Neutrosophic Sets and Systems, Vol. 99, 2026
University of New Mexico

On the Theory of Fuzzy Neutrosophic Strongly μ- Category Sets and Their Associated μ- Baire spaces

Poongothai Eswaran [1,*] , and Balaganesan Palanivelu [2]

[1]Department of Mathematics, AMET Deemed to be University, Chennai, Tamil Nadu, India; epoongothai@ametuniv.ac.in
[2] Department of Mathematics, AMET Deemed to be University, Chennai, Tamil Nadu, India; balaganesan.p@ametuniv.ac.in

* Correspondence: epoongothai@ametuniv.ac.in

Abstract: In this paper, we introduced some types of fuzzy neutrosophic strongly category sets and their characterizations and examples are obtained. Also we discuss fuzzy neutrosophic μ - strongly Baire Spaces and its properties are to be described.

Keywords: Fuzzy neutrosophic sets; fuzzy neutrosophic μ-strongly nowhere dense; fuzzy neutrosophic μ-strongly category sets and fuzzy neutrosophic μ-strongly Baire Space.

1. Introduction

The fuzzy idea was invaded all branches of science as far back as the presentation of fuzzy sets by L. A. Zadeh [22]. The important concept of fuzzy topological space was offered by C.L.Chang [3]. The idea of fuzzy σ- Baire Spaces was introduced by G. Thangaraj and E. Poongothai [14]. The concept of neutrosophic sets was defined with membership, non-membership and indeterminacy degrees. In 2017, Veereswari [21] introduced fuzzy neutrosophic topological spaces. The idea of fuzzy neutrosophic Baire spaces was introduced by E. Poongothai and E.Padmavathi [11]. In this paper, we introduced some types of fuzzy neutrosophic sets such as μ-nowhere dense set, μ-first and second category, μ-residual, μ-strongly nowhere dense, μ-strongly first and second category, μ-strongly residual and their characterizations and examples are explained. Also we discuss about fuzzy neutrosophic μ-strongly Baire Space and its properties are to be described.

2. Preliminaries

Definition 2.1. [2] A fuzzy neutrosophic set A on the universe of discourse X is defined as $\boldsymbol{A} = \langle \boldsymbol{x}, \boldsymbol{T_A(x)}, \boldsymbol{I_A(x)}, \boldsymbol{F_A(x)} \rangle, \boldsymbol{x} \in \boldsymbol{X}$ where $\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F}: \boldsymbol{X} \to [\boldsymbol{0}, \boldsymbol{1}]$ and $\boldsymbol{0} \le \boldsymbol{T_A(x)} + \boldsymbol{I_A(x)} + \boldsymbol{F_A(x)} \le \boldsymbol{3}$.

Definition 2.2. [2] A fuzzy neutrosophic set A isa subset of a fuzzy neutrosophic set B (i.e.,) $\boldsymbol{A} \subseteq \boldsymbol{B}$ for all x if $\boldsymbol{T_A(x)} \le \boldsymbol{T_B(x)}, \boldsymbol{I_A(x)} \le \boldsymbol{I_B(x)}, \boldsymbol{F_A(x)} \ge \boldsymbol{F_B(x)}$.

Definition 2.3. [2] Let X be a non-empty set, and $\boldsymbol{A} = \langle \boldsymbol{x}, \boldsymbol{T_A(x)}, \boldsymbol{I_A(x)}, \boldsymbol{F_A(x)} \rangle$, $\boldsymbol{B} = \langle \boldsymbol{x}, \boldsymbol{T_B(x)}, \boldsymbol{I_B(x)}, \boldsymbol{F_B(x)} \rangle$ be two fuzzy neutrosophic sets. Then

$$\boldsymbol{A} \cup \boldsymbol{B} = \langle \boldsymbol{x}, \boldsymbol{max}(\boldsymbol{T_A(x)}, \boldsymbol{T_B(x)}), \boldsymbol{max}(\boldsymbol{I_A(x)}, \boldsymbol{I_B(x)}), \boldsymbol{min}(\boldsymbol{F_A(x)}, \boldsymbol{F_B(x)}) \rangle$$

$$\boldsymbol{A} \cap \boldsymbol{B} = \langle \boldsymbol{x}, \boldsymbol{min}(\boldsymbol{T_A(x)}, \boldsymbol{T_B(x)}), \boldsymbol{min}(\boldsymbol{I_A(x)}, \boldsymbol{I_B(x)}), \boldsymbol{max}(\boldsymbol{F_A(x)}, \boldsymbol{F_B(x)}) \rangle$$

Definition 2.4. [2] The difference between two fuzzy neutrosophic sets A and B is defined as $A \backslash B(x) = \langle x, min(T_A(x), F_B(x)), min(I_A(x), 1 - I_B(x)), max(F_A(x), T_B(x)) \rangle$.

Definition 2.5. [2] A fuzzy neutrosophic set A over the universe X is said to be null or empty fuzzy neutrosophic set if $T_A(x) = 0, I_A(x) = 0, F_A(x) = 1$ for all $x \in X$. It is denoted by $0_{N.}$

Definition 2.6. [2] A fuzzy neutrosophic set A over the universe X is said to be absolute (universe) fuzzy neutrosophic set if $T_A(x) = 1, I_A(x) = 1, F_A(x) = 0$ for all $x \in X$. It is denoted by $1_{N.}$

Definition 2.7. [2] The complement of a fuzzy neutrosophic set A is denoted by A^c and is defined as $A^c = \langle x, T_{A^c}(x), I_{A^c}(x), F_{A^c}(x)\rangle$ where $T_{A^c}(x) = F_A(x), I_{A^c}(x) = 1 - I_A(x), F_{A^c}(x) = T_A(x)$. The complement of fuzzy neutrosophic set A can also be defined as $A^c = 1_N - A$.

Definition 2.8. [1] A fuzzy neutrosophic topology on a non-empty set X is a τ of fuzzy neutrosophic sets in X saisfying the following axioms.
(i) $0_N, 1_N \in \tau$
(ii) $A_1 \cap A_2 \in \tau$ *for any* $A_1, A_2 \in \tau$
(iii) $\cup A_i \in \tau$ *for any arbitrary family* $\{A_i: i \in J\} \in \tau$
In this case the pair (X, τ) is called fuzzy neutrosophic topological space and any fuzzy neutrosophic set in τ is known as fuzzy neutrosophic open set in X.

Definition 2.9. [1] The complement A^c of a fuzzy neutrosophic set A in a fuzzy neutrosophic topological space (X, τ) is called fuzzy neutrosophic closed set in X.

Definition 2.10. [1] Let (X, τ) be a fuzzy neutrosophic topological space and $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$ be a fuzzy neutrosophic set in X. Then the closure and interior of A are defined by

$$int\,(A) = \cup\{G: G \text{ is a fuzzy neutrosophic open set in } X \text{ and } G \subseteq A\}$$
$$cl(A) = \cap\{G: G \text{ is a fuzzy neutrosophic closed set in } X \text{ and } A \subseteq G\}$$

Definition 2.11. [1] Let (X, τ) be a fuzzy neutrosophic topological space over X. Then the following properties hold. $(i)\ cl(A^c) = (int\ A)^c$, $(ii)\ int\ (A)^c = (cl\ A)^c$.

Definition 2.12. [11] A fuzzy neutrosophic set A_N in a fuzzy neutrosophic topological space (P, τ_N) is called a fuzzy neutrosophic F_σ −set if $A_N = \vee_{i=1}^{\infty} A_{N_i}$, where $\overline{A_{N_i}} \in \tau_N$ for $i \in I$.

Definition 2.13. [11] A fuzzy neutrosophic set A_N in a fuzzy neutrosophic topological space (P, τ_N) is called a fuzzy neutrosophic G_δ −set if $A_N = \wedge_{i=1}^{\infty} A_{N_i}$, where $A_{N_i} \in \tau_N$ for $i \in I$.

Definition 2.14. [11] A fuzzy neutrosophic set A_N in a fuzzy neutrosophic topological space (P, τ_N) is called a fuzzy neutrosophic dense if there exist no fnCS B_N in (P, τ_N) s.t $A_N \subset B_N \subset 1_X$. That is, $fn(A_N)^- = 1_N$.

Definition 2.15. [11] A fuzzy neutrosophic set A_N in a fuzzy neutrosophic topological space (P, τ_N) is called a fuzzy neutrosophic nowhere dense set if there exist no non zero fnOS B_N in (P, τ_N) s.t $B_N \subset fn(A_N)^-$. That is, $fn(((A_N)^-)^+) = 0_N$.

Definition 2.16. [11] Let (P, τ_N) be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set A_N in (P, τ_N) is called fuzzy neutrosophic one category set if $A_N = \vee_{i=1}^{\infty} A_{N_i}$, where A_{N_i}'s are fuzzy neutrosophic nowhere dense sets in (P, τ_N). Any other fuzzy neutrosophic set in (P, τ_N) is said to be of fuzzy neutrosophic two category.

Definition 2.17. [11] A fuzzy neutrosophic topological space (P, τ_N) is called fuzzy neutrosophic one category space if the fuzzy neutrosophic set 1_x is a fuzzy neutrosophic one category set in (P, τ_N). That is, $1_x = \vee_{i=1}^{\infty} A_{N_i}$, where A_{N_i}'s are fuzzy neutrosophic nowhere dense sets in (P, τ_N). Otherwise (P, τ_N) will be called a fuzzy neutrosophic two category space.

Definition 2.18. [11] Let A_N be a fuzzy neutrosophic one category set in (P, τ_N). Then $\overline{A_N}$ is called fuzzy neutrosophic residual set in (P, τ_N).

Definition 2.19. [11] A fuzzy neutrosophic topological space (P, τ_N) is called fuzzy neutrosophic Baire space if $\mathrm{fn}\left(\vee_{i=1}^{\infty} (A_{N_i})\right)^{+} = 0_N$,where A_{N_i}'s are fuzzy neutrosophic nowhere dense sets in (P, τ_N).

Definition 2.20. [11] Let (P, τ_N) be a fuzzy neutrosophic topological space. Then the following are equivalent.

(1) (P, τ_N) is a fuzzy neutrosophic Baire Space.
(2) $fn(A_N)^{+} = 0_N$, for every fuzzy neutrosophic one category set A_N in (P, τ_N).
(3) $fn(B_N)^{+} = 1_N$, for every fuzzy neutrosophic residual set B_N in (P, τ_N).

Definition 2.21.[5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set λ_N in (X_N, T_N) is called a fuzzy neutrosophic σ −nowhere dense set if λ_N is a fuzzy neutrosophic F_σ −set in (X_N, T_N) such that $int(\lambda_N) = 0_N$.

Definition 2.22. [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set λ_N in (X_N, T_N) is called a fuzzy neutrosophic σ −first category set if $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic σ −nowhere dense sets in (X_N, T_N). Any other fuzzy neutrosophic set in (X_N, T_N) is said to be fuzzy neutrosophic σ −second category sets in (X_N, T_N).

Definition 2.23. [5] Let λ_N be a fuzzy neutrosophic σ −first category set in (X_N, T_N). Then $1_N - \lambda_N$ is called a fuzzy neutrosophic σ −residual set in (X_N, T_N).

Definition 2.24. [5] A fuzzy neutrosophic topological space (X_N, T_N) is called fuzzy neutrosophic σ −first category space if the fuzzy neutrosophic set 1_{X_N} isa fuzzy neutrosophic σ −first category set in (X_N, T_N). That is $1_{X_N} = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic σ −nowhere dense sets in (X_N, T_N). Otherwise (X_N, T_N) will be called a fuzzy neutrosophic σ −second category space.

Definition 2.25. [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. Then (X_N, T_N) is called a fuzzy neutrosophic σ −Baire Space if $int(\vee_{i=1}^{\infty} (\lambda_{N_i})) = 0_N$, where (λ_{N_i})'s are fuzzy neutrosophic σ −nowhere dense sets in (X_N, T_N).

Theorem 2.26. [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. Then the following are equivalent.

(1) (X_N, T_N) is a fuzzy neutrosophic σ −Baire Space.
(2) $int(\lambda_N) = 0_N$, for every fuzzy neutrosophic σ −first category set λ_N in (X_N, T_N).
(3) $cl(\mu_N) = 1_N$, for every fuzzy neutrosophic σ −residual set μ_N in (X_N, T_N).

3. Fuzzy Neutrosophic μ-Strongly Nowhere Dense Sets

Definition 3.1. Let $(\boldsymbol{X_N}, \boldsymbol{T_N})$ be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set $\boldsymbol{\lambda_N}$ is called fuzzy neutrosopic μ-nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$ if $[\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})]$ is a fuzzy neutrosophic nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Definition 3.2. Let $(\boldsymbol{X_N}, \boldsymbol{T_N})$ be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set $\boldsymbol{\lambda_N}$ is called fuzzy neutrosophic μ-first category in $(\boldsymbol{X_N}, \boldsymbol{T_N})$ if $\boldsymbol{\lambda_N} = \vee_{i=1}^{\infty} (\boldsymbol{\lambda_{N_i}})$, where $(\boldsymbol{\lambda_{N_i}})$'s are fuzzy neutrosophic μ-nowhere dense sets in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. Otherwise are called as fuzzy neutrosophic μ-second category set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Definition 3.3. The complement of fuzzy neutrosophic μ-first category sets are called fuzzy neutrosophic μ-residual set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Definition 3.4. Let $(\boldsymbol{X_N}, \boldsymbol{T_N})$ be a fuzzy neturosophic topological space. A fuzzy neutrosophic set $\boldsymbol{\lambda_N}$ is called fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$ if $[\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})]$is a fuzzy neutrosophic μ-nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. That is, $\boldsymbol{\lambda_N}$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$ if μ – {Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$]} = $\boldsymbol{0_N}$ in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Example 3.5. Let $\boldsymbol{X_N}$ = {a, b, c} the fuzzy neutrosophic sets $\boldsymbol{A_N}$,$\boldsymbol{B_N}$ and $\boldsymbol{C_N}$ are defined on $\boldsymbol{X_N}$ as follows:

$\boldsymbol{A_N} : \boldsymbol{X_N} \to [\boldsymbol{0_N}, \boldsymbol{1_N}]$ is defined as, $\boldsymbol{A_N}$ = {⟨a,(0.6, 0.7, 0.6)⟩,⟨b,(0.5, 0.6, 0.7)⟩,⟨c,(0.7, 0.6, 0.6)⟩}

$\boldsymbol{B_N} : \boldsymbol{X_N} \to [\boldsymbol{0_N}, \boldsymbol{1_N}]$ is defined as, $\boldsymbol{B_N}$ = {⟨a,(0.5, 0.6, 0.7)⟩,⟨b,(0.8, 0.6, 0.6)⟩,⟨c,(0.6, 0.5, 0.5)⟩}

$\boldsymbol{C_N} : \boldsymbol{X_N} \to [\boldsymbol{0_N}, \boldsymbol{1_N}]$ is defined as, $\boldsymbol{C_N}$ = {⟨a,(0.7, 0.6, 0.8)⟩,⟨b,(0.8, 0.8, 0.6)⟩,⟨c,(0.5, 0.5, 0.5)⟩}

Then, $\boldsymbol{T_N} = \{\boldsymbol{0_N}, \boldsymbol{A_N}, \boldsymbol{B_N}, \boldsymbol{C_N}, \boldsymbol{A_N} \vee \boldsymbol{B_N}, \boldsymbol{B_N} \vee \boldsymbol{C_N}, \boldsymbol{A_N} \wedge \boldsymbol{B_N}, \boldsymbol{B_N} \wedge \boldsymbol{C_N}, \boldsymbol{A_N} \wedge \boldsymbol{C_N} \wedge (\boldsymbol{A_N} \wedge \boldsymbol{B_N} \wedge \boldsymbol{C_N}), \boldsymbol{1_N}\}$ is a fuzzy neutrosophic topology on $\boldsymbol{X_N}$. We computation see that μ-Intcl[$(\boldsymbol{A_N} \wedge \boldsymbol{B_N} \wedge \boldsymbol{C_N}) \wedge (\boldsymbol{1_N} - (\boldsymbol{A_N} \wedge \boldsymbol{B_N} \wedge \boldsymbol{C_N}))$] = $\boldsymbol{0_N}$. Hence, $(\boldsymbol{A_N} \wedge \boldsymbol{B_N} \wedge \boldsymbol{C_N})$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Proposition 3.6. If $\boldsymbol{\lambda_N}$ is a fuzzy neutrosophic μ-nowhere dense set in a fuzzy neutrosophic topological space $(\boldsymbol{X_N}, \boldsymbol{T_N})$, then $\boldsymbol{\lambda_N}$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Proof. Let $\boldsymbol{\lambda_N}$ be a fuzzy neutrosophic μ-nowhere dense set in a fuzzy neutrosophic topological space $(\boldsymbol{X_N}, \boldsymbol{T_N})$, then μ − Intcl($\boldsymbol{\lambda_N}$) = $\boldsymbol{0_N}$ in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. Since $[\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})] \leq \boldsymbol{\lambda_N}$ in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. We obtain that μ – Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$] ≤ μ – Intcl($\boldsymbol{\lambda_N}$) and hence μ –{Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$]} ≤ $\boldsymbol{0_N}$. That is μ – {Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$]} = $\boldsymbol{0_N}$. Hence $\boldsymbol{\lambda_N}$ is afuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Proposition 3.7. If $(\boldsymbol{1_N} - \boldsymbol{\lambda_N})$ is a fuzzy neutrosophic μ-nowhere dense set in a fuzzy neutrosophic topological space in $(\boldsymbol{X_N}, \boldsymbol{T_N})$, then $\boldsymbol{\lambda_N}$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Proof. Suppose that $(\boldsymbol{1_N} - \boldsymbol{\lambda_N})$ is a fuzzy neutrosophic μ-nowhere dense set in a fuzzy neutrosophic topological space $(\boldsymbol{X_N}, \boldsymbol{T_N})$, then μ – [Intcl($\boldsymbol{1_N} - \boldsymbol{\lambda_N}$)] = $\boldsymbol{0_N}$ in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. Since $[\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})] \leq (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$, μ – {Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$]} ≤ μ – [Intcl($\boldsymbol{1_N} - \boldsymbol{\lambda_N}$)] and hence μ – {Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$]} ≤ $\boldsymbol{0_N}$. Implies that μ – {Intcl[$\boldsymbol{\lambda_N} \wedge (\boldsymbol{1_N} - \boldsymbol{\lambda_N})$]} = $\boldsymbol{0_N}$. Hence $\boldsymbol{\lambda_N}$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Proposition 3.8. If μ – [Clint($\mathbf{1}_N - \boldsymbol{\lambda}_N$)] = $\mathbf{1}_N$, for a fuzzy neutrosophic set $\boldsymbol{\lambda}_N$ defined on $(\boldsymbol{X}_N, \boldsymbol{T}_N)$, then $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proof. Suppose that μ – [Clint($\mathbf{1}_N - \boldsymbol{\lambda}_N$)] = $\mathbf{1}_N$ in$(\boldsymbol{X}_N, \boldsymbol{T}_N)$. Then $\mathbf{1}_N$ – {μ – [Clint($\mathbf{1}_N - \boldsymbol{\lambda}_N$)]} = $\mathbf{0}_N$ which implies that $\mathbf{1}_N - \mathbf{1}_N$ – {μ – [Intcl($\mathbf{1}_N - \mathbf{1}_N - \boldsymbol{\lambda}_N$)]} = $\mathbf{0}_N$. We obtain μ – [Intcl($\boldsymbol{\lambda}_N$)] = $\mathbf{0}_N$. Thus $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$. By proposition 3.2, $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proposition 3.9. If $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-strongly nowhere dense in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$, then $(\mathbf{1}_N - \boldsymbol{\lambda}_N)$ is also a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proof. Let $\boldsymbol{\lambda}_N$ be a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$ which implies that μ – {Intcl[$\boldsymbol{\lambda}_N \wedge (\mathbf{1}_N - \boldsymbol{\lambda}_N)$]} = $\mathbf{0}_N$ in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$. Now, μ – {[Intcl[$(\mathbf{1}_N - \boldsymbol{\lambda}_N) \wedge (\mathbf{1}_N - \mathbf{1}_N - \boldsymbol{\lambda}_N)$]]} = μ – {[Intcl[$(\mathbf{1}_N - \boldsymbol{\lambda}_N) \wedge \boldsymbol{\lambda}_N$]]} = $\mathbf{0}_N$. This implies that $(\mathbf{1}_N - \boldsymbol{\lambda}_N)$ is also a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proposition 3.10. If $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-nowhere dense set in fuzzy neutrosophic topological space $(\boldsymbol{X}_N, \boldsymbol{T}_N)$ then $(\mathbf{1}_N - \boldsymbol{\lambda}_N)$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proof. Let $\boldsymbol{\lambda}_N$ be a fuzzy neutrosophic μ-nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$. Now, by proposition 3.2, we get $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$ and by proposition 3.4, we obtain that $(\mathbf{1}_N - \boldsymbol{\lambda}_N)$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proposition 3.11. If $\boldsymbol{\lambda}_N$ is a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$ then μ – cl[$\boldsymbol{\lambda}_N \wedge (\mathbf{1}_N - \boldsymbol{\lambda}_N)$] = $\mathbf{1}_N$ in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Proof. Let $\boldsymbol{\lambda}_N$ be a fuzzy neutrosophic μ-strongly nowhere dense set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$. Then μ – cl[$\boldsymbol{\lambda}_N \wedge (\mathbf{1}_N - \boldsymbol{\lambda}_N)$] = $\mathbf{0}_N$ implies that μ – cl[$\mathbf{1}_N$ – Int[$\boldsymbol{\lambda}_N \wedge (\mathbf{1}_N - \boldsymbol{\lambda}_N)$]] = $\mathbf{1}_N$ which implies that μ – cl[Int($\mathbf{1}_N - \boldsymbol{\lambda}_N$) ∧ $(\mathbf{1}_N - \mathbf{1}_N - \boldsymbol{\lambda}_N)$] = $\mathbf{1}_N$. But μ – [Clint[$\mathbf{1}_N - \boldsymbol{\lambda}_N$) ∧ $\boldsymbol{\lambda}_N$] ≤ μ – [cl[$(\mathbf{1}_N - \boldsymbol{\lambda}_N) \vee \boldsymbol{\lambda}_N$]], this implies that $\mathbf{1}_N$ ≤ μ – cl[$(\mathbf{1}_N - \boldsymbol{\lambda}_N) \vee \boldsymbol{\lambda}_N$]. Hence we get μ – cl[$\boldsymbol{\lambda}_N \vee (\mathbf{1}_N - \boldsymbol{\lambda}_N)$] = $\mathbf{1}_N$ in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

4. Fuzzy Neutrosophic μ-strongly first(second) category sets

Definition 4.1. A fuzzy neturosophic set $\boldsymbol{\lambda}_N$ is said to be fuzzy neutrosophic μ-strongly first category set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$ if $\boldsymbol{\lambda}_N = \vee_{i=1}^{\infty} (\boldsymbol{\lambda}_{N_i})$, where $(\boldsymbol{\lambda}_{N_i})$'s are fuzzy neutrosophic μ-strongly nowhere dense sets in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$. Otherwise are called as fuzzy neutrosophic μ-strongly second category set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Definition 4.2. The complement of fuzzy neutrosophic μ-strongly first category sets are called fuzzy neutrosophic μ-strongly residual set in $(\boldsymbol{X}_N, \boldsymbol{T}_N)$.

Example 4.3. Let $(\boldsymbol{X}_N, \boldsymbol{T}_N)$ be a fuzzy neutrosophic topological space, where $\boldsymbol{X}_N$ = {a, b} and defined the fuzzy neutrosophic sets are $\boldsymbol{A}_N$, $\boldsymbol{B}_N$, $\boldsymbol{C}_N$, $\boldsymbol{D}_N$, $\boldsymbol{E}_N$ as following

$\boldsymbol{A}_N : \boldsymbol{X}_N \rightarrow [\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{A}_N$ = {⟨a,(0.6, 0.4, 0.8)⟩,⟨b,(0.8, 0.6, 0.9)⟩}

$\boldsymbol{B}_N : \boldsymbol{X}_N \rightarrow [\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{B}_N$ = {⟨a,(0.6, 0.3, 0.8)⟩,⟨b,(0.9, 0.2, 0.7)⟩}

$\boldsymbol{C}_N : \boldsymbol{X}_N \rightarrow [\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{C}_N$ = {⟨a,(0.5, 0.4, 0.9)⟩,⟨b,(0.7, 0.8, 0.9)⟩}

$\boldsymbol{D}_N : \boldsymbol{X}_N \rightarrow [\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{D}_N$ = {⟨a,(0.4, 0.6, 0.9)⟩,⟨b,(0.6, 0.8, 0.9)⟩}

$E_N : X_N \rightarrow [0_N, 1_N]$ is defined as E_N = {⟨a,(0.3, 0.7, 0.9)⟩,⟨b,(0.5, 0.9, 0.9)⟩}

Then, T_N = {0_N , A_N , B_N , C_N , D_N , E_N , 1_N } is a fuzzy neutrosophic topology on X_N . We computation see that, fuzzy neutrosophic μ-strongly first category set is B_N and fuzzy neutrosophic μ-strongly second category sets are {0_N , A_N , C_N , D_N , E_N , 1_N }. Also, the fuzzy neutrosophic μ-strongly residual set is $1_N - B_N$. .

Proposition 4.4. If λ_N is a fuzzy neutrosophic μ-first category set in (X_N, T_N),then λ_N is a fuzzy neutrosophic μ-strongly first category set in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic μ-first category set in a fuzzy neutrosophic topological space (X_N, T_N). Then $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic μ-strongly nowhere dense sets in (X_N, T_N). The fact "Every fuzzy neutrosophic μ-nowhere dense set is fuzzy neutrosophic μ-strongly nowhere dense set" in (X_N, T_N). We deduce that (λ_{N_i})'s are fuzzy neutrosophic μ-strongly nowhere dense sets and hence $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic μ-strongly nowhere dense sets in (X_N, T_N). Hence λ_N is a fuzzy neutrosophic μ–strongly first category set in (X_N, T_N).

Remark 4.5. A fuzzy neutrosophic μ-strongly first category sets need not be a fuzzy neutrosophic μ-first category set in (X_N, T_N). For, consider the following example.

Example 4.6. Let (X_N, T_N) be a fuzzy neutrosophic topological space. X_N = {a} and define the fuzzy neutrosophic sets are A_N ,B_N ,C_N as follows:

$A_N : X_N \rightarrow [0_N, 1_N]$ is defined as A_N = {a,(0.6, 0.4, 0.6)}

$B_N : X_N \rightarrow [0_N, 1_N]$ is defined as B_N = {a,(0.3, 0.4, 0.6)}

$C_N : X_N \rightarrow [0_N, 1_N]$ is defined as C_N = {a,(0.9, 0.7, 0.6)}

Then, T_N = {0_N , A_N , B_N , C_N , 1_N } is a fuzzy neutrosophic topology on X_N . We computation see that, A_N and B_N fuzzy neutrosophic μ-strongly nowhere dense sets in (X_N, T_N). Also($A_N \vee B_N$) = A_N this implies that, A_N is a fuzzy neutrosophic μ-strongly first category set in (X_N, T_N). Now, $(1_N - B_N)$ and $(1_N - C_N)$ are fuzzy neutrosophic μ-nowhere dense sets in(X_N, T_N). Then $[(1_N - B_N) \vee (1_N - C_N)] = D_N$ is not a fuzzy neutrosophic μ-first category set in (X_N, T_N).

Proposition 4.7. If $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic closed sets with μ − Int(λ_N) = 0_N , then λ_N is a fuzzy neutrosophic μ-strongly first category set in (X_N, T_N).

Proof. Suppose $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic closed sets with μ − Int(λ_N) = 0_N in (X_N, T_N). If λ_N is a fuzzy neutrosophic closed in fuzzy neutrosophic topological space (X_N, T_N) with μ − Int(λ_{N_i}) = 0_N, then λ_N is a fuzzy neutrosophic μ-strongly nowhere dense set. Then use this concept λ_N is a fuzzy neutrosophic closed in fuzzy neutrosophic topological space (X_N, T_N) with μ − Int(λ_{N_i}) = 0_N . Thus, λ_N is a fuzzyneutrosophic μ-strongly nowhere dense set and then we have $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic μ-strongly nowhere dense sets in (X_N, T_N). Therefore λ_N is a fuzzyneutrosophic μ-strongly first category set in (X_N, T_N).

5. Fuzzy Neutrosophic μ-strongly Baire Spaces

Definition 5.1. A fuzzy neutrosophic topological space $(\boldsymbol{X_N}, \boldsymbol{T_N})$ is called fuzzy neutrosophic μ-strongly Baire Space if μ − cl[$\vee_{i=1}^{\infty}$ $(\boldsymbol{\lambda}_{N_i})$] = $\mathbf{1}_N$, where $(\boldsymbol{\lambda}_{N_i})$'s are fuzzy neutrosophic μ-strongly nowhere dense sets in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Example 5.2. Let $(\boldsymbol{X_N}, \boldsymbol{T_N})$ be a fuzzy neutrosophic topological space. Where $\boldsymbol{X_N}$ = {a} and define the fuzzy neutrosophic sets are $\boldsymbol{A_N}$, $\boldsymbol{B_N}$ and $\boldsymbol{C_N}$ as follows:

$\boldsymbol{A_N}$: $\boldsymbol{X_N}$ → $[\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{A_N}$ = {⟨a,(0.7, 0.8, 0.9)⟩}

$\boldsymbol{B_N}$: $\boldsymbol{X_N}$ → $[\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{B_N}$ = {⟨a,(0.3, 0.4, 0.6)⟩}

$\boldsymbol{C_N}$: $\boldsymbol{X_N}$ → $[\mathbf{0}_N, \mathbf{1}_N]$ is defined as $\boldsymbol{C_N}$ = {⟨a,(0.9, 0.7, 0.6)⟩}

Then, $\boldsymbol{T_N}$= {$\mathbf{0}_N$, $\boldsymbol{A_N}$, $\boldsymbol{B_N}$, $\boldsymbol{C_N}$, $\mathbf{1}_N$} is a fuzzy neutrosophic topology on $\boldsymbol{X_N}$. We computation see that, fuzzy neutrosophic μ-strongly nowhere dense sets are $\boldsymbol{A_N}$ and $\boldsymbol{B_N}$. Also, μ−cl[$\boldsymbol{A_N}$ ∨$\boldsymbol{B_N}$] = $\mathbf{1}_N$. Hence $(\boldsymbol{X_N}, \boldsymbol{T_N})$ is a fuzzy neutrosophic μ-strongly Baire Space.

Proposition 5.3. Let $(\boldsymbol{X_N}, \boldsymbol{T_N})$ be a fuzzy neutrosophic topological space. Then the following are equivalent

(i) $(\boldsymbol{X_N}, \boldsymbol{T_N})$ is a fuzzy neutrosophic μ-strongly Baire space.

(ii) μ − [cl($\boldsymbol{\lambda_N}$)] = $\mathbf{1}_N$, for every fuzzy neutrosopic μ-strongly first category set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

(iii) μ − [Int($\boldsymbol{\gamma_N}$)] = $\mathbf{0}_N$, for every fuzzy neutrosophic μ-strongly residual set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$.

Proof.

(i) ⟹ (ii)

Let $\boldsymbol{\lambda_N}$ be a fuzzy neutrosophic μ-strongly first category set in$(\boldsymbol{X_N}, \boldsymbol{T_N})$. Then $\boldsymbol{\lambda_N}$ = $\vee_{i=1}^{\infty}$ $(\boldsymbol{\lambda}_{N_i})$, where $(\boldsymbol{\lambda}_{N_i})$'s are fuzzy neutrosophic μ-strongly nowhere dense sets in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. Since$(\boldsymbol{X_N}, \boldsymbol{T_N})$ is a fuzzy neutrosophic μ-strongly Baire space implies that μ−cl[$\vee_{i=1}^{\infty}$ $(\boldsymbol{\lambda}_{N_i})$] =$\mathbf{1}_N$. Hence μ − [cl($\boldsymbol{\lambda_N}$)] = $\mathbf{1}_N$.

(ii) ⟹ (iii)

Let $\boldsymbol{\gamma_N}$ be a fuzzy neutrosophic μ-strongly residual set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. Then $\mathbf{1}_N$ −$\boldsymbol{\gamma_N}$ is a fuzzy neutrosophic μ-strongly first category set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. We obtain that μ−cl[($\mathbf{1}_N$ −$\boldsymbol{\gamma_N}$)] = $\mathbf{1}_N$ implies that $\mathbf{1}_N$ − [μ − Int($\boldsymbol{\gamma_N}$)] = $\mathbf{1}_N$. Hence μ − Int($\boldsymbol{\gamma_N}$) = $\mathbf{0}_N$.

(iii) ⟹ (i)

Let $\boldsymbol{\lambda_N}$ be a fuzzy neutrosophic μ-strongly first category set in$(\boldsymbol{X_N}, \boldsymbol{T_N})$. Then $\boldsymbol{\lambda_N}$ = $\vee_{i=1}^{\infty}$ $(\boldsymbol{\lambda}_{N_i})$, where $(\boldsymbol{\lambda}_{N_i})$'s are fuzzy neutrosophic μ-strongly nowhere dense sets in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. We have $\boldsymbol{\lambda_N}$ is a fuzzy neutrosophic μ-strongly first category set then ($\mathbf{1}_N$ − $\boldsymbol{\lambda_N}$) is a fuzzy neutrosophic μ−strongly residual set in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. We obtain that μ − Int($\mathbf{1}_N$ − $\boldsymbol{\lambda_N}$) =$\mathbf{0}_N$ which gives that $\mathbf{1}_N$ − [μ − cl($\boldsymbol{\lambda}_{N_i}$)] = $\mathbf{0}_N$. Therefore we get μ − cl($\boldsymbol{\lambda}_{N_i}$) = $\mathbf{1}_N$ and hence μ − cl[$\vee_{i=1}^{\infty}$ $(\boldsymbol{\lambda}_{N_i})$] = $\mathbf{1}_N$, where$(\boldsymbol{\lambda}_{N_i})$'s are fuzzy neutrosophic μ-strongly nowhere dense sets in $(\boldsymbol{X_N}, \boldsymbol{T_N})$. Hence $(\boldsymbol{X_N}, \boldsymbol{T_N})$ is a fuzzy neutrosophic μ-strongly Baire space.

"The following diagram illustrates the relationship among fuzzy neutrosophic strongly μ-nowhere dense sets, μ-category sets, and μ-Baire spaces."

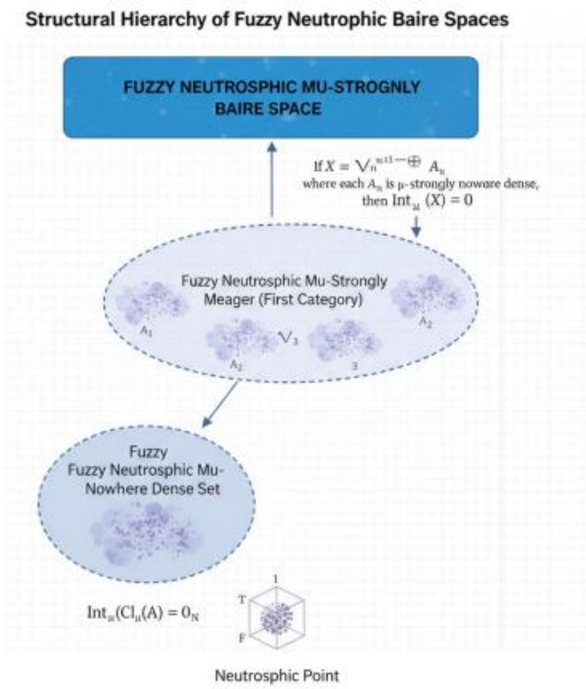

Figure 1. Structural Hierarchy of Fuzzy Neutrosophic Strongly μ- Baire Space

6. Applications

Fuzzy neutrosophic strongly μ −nowhere dense and Baire Spaces play an important role in handling uncertainty, indeterminacy, and incomplete information in various real-world problems. In data analysis and information systems, these concepts help in identifying insignificant or negligible data regions within large datasets where uncertainty and indeterminacy are present. In medical diagnosis, fuzzy neutrosophic μ −Baire Spaces can be used to model complex patient data where symptoms, test results, and diagnoses may contain vagueness and indeterminate values.

7. Conclusions

In this study, we introduced and explored the concept of fuzzy neutrosophic μ-strongly various sets and fuzzy neutrosophic μ-strongly Baire space. We have examined the foundational properties of fuzzy neutrosophic μ-strongly various sets and fuzzy neutrosophic μ- strongly Baire space. Future work may include the study of other functions, compactness and connectedness in these sets and spaces as well as potential Generalizations and real-world applications.

Funding: No funding was received for this research work.

Acknowledgments: The author thanks the AMET Deemed to be University, Kanathur, Chennai, Tamil Nadu, India for providing the necessary facilities and encouragement to carry out this research.

Conflicts of Interest: .The author declares no conflict of interest.

References

[1]Arockiarani I.; J. Martina Jency. More on Fuzzy Neutrosophic Sets and Fuzzy Neutrosophic Topological Spaces., International Journal of Innovative Research & Studies, Volume 3(5), 2014, 643-652.

[2] Arockiarani I.; I. R. Sumathi.; J.Martina Jency. Fuzzy Neutrosophic Soft Topological Spaces. International

Journal of Mathematical archive, Volume 4(10), 2013., 225-238.
[3] Chang C.L, Fuzzy Topological Spaces, Journal of Mathematical Analysis and Applications, Volume 24(1), 1968., 182-190.
[4] Poongothai,E.; Kalaivani, E. Neutrosophic σ-Baire Spaces on Fuzzy setting, Journal of Xedian University, 2024, Volume 18(1,) pp. 253-264.
[5] Poongothai,E.; Kalaivani, E. A Characterzation of Neutrosophic σ-Baire Spaces on Fuzzy setting. Neutrosophic Sets and Systems, 2025, Volume 83, pp. 87-95.
[6] Poongothai, E.; Kalaivani, E. Some remarks on Fuzzy Neutrosophic σ-Baire Spaces and other Fuzzy Neutrosophic Spaces, Neutrosophic Sets and Systems, 2025, Volume 90, pp.998-1010.
[7] Poongothai, E.; Kalaivani, E. Fuzzy Neutrosophic Regular σ-Baire Space. Neutrosophic Sets and Systems,2025, Volume 90, pp. 538-551.
[8] Ganesan Thangaraj.; Eswaran Poongothai. A Note on Fuzzy σ-Baire Spaces, Annals of Fuzzy Mathematicsand Informatics, 2015, Volume 9(2), pp. 197-203.
[9] Poongothai,E.; Padmavathi, E. Baire Spaces on Fuzzy Neutrosophic Topological Spaces. Neutrosophic Sets and Systems, 2022, Volume 51(1), pp. 708-723.
[10] Thangaraj, G.; Poongothai, E. On Fuzzy σ-Baire Spaces, International Journal of Fuzzy Math. and Sys.,2013, Volume 3(4), pp. 275-283.
[11] Thangaraj, G.; Poongothai, E., On Fuzzy Regular σ-Baire Spaces, Advances in Fuzzy Mathematics, 2014,Volume 9(1), pp. 95-104.
[12] Smarandache fuzzy, Neutrosophic Set, A Generalization of the Intuitionistic Fuzzy Sets, Inter.J.PureAppl.Math., Volume 24, 2005., 287-297.
[13] Veereswari V., An Introduction to fuzzy Neutrosophic Topological spaces, IJMA, Volume 8(3), 2017.,144-149.
[14] Zadeh L.A., fuzzy sets, Information and control, Volume 8, 1965, 338-353.
[15] Poongothai E., Balaganesan P., and Florentin Smarandache., A Study of Fuzzy σ- Baire, Supra-Baire andSoft Baire Spaces, Neutrosophic Science International Association Publishing House, 2026, Book pages-169

Received: Oct 5, 2025. Accepted: March 13, 2026