q., Wang, J., Zhang, H. Y., & Chen, X. H. (2016). Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. International Journal of Systems Science, 47(10), 2342–2358. https://doi.org/10.1080/00207721.2014.994050
24. Park, J. H. (2004). Intuitionistic fuzzy metric spaces. Chaos, Solitons & Fractals, 22(5), 1039–1046. https://doi.org/10.1016/j.chaos.2004.02.057
25. Phanikanth, T., Sivakrishna, T., & Vijaya Kumar, M. (2019). Common fixed point theorems in intuitionistic fuzzy metric space using general contractive condition of integral type. IOSR Journal of Mathematics, 15(5), 35–45.
26. Rhoades, B. E. (2003). Two fixed-point theorems for mappings satisfying a general contractive condition of integral type. International Journal of Mathematics and Mathematical Sciences, 2003(63), 4007–4013. https://doi.org/10.1155/S0161171203206073
27. Saini, R. K., Kumar, S., & Mohamad, P. (2010). Common fixed point theorem for hybrid pairs of R-weakly commuting maps. Surveys in Mathematics and Its Applications, 5, 191–199. http://www.utgjiu.ro/math/sma
28. Rajesh Kumar Saini, Mukesh Kushwaha, Hybrid Fixed Point Theorems for Integral Type Implicit Relations in Hausdorff Fuzzy Metric, Advances and Applications in Mathematical Sciences, Volume 21, Issue 8, June 2022, Pages 4847-486 (ESCI, web of science) DOI.org/10.1142/9789814261302_0021
29. Smarandache, F. (1998). A unifying field in logics: Neutrosophy, neutrosophic probability, set, and logic. American Research Press. ISBN: 978-1599730806
30. Smarandache, F. (2005). Neutrosophic set: A generalization of the intuitionistic fuzzy set. International Journal of Pure and Applied Mathematics, 24(3), 287–297.
31. Turkoglu, D., Alaca, C., Cho, Y. J., & Yildiz, C. (2006). Common fixed point theorems in intuitionistic fuzzy metric spaces. Journal of Applied Mathematics and Computation, 22(3), 411–424. https://doi.org/10.1007/BF02896489
32. Vijayaraju, P., Rhoades, B. E., & Mohanraj, R. (2005). A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type. International Journal of Mathematics and Mathematical Sciences, 2005(15), 2359–2364. https://doi.org/10.1155/IJMMS.2005.2359
33. Wang, H., Smarandache, F., Zhang, Y. Q., & Sunderraman, R. (2010). Single valued neutrosophic sets. Multispace and Multistructure, 4, 410–413.
34. Wang, H., Smarandache, F., Zhang, Y. Q., & Sunderraman, R. (2010). Single valued neutrosophic sets and logic: Theory and applications in computing. Journal of Computers, 3(2), 116–123. https://doi.org/10.4304/jcp.3.2.116-123
35. Deli, I., & Şuba, S. (2010). A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. International Journal of Fuzzy Systems, 12(3), 45–56.
36. Ye, J. (2014). Simplified neutrosophic sets: Basic concepts and fundamental operations. Journal of Information and Computational Science, 11(14), 5179–5186.
37. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

Received: July 22, 2025. Accepted: Jan 5, 2026