NSS

Neutrosophic Sets and Systems, Vol. 99, 2026

University of New Mexico

Best Proximity Point Results in Neutrosophic Fuzzy Metric Space for Ciric type α-ψ proximal contractive mappings

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Abstract. The present manuscript delves into the ideology of best proximity point under the purview of neutrosophic fuzzy metric space. For this purpose, the concept of a new type of mapping named α-proximal admissible mapping has been defined under the realm of the said space. Also for the contractive condition, the pre-existent ideologies of Ciric and rational contraction has been combined. Notably, for the sake of best proximity, concept of α-ψ proximal contraction has also been considered. Here α and ψ can also be regarded as control functions. Additionally, some examples and applications are also deduced in support of the established results, for some particular conditions. To the best of known literatures, the results are established for the first time in the mentioned area.

Keywords: Neutrosophic sets (NS); Neutrosophic fuzzy sets (NFS); Continuous t-norm (CTN); Continuous t-conorm (CTCN); Neutrosophic fuzzy metric space (NFMS); Best proximity point (BPP); α-proximal admissible (α-P_A); α-ψ-proximal contractive (α-ψ-P_{CV}); α-ψ-contractive (α-ψ-C_V); t-uniformly continuous (t-unif. cont.); α admissible (α adm).

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1. Introduction

An advanced approach to handle uncertainity and vagueness is neutrosophic sets. It uses three types of membership grades related to truth, indeterminacy and falsity to establish its purpose. And that's the most important part which makes it more efficient than the conventional fuzzy sets in dealing with cases related to uncertainity. This kind of set was first introduced in a book by Smarandache [2], where the concept of neutrosophy is described as a philosophical study of neutrality. Some later developments of this concepts include, neutrosophic fuzzy set, which was established by Das et. al. [17] in 2020, and neutrosophic metric space which was given by Kirisci et. al. [12] in 2020. Neutrosophic metric space is a framework which is also used to model various real life situations related to vagueness. It combines the concepts of neutrosophic sets and metric space altogether to fulfill it's name and purpose. After this development, various fixed point results has been established in this ideology by several mathematicians such as [13–16].

Also, the notion of fuzzy set was first given by Zadeh [1] in 1965, through one of his groundbreaking researches. It is the ideology that describes the extent to which a random element belonging to any random universal set becomes an element of a more specified set, which is itself a subset of that universal set. The establishment of this theory, allowed it to be used as a base for the introduction of another concept named fuzzy metric space. Then, the ideology of fuzzy sets and metric space was amalgamated together by Kramosil et. al. [3] in 1975, to introduce a new generalization of the traditional metric space, and they called it fuzzy metric space.

Simultaneously many other recent developments in classical metric space concept has taken place. Such as the concept of rough metric space by Biswas [27] in 1996 evolved from the concept of rough sets given by a computer scientist Pawlak [28] in 1982. Then the theory of soft sets also took shape in the hands of a renowned mathematician named Molodstov [29] in 1999. It was designed to handle real life situations related to vagueness and uncertainity, where the classical set theory falls short of. It is usually characterized by a set of specifications, and a mapping that links each of the specifications from the set to another set which is a subset of a random universal set. After this discovery, the concept of soft metric space also developed under the hands of Das et. al. [30] in 2013. It also turned out to be an extention of the traditional theory for metric space. Then concept of NFMS was also introduced by Ghosh et. al. [19] in 2024, by extending the previously established concept of neutrosophic fuzzy sets. Inspired by Das et. al.'s idea, NFMS uses an extra fuzzy membership grade along with the pre-existing neutrosophic ones.

Lately, some developments in fixed point results in the realm of different metric space generalizations, has been showcased in [8, 9, 14, 18]. Remarkably after [19], not only several conventional fpt established in framework for the said space but also, the ideology of semi metric has been defined in it's realm. Lastly, all the established concepts has been brought together so as to charecterize the completeness of the said space. And also in [10], Sonam et. al. laid the foundation of soft rectangular b-metric space, as well as deduced some fixed point results under it's purview.

On the other hand, the terminology best proximity point puts forward a procedure to deal with minimum distance between two sets. Mathematically, for a non self mapping $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$, an element $\mathfrak{m} \in \mathfrak{J}$ is recognized as the BPP of $\mathfrak{T}$ if $\tau(\mathfrak{m}, \mathfrak{T}(\mathfrak{m})) = \inf\{\tau(\mathfrak{p}, \mathfrak{q}) : \mathfrak{p} \in \mathfrak{J}, \mathfrak{q} \in \mathfrak{K}\}$, where τ denotes the accustomed distance metric. This kind of result was first established by Basha [25], and further developments in the concept with respect to different other generalisations of metric space, has been brought by several other mathematicians such as [22–24,26,31]. More specifically, [22] describes some best proximity results in metric space and G-metric space, where control functions has been brought to use.

To the best of known results, the literature is unavailable for best proximity point in neutrosophic fuzzy metric space. However fixed point results have already been established in NMS and NFMS. The present paper deals with the concept of best proximity point in neutrosophic fuzzy metric space, where Ciric type mappings which are also $\alpha - \psi$ proximal, have been taken into consideration. Applications of the established results have also been discussed for some specified conditions.

2. Preliminaries

Definition 2.1. (NFMS) [19,20] A 7-tuple $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is known as an NFMS if $\mathfrak{S}$ is any random set, $\star$ is a CTN, $\diamond$ is a CTCN and $\mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}$ are fuzzy sets on $\mathfrak{S}^2 \times (0, \infty)$, which are similar to the notations $\mu(x), T_B(x, \mu), I_B(x, \mu)$ and $F_B(x, \mu)$ in Definition 3 in [19], satisfying the following conditions $\forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{p} \in \mathfrak{S}$ and $\mathfrak{z}, \mathfrak{a} > 0$,

(1) $0 \leq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq 1,\ 0 \leq \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq 1,\ 0 \leq \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq 1,\ 0 \leq \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq 1$,
(2) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) + \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq 4$,
(3) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})$,
(4) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$ iff $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$,
(5) $\lim_{\mathfrak{z} \to \infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$,
(6) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{m}, \mathfrak{p}, \mathfrak{a}) \leq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z} + \mathfrak{a})$,
(7) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .) : \mathbb{R} - (-\infty, 0) \to \mathbb{R}^+ - (1, \infty)$ is continuous,
(8) $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})$,
(9) $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$ iff $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$,
(10) $\lim_{\mathfrak{z} \to \infty} \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$,
(11) $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \star \mathfrak{D}(\mathfrak{m}, \mathfrak{p}, \mathfrak{a}) \leq \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z} + \mathfrak{a})$,

(12) $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .) : \mathbb{R} - (-\infty, 0) \to \mathbb{R}^+ - (1, \infty)$ is continuous,

(13) $\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{F}(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})$,

(14) $\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 0$ iff $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$,

(15) $\lim_{\mathfrak{z}\to\infty} \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 0$,

(16) $\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \diamond \mathfrak{F}(\mathfrak{m}, \mathfrak{p}, \mathfrak{a}) \geq \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z} + \mathfrak{a})$,

(17) $\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .) : \mathbb{R} - (-\infty, 0) \to \mathbb{R}^+ - (1, \infty)$ is continuous,

(18) $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{H}(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})$,

(19) $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 0$ iff $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$,

(20) $\lim_{\mathfrak{z}\to\infty} \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 0$,

(21) $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \diamond \mathfrak{H}(\mathfrak{m}, \mathfrak{p}, \mathfrak{a}) \geq \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z} + \mathfrak{a})$,

(22) $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .) : \mathbb{R} - (-\infty, 0) \to \mathbb{R}^+ - (1, \infty)$ is continuous,

(23) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 0,\ \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 0,\ \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$ and $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$ if $\mathfrak{z} \leq 0$.

Where $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$ denotes the surety that the distance between $\mathfrak{O}_{\mathfrak{nf}}$ and $\mathfrak{m}$ is less than $\mathfrak{z}$, and $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$ denotes the degree of nearness, neautralness, non-nearness respectively of $\mathfrak{O}_{\mathfrak{nf}}$ and $\mathfrak{m}$ w.r.t $\mathfrak{z}$.

Remark 2.2. [19, 20] From the above definition, it is clear that $\forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S},\ \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .)$ and $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .)$ are non-decreasing functions on $[0, \infty)$, and $\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .)$ and $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, .)$ are non-increasing functions on $[0, \infty)$.

Remark 2.3. [19, 20] It can be concluded easily from Remark 2.2 that, given $\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$, if $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}),\ \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) > 1 - \mathfrak{z}$ and $\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}),\ \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) < \mathfrak{z}\ \forall\ \mathfrak{z} \in (0, \infty)$, then $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$.

Definition 2.4. [20] $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS. Then,

(a) A sequence $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} \in \mathfrak{S}$ is known to be a Cauchy sequence if for each $\mathfrak{z}, \mathfrak{a} > 0$,

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+\mathfrak{a}}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = \lim_{\mathfrak{n}\to\infty} \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+\mathfrak{a}}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = 1 \text{ and},$$
$$\lim_{\mathfrak{n}\to\infty} \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+\mathfrak{a}}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = \lim_{\mathfrak{n}\to\infty} \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+\mathfrak{a}}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = 0.$$

(b) A sequence $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} \in \mathfrak{S}$ is converging to some $\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{S}$ if $\forall\ \mathfrak{z} > 0$,

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \lim_{\mathfrak{n}\to\infty} \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = 1 \text{ and},$$
$$\lim_{\mathfrak{n}\to\infty} \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \lim_{\mathfrak{n}\to\infty} \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = 0.$$

A NFMS is said to be complete iff every Cauchy sequence is convergent, and it is known to be compact if every sequence contains a convergent subsequence.

Definition 2.5. [21] Let $(\mathfrak{S}, \tau)$ be a metric space, and $\mathfrak{T}$ be a self mapping on $\mathfrak{S}$. Then $\mathfrak{T}$ is a Ciric contraction or quasi-contraction if and only if,

$$\tau(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}) \leq \mathfrak{q}.\max\{\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}), \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}), \tau(\mathfrak{m}, \mathfrak{T}\mathfrak{m}), \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}), \tau(\mathfrak{m}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}})\}$$

$\forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$, where $\mathfrak{q} \in [0, 1)$.

3. Main Results

In this section, some new concepts and BPP results have been established in a complete NFMS.

Definition 3.1. For a NFMS $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$, a mapping $\mathfrak{T} : \mathfrak{S} \to \mathfrak{S}$ is t-unif. cont. if for every $0 < \epsilon < 1$, $\exists\, \delta \in (0,1) :\ \forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$ and $\mathfrak{z} > 0$,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq 1-\delta,\ \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq 1-\delta,\ \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq \delta, \text{ and, } \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq \delta$$

gives,

$$\mathfrak{B}(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) \geq 1-\epsilon,\ \mathfrak{D}(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) \geq 1-\epsilon,\ \mathfrak{F}(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) \leq \epsilon, \text{ and, } \mathfrak{H}(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) \leq \epsilon.$$

Remark 3.2. Suppose $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS and $\mathfrak{T}$ is a t-unif. cont. mapping on $\mathfrak{S}$. Then for a sequence $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} \in \mathfrak{S}$ if $\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to \mathfrak{O}_{\mathfrak{nf}}$ as $\mathfrak{n} \to \infty$, $\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}$ as $\mathfrak{n} \to \infty$.

Remark 3.3. Suppose $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS where $\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to \mathfrak{O}_{\mathfrak{nf}}$ and $\mathfrak{m}_\mathfrak{n} \to \mathfrak{m}$ as $\mathfrak{n} \to \infty$. Then, $\forall\ \mathfrak{z} > 0$ as $\mathfrak{n} \to \infty$,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{m}_\mathfrak{n}, \mathfrak{z}) \to \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}),\ \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{m}_\mathfrak{n}, \mathfrak{z}) \to \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}),\ \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{m}_\mathfrak{n}, \mathfrak{z}) \to \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}),$$
$$\text{and, } \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{m}_\mathfrak{n}, \mathfrak{z}) \to \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}).$$

Definition 3.4. For a NFMS $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$, the neutrosophic fuzzy metric $(\mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H})$ is triangular $\forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{p} \in \mathfrak{S}$ and $\mathfrak{z} > 0$ iff,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})} - 1 \leq \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z})} - 1 + \frac{1}{\mathfrak{B}(\mathfrak{p}, \mathfrak{m}, \mathfrak{z})} - 1,$$
$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})} - 1 \leq \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z})} - 1 + \frac{1}{\mathfrak{D}(\mathfrak{p}, \mathfrak{m}, \mathfrak{z})} - 1,$$
$$\mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z}) + \mathfrak{F}(\mathfrak{p}, \mathfrak{m}, \mathfrak{z}),$$
$$\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \leq \mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z}) + \mathfrak{H}(\mathfrak{p}, \mathfrak{m}, \mathfrak{z}).$$

Now for the results, it has been considered that,

$$\Psi = \left\{ \psi : [0,\infty) \to [0,\infty) \text{ s.t } \psi \text{ is increasing and } \sum_{\mathfrak{n}=1}^{+\infty} \psi^{\mathfrak{n}}(\mathfrak{z}) < +\infty\ \forall\ \mathfrak{z} > 0 \right\}$$

where $\psi^{\mathfrak{n}}$ denotes the value obtained after $\mathfrak{n}$ repeatations of ψ.

Also, let $\mathfrak{J} \neq \phi, \mathfrak{K} \neq \phi \subset (\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ which is an NFMS, and,

$$\begin{aligned} \mathfrak{J}_0(\mathfrak{z}) &= \{\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) \text{ for some } \mathfrak{m} \in \mathfrak{K}\}, \\ \mathfrak{K}_0(\mathfrak{z}) &= \{\mathfrak{m} \in \mathfrak{K} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) \text{ for some } \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}\}, \end{aligned} \quad (1)$$

where,

$$\mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \sup \{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) : \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}, \mathfrak{m} \in \mathfrak{K}\},$$
$$\mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \sup \{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) : \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}, \mathfrak{m} \in \mathfrak{K}\}.$$

Definition 3.5. Suppose $\mathfrak{J} \neq \phi, \mathfrak{K} \neq \phi \subset (\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ which is an NFMS. Let $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ and $\alpha : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$. Then, $\mathfrak{T}$ is an α-P_A mapping if, for $\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{J}$,

$$\begin{cases} \alpha(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) \geq \mathfrak{z}, \\ \mathfrak{B}(\mathfrak{u}_1, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{u}_2, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}_1, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}_2, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) \end{cases}$$

gives $\alpha(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{z}) \geq \mathfrak{z} \ \forall \ \mathfrak{z} > 0$.

Suppose $\mathfrak{J} \neq \phi, \mathfrak{K} \neq \phi \subset (\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ which is an NFMS, and $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ be such that, $\mathfrak{J} \neq \mathfrak{K}$. Now stipulate $\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}), \mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}), \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z})$, and $\mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z})$ as:

$$\begin{aligned} \mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) = \max \Bigg\{ & \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{u}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{m}, \mathfrak{v}, \mathfrak{z})}\right], \\ & \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{m}, \mathfrak{u}, \mathfrak{z})} - 1\right]\Bigg\} \\ \mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) = \max \Bigg\{ & \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{u}, \mathfrak{z})} + \frac{1}{\mathfrak{D}(\mathfrak{m}, \mathfrak{v}, \mathfrak{z})}\right], \\ & \frac{1}{2}\left[\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v}, \mathfrak{z})} + \frac{1}{\mathfrak{D}(\mathfrak{m}, \mathfrak{u}, \mathfrak{z})} - 1\right]\Bigg\} \\ \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) = \max \Bigg\{ & \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{u}, \mathfrak{z}), \mathfrak{B}(\mathfrak{m}, \mathfrak{v}, \mathfrak{z}), \frac{\mathfrak{B}(\mathfrak{m}, \mathfrak{v}, \mathfrak{z})\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{u}, \mathfrak{z})}{1 + \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v}, \mathfrak{z})}, \\ & \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v}, \mathfrak{z}), \mathfrak{B}(\mathfrak{m}, \mathfrak{u}, \mathfrak{z})\} \\ \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) = \max \Bigg\{ & \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{u}, \mathfrak{z}), \mathfrak{D}(\mathfrak{m}, \mathfrak{v}, \mathfrak{z}), \frac{\mathfrak{D}(\mathfrak{m}, \mathfrak{v}, \mathfrak{z})\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{u}, \mathfrak{z})}{1 + \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v}, \mathfrak{z})}, \\ & \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v}, \mathfrak{z}), \mathfrak{D}(\mathfrak{m}, \mathfrak{u}, \mathfrak{z})\} \end{aligned}$$

Definition 3.6. Suppose $\mathfrak{J} \neq \phi, \mathfrak{K} \neq \phi \subset (\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ which is an NFMS, and $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ be such that, $\mathfrak{J} \neq \mathfrak{K}$. Also, let $\alpha : \mathfrak{J} \times \mathfrak{J} \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$. Then, $\mathfrak{T}$ is an α-ψ-P_{CV} mapping of Ciric type if, for $\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v} \in \mathfrak{J}$,

$$\begin{cases} \alpha(\mathfrak{O}_{\mathfrak{nf}}, m, \mathfrak{z}) \geq \mathfrak{z}, \\ \mathfrak{B}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) \end{cases}$$

gives,

$$\begin{aligned} \frac{1}{\mathfrak{B}(\mathfrak{u},\mathfrak{v},\mathfrak{z})} - 1 \leq \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m},\mathfrak{u},\mathfrak{v},\mathfrak{z}) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m},\mathfrak{u},\mathfrak{v},\mathfrak{z})) \\ \frac{1}{\mathfrak{D}(\mathfrak{u},\mathfrak{v},\mathfrak{z})} - 1 \leq \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m},\mathfrak{u},\mathfrak{v},\mathfrak{z}) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m},\mathfrak{u},\mathfrak{v},\mathfrak{z})) \end{aligned} \tag{2}$$

$\forall\ \mathfrak{z} > 0$, where $\psi \in \Psi$.

Theorem 3.7. *Suppose $\mathfrak{J} \neq \phi, \mathfrak{K} \neq \phi \subset (\mathfrak{S},\mathfrak{B},\mathfrak{D},\mathfrak{F},\mathfrak{H},\star,\diamond)$ which is a complete triangular neutrosophic fuzzy metric space, such that $\mathfrak{J}_0(\mathfrak{z}) \neq \phi\ \forall\ \mathfrak{z} > 0$. And $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ be a t-unif. cont. mapping where $\mathfrak{J} \neq \mathfrak{K}$:*

1. *$\mathfrak{T}$ is an α-P_A mapping and $\mathfrak{T}(\mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})) \subseteq \mathfrak{K}_{\mathfrak{o}}(\mathfrak{z})\ \forall\ \mathfrak{z} > 0$.*
2. *$\mathfrak{T}$ is an α-ψ-P_{CV} mapping of Ciric type.*
3. *If a sequence $\mathfrak{m}_{\mathfrak{n}} \in \mathfrak{K}_{\mathfrak{o}}(\mathfrak{z})$ and $\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}$ satisfies $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m}_{\mathfrak{n}},\mathfrak{z}) \to \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z})$ as $\mathfrak{n} \to +\infty$, then $\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})\ \forall\ \mathfrak{z} > 0$.*
4. *$\exists\ \mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{O}_{\mathfrak{nf}1} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z}) : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}},\mathfrak{z}) = \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z})$, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}},\mathfrak{z}) = \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z})$ and $\alpha(\mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{O}_{\mathfrak{nf}1},\mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$.*
   Then, $\exists\ \mathfrak{O}_{\mathfrak{nf}}{}^* \in \mathfrak{J} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*,\mathfrak{z}) = \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z})$, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*,\mathfrak{z}) = \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z})\ \forall\ \mathfrak{z} > 0$. Implies that, $\mathfrak{T}$ has a BPP $\mathfrak{O}_{\mathfrak{nf}}{}^* \in \mathfrak{J}$.
5. *Furthermore, if,*

$$\begin{aligned} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}},\mathfrak{z}) &= \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \\ \mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{m},\mathfrak{z}) &= \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \\ \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}},\mathfrak{z}) &= \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \\ \mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{m},\mathfrak{z}) &= \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \end{aligned}$$

   gives, $\alpha(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m},\mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$, then $\mathfrak{T}$ has a unique BPP.

Proof. By condition (4) of the theorem above, it can be said that, $\exists\ \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}}, \mathfrak{O}_{\mathfrak{nf}_1} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})$: $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}},\mathfrak{z}) = \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z})$, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}},\mathfrak{z}) = \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z})$ and $\alpha(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}}, \mathfrak{O}_{\mathfrak{nf}_1},\mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$. Also, $\mathfrak{T}(\mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})) \subseteq \mathfrak{K}_{\mathfrak{o}}(\mathfrak{z})$, so $\exists\ \mathfrak{O}_{\mathfrak{nf}2} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})$:

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1},\mathfrak{z}) = \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1},\mathfrak{z}) = \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z})\ \forall\ \mathfrak{z} > 0.$$

Since $\mathfrak{T}$ is also a α-P_A mapping, $\alpha(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{O}_{\mathfrak{nf}_2},\mathfrak{z}) \geq \mathfrak{z}$.
Implies that,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1},\mathfrak{z}) = \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1},\mathfrak{z}) = \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z}) \text{ and } \alpha(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{O}_{\mathfrak{nf}_2},\mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0.$$

Again, since $\mathfrak{T}(\mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})) \subseteq \mathfrak{K}_{\mathfrak{o}}(\mathfrak{z})$, $\exists\ \mathfrak{O}_{\mathfrak{nf}3} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})$:

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_3}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2},\mathfrak{z}) = \mathfrak{B}(\mathfrak{J},\mathfrak{K},\mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_3}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2},\mathfrak{z}) = \mathfrak{D}(\mathfrak{J},\mathfrak{K},\mathfrak{z})\ \forall\ \mathfrak{z} > 0.$$

Implies that, $\forall\ \mathfrak{z} > 0$,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_3}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_3}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\alpha(\mathfrak{O}_{\mathfrak{nf}1}, \mathfrak{O}_{\mathfrak{nf}2}, \mathfrak{z}) \geq z.$$

Since $\mathfrak{T}$ is also a α-P_A mapping, $\alpha(\mathfrak{O}_{\mathfrak{nf}2}, \mathfrak{O}_{\mathfrak{nf}3}, \mathfrak{z}) \geq \mathfrak{z}$. Therefore,

$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_3}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_3}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_2}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$, and $\alpha(\mathfrak{O}_{\mathfrak{nf}2}, \mathfrak{O}_{\mathfrak{nf}3}, \mathfrak{z}) \geq z \ \forall\ \mathfrak{z} > 0$.

Continuing in this way, it is procured that, $\forall\ \mathfrak{z} > 0$ and $\mathfrak{n} \in \mathbb{W}$,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) \text{ and } \alpha(\mathfrak{O}_{\mathfrak{nf}n}, \mathfrak{O}_{\mathfrak{nf}n+1}, \mathfrak{z}) \geq z \tag{3}$$

Now putting $\mathfrak{u} = \mathfrak{m} = \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{v} = \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}$ and $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}$ in equation (3.2), it is procured that,

$$\begin{aligned} &\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \\ &\qquad \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})) \\ &\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \\ &\qquad \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})) \end{aligned} \tag{4}$$

$\forall\ \mathfrak{z} > 0$ and $\mathfrak{n} \in \mathbb{N}$ where,

$$\begin{aligned} &\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \\ &\quad = \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})}\right],\right. \\ &\qquad \left.\frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})} - 1\right]\right\} \\ &\quad = \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})}\right],\right. \\ &\qquad \left.\frac{1}{2\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})}\right\} \\ &\quad \leq \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})}\right],\right. \\ &\qquad \left.\frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z})} - 1 + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1\right] + \frac{1}{2}\right\} \end{aligned}$$

$$\leq \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})},\frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})}+\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})}\right],\right.$$
$$\left.\frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})}+\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})}\right]-\frac{1}{2}\right\}$$
$$=\max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})},\frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})}+\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})}\right]\right\}$$
$$\leq \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})},\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})}\right\}.$$

This indicates that,

$$\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})\leq \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})},\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})}\right\}. \quad (5)$$

Similarly,

$$\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})\leq \max\left\{\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})},\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})}\right\}. \quad (6)$$

Also,

$$\mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})$$
$$=\max\left\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z}),\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z}),\frac{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})}{1+\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})},\right.$$
$$\left.\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z}),\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z}),\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})\right\}$$
$$=\max\left\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z}),\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z}),\frac{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{z})}{1+\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})},\right.$$
$$\left.\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z}),1\right\}$$
$$=1.$$

Thus,

$$\mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})=1. \quad (7)$$

Similarly,

$$\mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}},\mathfrak{z})=1. \quad (8)$$

Therefore, from equations 5-8:

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi \Big(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z}) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z}) \Big)$$
$$\leq \psi \left(\max \left\{ \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{z})}, \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} \right\} - 1 \right).$$

Similarly,

$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi \Big(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z}) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z}) \Big)$$
$$\leq \psi \left(\max \left\{ \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{z})}, \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} \right\} - 1 \right).$$

Now if $\max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}},\mathfrak{z})}, \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}},\mathfrak{z})}\right\} = \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}},\mathfrak{z})}$,
and $\max\left\{\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}},\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}},\mathfrak{z})}, \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}},\mathfrak{z})}\right\} = \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}},\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}},\mathfrak{z})}$, then it is obtained that,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi \left(\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \right) < \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1$$

And similarly,

$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 < \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1$$

which results in a contradiction. Thus, $\forall\ n \in \mathbb{N}$ and $\mathfrak{z} \in (0, \infty)$,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi \left(\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{z})} - 1 \right),$$
$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi \left(\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}-1}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{z})} - 1 \right).$$

From here, it can be established that, $\forall\ n \in \mathbb{N}$ and $\mathfrak{z} \in (0, \infty)$,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi^{\mathfrak{n}} \left(\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{o}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{1}}, \mathfrak{z}) - 1} \right),$$
$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{n}\mathfrak{f}_{\mathfrak{n}+1}}, \mathfrak{z})} - 1 \leq \psi^{\mathfrak{n}} \left(\frac{1}{\mathfrak{D}(\mathfrak{Q}_{\mathfrak{o}}, \mathfrak{Q}_{1}, \mathfrak{z}) - 1} \right).$$

Now setting $\epsilon > 0,\ \exists\ P \in \mathbb{N}:$

$$\sum_{\mathfrak{n} \geq P} \psi^{\mathfrak{n}} \left(\frac{1}{\mathfrak{B}(\mathfrak{Q}_{\mathfrak{o}}, \mathfrak{Q}_{\mathfrak{1}}, \mathfrak{z}) - 1} \right) < \epsilon,$$

$$\sum_{\mathfrak{n} \geq P} \psi^{\mathfrak{n}} \left(\frac{1}{\mathfrak{D}(\mathfrak{Q}_{\mathfrak{o}}, \mathfrak{Q}_{\mathfrak{1}}, \mathfrak{z}) - 1} \right) < \epsilon.$$

Suppose $\mathfrak{r}, \mathfrak{n} \in \mathbb{N}$ where $\mathfrak{r} > \mathfrak{n} \geq P$. Then by using triangular inequality it can be said that,

$$\begin{aligned} \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{r}}}, \mathfrak{z})} - 1 &\leq \sum_{\mathfrak{j}=\mathfrak{n}}^{\mathfrak{r}-1} \left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{1}}}, \mathfrak{z}) - 1} \right] \\ &\leq \sum_{\mathfrak{n} \geq P} \psi^{\mathfrak{n}} \left(\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{1}}}, \mathfrak{z}) - 1} \right) < \epsilon, \end{aligned}$$

and similarly,

$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{r}}}, \mathfrak{z})} - 1 < \epsilon.$$

As a result,

$$\lim_{\mathfrak{r},\mathfrak{n} \to \infty} \left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{r}}}, \mathfrak{z})} - 1 \right] = 0, \text{ and } \lim_{\mathfrak{r},\mathfrak{n} \to \infty} \left[\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{r}}}, \mathfrak{z})} - 1 \right] = 0.$$

That is,

$$\lim_{\mathfrak{r},\mathfrak{n} \to \infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{r}}}, \mathfrak{z}) = 1, \text{ and } \lim_{\mathfrak{r},\mathfrak{n} \to \infty} \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{r}}}, \mathfrak{z}) = 1.$$

Hence $\left\{\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}\right\}$ is a Cauchy sequence. Now since $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a complete neutrosophic fuzzy metric space, $\exists\ \mathfrak{O}_{\mathfrak{nf}}{}^* \in \mathfrak{S} : \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}} \to \mathfrak{O}_{\mathfrak{nf}}{}^*$ as $\mathfrak{n} \to \infty$.

Now since, $\mathfrak{T}$ is t-unif. cont., by Remark 3.2 and Remark 3.3 it can be concluded that,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) = \lim_{n \to \infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+\mathfrak{1}}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$

$$\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) = \lim_{n \to \infty} \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+\mathfrak{1}}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}).$$

Implies that, $\mathfrak{O}_{\mathfrak{nf}}{}^*$ is a BPP of $\mathfrak{T}$.

Now it remains to prove the uniqueness of the best proximity point hence obtained.

Let to the contrary, $\exists\ \mathfrak{z}_0 > 0 : 0 < \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{w}, \mathfrak{z}_0) < 1$ and $\mathfrak{w} \neq \mathfrak{O}_{\mathfrak{nf}}{}^*$ be another BPP of $\mathfrak{T}$. Which means, $\forall\ \mathfrak{z} > 0$,

$$\begin{aligned} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) &= \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{w}, \mathfrak{T}\mathfrak{w}, \mathfrak{z}) &= \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) &= \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{w}, \mathfrak{T}\mathfrak{w}, \mathfrak{z}) &= \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{aligned}$$

Now given that condition (5) is satisfied, using Equation 2 it can be procured that,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_{o})} - 1 \leq \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0))$$

$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1 \leq \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0))$$

Where,

$$\begin{aligned}
&\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) \\
&= \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{z}_0)} + \frac{1}{\mathfrak{B}(\mathfrak{w}, \mathfrak{w}, \mathfrak{z}_0)}\right],\right. \\
&\left.\frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} + \frac{1}{\mathfrak{B}(\mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{z}_0)} - 1\right]\right\} \\
&= \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)}, 1, \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - \frac{1}{2}\right\} \\
&= \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)}.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) \\
&= \max\left\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{z}_0), \mathfrak{B}(\mathfrak{w}, \mathfrak{w}, \mathfrak{z}_0), \frac{\mathfrak{B}(\mathfrak{w}, \mathfrak{w}, \mathfrak{z}_0)\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{z}_0)}{1 + \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)}\right. \\
&\left.\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0), \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0), \mathfrak{B}(\mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{z}_0)\right\} \\
&= 1.
\end{aligned}$$

Similarly,

$$\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) = \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)}, \text{ and, } \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) = 1.$$

As a result,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1 \leq \psi\left(\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1\right) < \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1$$

and,

$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1 \leq \psi\left(\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1\right) < \frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0)} - 1.$$

Which is a contradiction. Therefore, $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}_0) = 1$ and $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^{*}, \mathfrak{w}, \mathfrak{z}) = 1 \ \forall \ \mathfrak{z} > 0$. That gives, $\mathfrak{O}_{\mathfrak{nf}}{}^{*} = \mathfrak{w}$.

Hence, $\mathfrak{T}$ has a unique BPP. □

Theorem 3.8. *Suppose $\mathfrak{J} \neq \phi, \mathfrak{K} \neq \phi \subset (\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ which is a complete triangular neutrosophic fuzzy metric space, such that $\mathfrak{J}_0(\mathfrak{z}) \neq \phi \ \forall \ \mathfrak{z} > 0$. And $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ be a mapping where $\mathfrak{J} \neq \mathfrak{K}$:*

(1) $\mathfrak{T}$ *is an* α-P_A *mapping and* $\mathfrak{T}(\mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})) \subseteq \mathfrak{K}_{\mathfrak{o}}(\mathfrak{z})\ \forall\ \mathfrak{z} > 0$.

(2) $\mathfrak{T}$ *is an* α-ψ-P_{CV} *mapping of Ciric type.*

(3) *If a sequence* $\mathfrak{m}_{\mathfrak{n}} \in \mathfrak{K}_{\mathfrak{o}}(\mathfrak{z})$ *and* $\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}$ *satisfies* $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}_{\mathfrak{n}}, \mathfrak{z}) \to \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$ *as* $\mathfrak{n} \to +\infty$, *then* $\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z})\ \forall\ \mathfrak{z} > 0$.

(4) $\exists\ \mathfrak{O}_{\mathfrak{nf}_0}, \mathfrak{O}_{\mathfrak{nf}_1} \in \mathfrak{J}_{\mathfrak{o}}(\mathfrak{z}) : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{o}}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$ *and* $\alpha(\mathfrak{O}_{\mathfrak{nf}_0}, \mathfrak{O}_{\mathfrak{nf}_1}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$.

(5) *For any sequence* $\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}} \in \mathfrak{S} : \alpha(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$ *and* $\mathfrak{n}$ *with* $\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}} \to \mathfrak{O}_{\mathfrak{nf}}$ *as* $\mathfrak{n} \to \infty$, *then* $\alpha(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$ *and* $\mathfrak{n}$.

And also, $\exists\ \mathfrak{O}_{\mathfrak{nf}}{}^* \in \mathfrak{J} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})\ \forall\ \mathfrak{z} > 0$. *i.e,* $\mathfrak{T}$ *has a BPP* $\mathfrak{O}_{\mathfrak{nf}}{}^* \in \mathfrak{J}$.

(6) *Furthermore, if,*

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$
$$\mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$

gives, $\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$, *then* $\mathfrak{T}$ *has a unique BPP.*

Proof. Following the aforementioned lines in the proof of Theorem 3.7, a sequence $\{\mathfrak{O}_{\mathfrak{nf}_n}\} \in \mathfrak{J}_0(\mathfrak{z})$ can be constucted such that, $\forall\ \mathfrak{n} \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) =& \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ & \alpha(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \geq \mathfrak{z}. \end{aligned} \tag{9}$$

And $\{\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}\} \to \mathfrak{O}_{\mathfrak{nf}}{}^*$ as $\mathfrak{n} \to \infty$, i.e, $\lim_{\mathfrak{n}\to\infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) = 1,\ \forall\ \mathfrak{z} > 0$. Over and above that,

$$\begin{aligned} \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) &= \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) \\ &\geq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) \\ &\geq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) \\ &= \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{aligned}$$

Implies that,

$$\begin{aligned} \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) &\geq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) \\ &\geq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{aligned}$$

Considering the limit as $\mathfrak{n} \to \infty$ in the above inequality, it is procured that,

$$\mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) \geq 1 \star \lim_{\mathfrak{n}\to\infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}}}, \mathfrak{z}) \geq 1 \star 1 \star \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}),$$

which means,

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}).$$

Similarly,

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}).$$

And therefore, by condition (3), $\mathfrak{O}_{\mathfrak{nf}}{}^* \in \mathfrak{J}_0(\mathfrak{z})$. Now since $\mathfrak{T}(\mathfrak{J}_\circ(\mathfrak{z})) \subseteq \mathfrak{K}_\circ(\mathfrak{z})$, $\exists\ \mathfrak{p} \in \mathfrak{J}_\circ(\mathfrak{z})$: $\mathfrak{B}(\mathfrak{p}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$ and $\mathfrak{D}(\mathfrak{p}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$. Also using condition (4) it can be obtained that, $\alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{n} \in \mathbb{W}$.

Let $\exists\ \mathfrak{z}_0 > 0 : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0) < 1$ and $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0) < 1$. Then using equation (3.2), by taking $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{m} = \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{u} = \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}$ and $\mathfrak{v} = \mathfrak{p}$ it is obtained that,

$$\begin{aligned} &\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0)} - 1 \leq \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0)) \\ &\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0)} - 1 \leq \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0)) \end{aligned} \tag{10}$$

Moreover, it is known that,

$$\begin{aligned} &\lim_{\mathfrak{n}\to\infty} \mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0) \\ &\quad = \lim_{\mathfrak{n}\to\infty} \left(\max\left\{ \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}_0)}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}_0)} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right], \right.\right. \\ &\quad\quad \left.\left. \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{p}, \mathfrak{z}_0)} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}_0)} - 1\right] \right\}\right) \\ &\quad = \max\left\{ \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}_0)}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}_0)} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right], \right. \\ &\quad\quad \left. \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{p}, \mathfrak{z}_0)} + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}_0)} - 1\right] \right\} \\ &\quad = \max\left\{1, \frac{1}{2}\left[1 + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right], \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right]\right\} \\ &\quad = \frac{1}{2}\left[1 + \frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right] \end{aligned}$$

and,

$$\lim_{\mathfrak{n}\to\infty} \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0)$$
$$= \lim_{\mathfrak{n}\to\infty} \max\left\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}_0), \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0), \frac{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}_0)}{1+\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{p}, \mathfrak{z}_0)},\right.$$
$$\left.\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{z}_0), \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{p}, \mathfrak{z}_0), \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}_0)\right\}$$
$$= \max\left\{1, \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0), \frac{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}{1+\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}, 1, \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0), 1\right\}$$
$$= 1.$$

Similarly,

$$\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0) = \frac{1}{2}\left[1+\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right], \text{ and } \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{p}, \mathfrak{z}_0) = 1.$$

Now considering the limit as $\mathfrak{n} \to \infty$ in Equation 8, it is procured that,

$$\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)} - 1 \le \psi\left(\frac{1}{2}\left[1+\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right]-1\right) < \frac{1}{2}\left[1+\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right]-1.$$

And,

$$\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)} - 1 \le \psi\left(\frac{1}{2}\left[1+\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right]-1\right) < \frac{1}{2}\left[1+\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)}\right]-1.$$

Implies that, $\frac{1}{2\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)} < \frac{1}{2}$, and, $\frac{1}{2\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0)} < \frac{1}{2}$. That is, $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0) > 1$, and, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}_0) > 1$. Which gives a contradiction.

Hence $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}) = 1$, and, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}{}^*, \mathfrak{p}, \mathfrak{z}) = 1 \ \forall\ \mathfrak{z} > 0$. As a result, $\mathfrak{O}_{\mathfrak{nf}}{}^* = p$. Therefore, $\mathfrak{T}$ has a point of best proximity. □

Example 3.9. Suppose $\mathfrak{S} = (-\infty, \infty)$ be endowed with $\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}) = |\mathfrak{O}_{\mathfrak{nf}} - \mathfrak{m}|$, which represents the usual distance metric.

Now $\forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$ and $\mathfrak{z} > 0$, take,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\mathfrak{z}^2}{\mathfrak{z}^2 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^2}, \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\mathfrak{z}^6}{\mathfrak{z}^6 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^6}, \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})}{\mathfrak{z} + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})}$$

and,

$$\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^3}{\mathfrak{z}^3 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^3}\ .$$

Also, define the CTN $\star$ and CTCN $\diamond$ as,

$$a \star b = \min\{0, a+b\}, a \diamond b = \max\{1, ab\}\ \ \forall\ a, b \in \mathfrak{S}.$$

Then it can be noticed that,

(1) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \in [0, 1]$,
(2) As $\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}) = \tau(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}})$, it is evident that, $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})$,
(3) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$ when $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$,

(4) $\lim_{\mathfrak{z}\to\infty} \dfrac{\mathfrak{z}^2}{\mathfrak{z}^2+\tau(\mathfrak{O}_{\mathfrak{nf}},\mathfrak{m})^2} = 1 \ \forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$ and $\mathfrak{z} > 0$,

(5) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{m}, \mathfrak{p}, \mathfrak{a}) \leq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z}+\mathfrak{a}) \ \forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{p} \in \mathfrak{S}$ and $\mathfrak{z}, \mathfrak{a} > 0$.

In a similar way, all the conditions for $\mathfrak{D}, \mathfrak{F}$ and $\mathfrak{H}$ can be checked. Hence, the 7-tuple $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS.

Now let, $\mathfrak{J}, \mathfrak{K} \subset \mathfrak{S}$ be given as $\mathfrak{J} = (-\infty, -1]$ and $\mathfrak{K} = [1, \infty)$. And the mapping $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ be defined as,

$$\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}} = \begin{cases} 3{\mathfrak{O}_{\mathfrak{nf}}}^2 + 10, & \text{if } \mathfrak{O}_{\mathfrak{nf}} \in (-\infty, -18), \\ 26, & \text{if } \mathfrak{O}_{\mathfrak{nf}} \in [-18, -10), \\ |(\mathfrak{O}_{\mathfrak{nf}}+5)(\mathfrak{O}_{\mathfrak{nf}}-2)|, & \text{if } \mathfrak{O}_{\mathfrak{nf}} \in [-10, -2), \\ 1, & \text{if } \mathfrak{O}_{\mathfrak{nf}} \in [-2, -1]. \end{cases}$$

Also, let the mappings $\alpha : \mathfrak{S}^2 \times (0, \infty) \to [0, \infty)$ and $\psi : \mathbb{R} \cup \{0\} \to \mathbb{R} \cup \{0\}$ be defined as,

$$\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \begin{cases} 12\mathfrak{z}, & \text{if } \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in [-2, -1], \\ \frac{\mathfrak{z}}{6}, & \text{elsewhere} . \end{cases}$$

And,

$$\psi(\mathfrak{z}) = \frac{\mathfrak{z}}{15} \ \forall\ \mathfrak{z} > 0.$$

Then undoubtedly,

$$\mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \sup\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) : \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}, \mathfrak{m} \in \mathfrak{K}\} = \frac{\mathfrak{z}^2}{\mathfrak{z}^2+4},$$

and,

$$\mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \sup\{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) : \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}, \mathfrak{m} \in \mathfrak{K}\} = \frac{\mathfrak{z}^6}{\mathfrak{z}^6+64}.$$

Hence,

$$\begin{aligned} \mathfrak{J}_0(\mathfrak{z}) = \Big\{ & \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \frac{\mathfrak{z}^2}{\mathfrak{z}^2+4} \\ & \text{and } \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \frac{\mathfrak{z}^6}{\mathfrak{z}^6+64} \text{ for some } \mathfrak{m} \in \mathfrak{K} \Big\} \\ = \{-1\} & \end{aligned}$$

and,

$$\begin{aligned} \mathfrak{K}_0(\mathfrak{z}) = \Big\{ & \mathfrak{m} \in \mathfrak{K} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \frac{\mathfrak{z}^2}{\mathfrak{z}^2+4} \\ & \text{and } \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \frac{\mathfrak{z}^6}{\mathfrak{z}^6+64} \text{ for some } \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J} \Big\} \\ = \{1\} & \end{aligned}$$

It is then obvious that, $\mathfrak{T}(\mathfrak{J}_o(\mathfrak{z})) \subseteq \mathfrak{K}_o(\mathfrak{z})\ \forall\ \mathfrak{z} > 0$, also $\mathfrak{B}(-1, \mathfrak{T}(-1), \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$, $\mathfrak{D}(-1, \mathfrak{T}(-1), \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$ and, $\alpha(-1, -1, \mathfrak{z}) \geq \mathfrak{z}$.

Now suppose,

$$\begin{cases} \alpha(\mathfrak{O}_{\mathfrak{nf}}, m, \mathfrak{z}) \geq \mathfrak{z}, \\ \mathfrak{B}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{cases}$$

Then,

$$\begin{cases} \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in [-2, -1], \\ \mathfrak{B}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{cases}$$

Therefore, $\mathfrak{u} = \mathfrak{v} = -1$ i.e., $\alpha(\mathfrak{u}, \mathfrak{v}, \mathfrak{z}) \geq \mathfrak{z}$. Hence, $\mathfrak{T}$ is an α-P_A mapping.

Furthermore,

$$\frac{1}{\mathfrak{B}(\mathfrak{u}, \mathfrak{v}, \mathfrak{z})} - 1 = 0 \leq \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z})),$$
$$\frac{1}{\mathfrak{D}(\mathfrak{u}, \mathfrak{v}, \mathfrak{z})} - 1 = 0 \leq \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z})).$$

Hence $\mathfrak{T}$ is also an α-ψ-P_{CV} mapping of Ciric type.

Also if $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} : \alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{n} \in \mathbb{W}, \mathfrak{z} \in (0, \infty)$, where $\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to \mathfrak{O}_{\mathfrak{nf}}$ as $\mathfrak{n} \to \infty$, then $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} \subseteq [-2, -1]$. Hence $\mathfrak{O}_{\mathfrak{nf}} \in [-2, -1]$. As a result, $\alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{n} \in \mathbb{W}, \mathfrak{z} \in (0, \infty)$.

$\implies$ All the constraints of Theorem 3.8 are fulfilled and, $\mathfrak{p} = -1$ is the unique BPP, for $\mathfrak{T}$.

4. Applications to fixed point theory

As it is known that, for a self mapping BPP becomes a fixed point, in this section, some new ideologies and fixed point results have been discussed for a complete NFMS, where the equality of the two sets associated with the mapping, has already been assumed.

Definition 4.1. Suppose $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS, $\mathfrak{T}$ is a self mapping on $\mathfrak{S}$, and, $\alpha : \mathfrak{S}^2 \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$. Then $\mathfrak{T}$ is an α adm mapping if,

$$\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq \mathfrak{z} \implies \alpha(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) \geq \mathfrak{z}$$

$\forall\ \mathfrak{z} > 0$, where $\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$.

Suppose $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS, and $\mathfrak{T} : \mathfrak{S} \to \mathfrak{S}$. Now stipulate $\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$, $\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$, $\mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$, and $\mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$ as:

$$\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \max\left\{\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})}\right], \frac{1}{2}\left[\frac{1}{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})} + \frac{1}{\mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})} - 1\right]\right\}$$

$$\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \max\left\{\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})}, \frac{1}{2}\left[\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})} + \frac{1}{\mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})}\right], \frac{1}{2}\left[\frac{1}{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})} + \frac{1}{\mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})} - 1\right]\right\}$$

$$\mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \max\left\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}), \mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}), \frac{\mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})}{1 + \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})}, \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}), \mathfrak{B}(\mathfrak{m}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})\right\}$$

$$\mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \max\left\{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}), \mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}), \frac{\mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})}{1 + \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})}, \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}), \mathfrak{D}(\mathfrak{m}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})\right\}$$

Definition 4.2. Suppose $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS, and $\mathfrak{T} : \mathfrak{S} \to \mathfrak{S}$. Also, let $\alpha : \mathfrak{S}^2 \times \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$. Then, $\mathfrak{T}$ is an α-ψ-C_V mapping of Ciric type if, for $\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$, $\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq \mathfrak{z}$ implies,

$$\frac{1}{\mathfrak{B}(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})} - 1 \leq \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}))$$
$$\frac{1}{\mathfrak{D}(\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{T}\mathfrak{m}, \mathfrak{z})} - 1 \leq \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}))$$

$\forall\, \mathfrak{z} > 0$, where $\psi \in \Psi$.

Theorem 4.3. *Suppose* $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ *is a complete triangular NFMS, and* $\mathfrak{T} : \mathfrak{S} \to \mathfrak{S}$ *is a t-unif. cont. self mapping:*

(1) $\mathfrak{T}$ *is an* α *adm mapping,*

(2) $\mathfrak{T}$ *is an* α-ψ-C_V *mapping of Ciric type,*

(3) $\exists\, \mathfrak{O}_{\mathfrak{nf}0} \in \mathfrak{S} : \alpha(\mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{z}) \geq \mathfrak{Z}$.
Then $\mathfrak{T}$ *has a fixed point in* $\mathfrak{S}$.

(4) *Furthermore, if* $\mathfrak{O}_{\mathfrak{nf}}$ *and* $\mathfrak{m}$ *are two fixed points of* $\mathfrak{T}$*:* $\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq \mathfrak{z}$*, then* $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$*, i.e,* $\mathfrak{T}$ *has a unique fixed point.*

Theorem 4.4. *Suppose* $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ *is a complete triangular NFMS, and* $\mathfrak{T} : \mathfrak{S} \to \mathfrak{S}$ *is a self mapping:*

(1) $\mathfrak{T}$ *is an* α *adm mapping,*

(2) $\mathfrak{T}$ *is an* α-ψ-C_V *mapping of Ciric type,*

(3) $\exists\ \mathfrak{O}_{\mathfrak{nf}_0} \in \mathfrak{S} : \alpha(\mathfrak{O}_{\mathfrak{nf}_0}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}_0}, \mathfrak{z}) \geq \mathfrak{Z}$,

(4) *For any sequence* $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} \in \mathfrak{S} : \alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$ *and* $\mathfrak{n}$ *with* $\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to \mathfrak{O}_{\mathfrak{nf}}$ *as* $\mathfrak{n} \to \infty$, *then* $\alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{z} > 0$ *and* $\mathfrak{n} \in \mathbb{N}$.
Then $\mathfrak{T}$ *has a fixed point in* $\mathfrak{S}$.

(5) *Furthermore, if* $\mathfrak{O}_{\mathfrak{nf}}$ *and* $\mathfrak{m}$ *are two fixed points of* $\mathfrak{T}$: $\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \geq \mathfrak{z}$, *then* $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$, *i.e,* $\mathfrak{T}$ *has a unique fixed point.*

5. **Applications to climate forecasting**

Suppose $\mathfrak{S} = (-\infty, \infty)$ be endowed with $\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}) = |\mathfrak{O}_{\mathfrak{nf}} - \mathfrak{m}|$, which represents the usual distance metric.

Now $\forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$ and $\mathfrak{z} > 0$, take,

$$\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\mathfrak{z}^2}{\mathfrak{z}^2 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^2}, \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\mathfrak{z}^6}{\mathfrak{z}^6 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^6}, \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})}{\mathfrak{z} + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})}$$

and,

$$\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \frac{\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^3}{\mathfrak{z}^3 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^3}\ .$$

As given in Definition 7 in [19], the function $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$ represents the certainity that the distance between $\mathfrak{O}_{\mathfrak{nf}}$ and $\mathfrak{m}$ is less than $\mathfrak{z}$, it can also be compared to the fuzzy membership grade as well. Whereas the functions $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}), \mathfrak{F}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$ and $\mathfrak{H}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z})$ represents the degree of nearness, neutralness and non-nearness between $\mathfrak{O}_{\mathfrak{nf}}$ and $\mathfrak{m}$ with respect to $\mathfrak{z}$, respectively. Now following the logic of neutrosophic fuzzy sets as given in [17], one can also compare the above described functions to truth, indeterminacy and falsity membership grades respectively, as well.

Also, define the CTN $\star$ and CTCN $\diamond$ as,

$$a \star b = \min\{0, a + b\}, a \diamond b = \max\{1, ab\}\ \ \forall\ a, b \in \mathfrak{S}.$$

Then it can be noticed that,

(1) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \in [0, 1]$,

(2) As $\tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}) = \tau(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}})$, it is evident that, $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{m}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z})$,

(3) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = 1$ when $\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}$,

(4) $\lim_{\mathfrak{z}\to\infty} \dfrac{\mathfrak{z}^2}{\mathfrak{z}^2 + \tau(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m})^2} = 1\ \forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in \mathfrak{S}$ and $\mathfrak{z} > 0$,

(5) $\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) \star \mathfrak{B}(\mathfrak{m}, \mathfrak{p}, \mathfrak{a}) \leq \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{p}, \mathfrak{z} + \mathfrak{a})\ \forall\ \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{p} \in \mathfrak{S}$ and $\mathfrak{z}, \mathfrak{a} > 0$.

In a similar way, all the conditions for $\mathfrak{D}, \mathfrak{F}$ and $\mathfrak{H}$ can be checked. Hence, the 7-tuple $(\mathfrak{S}, \mathfrak{B}, \mathfrak{D}, \mathfrak{F}, \mathfrak{H}, \star, \diamond)$ is a NFMS.

Now let, $\mathfrak{J} = \mathfrak{K} = [0, 255] \subset \mathfrak{S}$, where $\mathfrak{J}$ represents the set of wind speeds (in km/hr) of a particular time period, for a particular place and $\mathfrak{K}$ represents the set of rainfall (in mm) of that particular place, and for the same particular time period . And the relation between them is given by the mapping $\mathfrak{T} : \mathfrak{J} \to \mathfrak{K}$ which is defined as,

$$\mathfrak{T}\mathfrak{O}_{\mathfrak{nf}} = \mathfrak{O}_{\mathfrak{nf}} - \log(1 + \mathfrak{O}_{\mathfrak{nf}}) \; \forall \; \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}.$$

Here the upper bound of the sets $\mathfrak{J}$ and $\mathfrak{K}$ are strictly considered to be 255, because following the Beaufort and Saffir-Simpson scales which classifies the strength of the winds and the conditions observed on land or sea, according to the wind speeds recorded, it can be inferred that, storms with wind speeds ≥ 252 km/hr are already category 5 hurricanes or cyclones, but such storms are very rare in this world. So to the best of our knowledge, there are not much storms till now which has exceeded the speed limit of 255 km/hr. But there are certainly some of them, such as Cyclone Amphan that hit Kolkata in 2020. Wind speeds during that time were recorded as high as 260 km/hr. Now if one considers the data of wind speeds in Kolkata for the year 2020, then the upper bound of the sets $\mathfrak{J}$ and $\mathfrak{K}$ can also be extended to 260.

Also, let the mappings $\alpha : \mathfrak{S}^2 \times (0, \infty) \to [0, \infty)$ and $\psi : \mathbb{R} \cup \{0\} \to \mathbb{R} \cup \{0\}$ be defined as,

$$\alpha(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \begin{cases} 12\mathfrak{z}, & \text{if } \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in [0, 2], \\ \frac{\mathfrak{z}}{6}, & \text{elsewhere .} \end{cases}$$

and,

$$\psi(\mathfrak{z}) = \frac{\mathfrak{z} + 1}{15} \; \forall \; \mathfrak{z} > 0.$$

Then undoubtedly,

$$\mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \sup\{\mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) : \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}, \mathfrak{m} \in \mathfrak{K}\} = 1,$$
$$\text{which is achievable only when } \mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}.$$

and,

$$\mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = \sup\{\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) : \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J}, \mathfrak{m} \in \mathfrak{K}\} = 1.$$
$$\text{which is achievable only when } \mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}.$$

Hence,

$$\begin{aligned} \mathfrak{J}_0(\mathfrak{z}) = \{\mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = 1 \\ \text{and } \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = 1 \text{ for some } \mathfrak{m} \in \mathfrak{K} : \mathfrak{m} = \mathfrak{O}_{\mathfrak{nf}}\} \\ = \mathfrak{J} \end{aligned}$$

and,

$$\mathfrak{K}_0(\mathfrak{z}) = \{\mathfrak{m} \in \mathfrak{K} : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = 1 \text{ and } \mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}) = 1 \text{ for some } \mathfrak{O}_{\mathfrak{nf}} \in \mathfrak{J} : \mathfrak{O}_{\mathfrak{nf}} = \mathfrak{m}\} = \mathfrak{K}$$

It is then obvious that, $\mathfrak{T}(\mathfrak{J}_o(\mathfrak{z})) \subseteq \mathfrak{K}_o(\mathfrak{z})\ \forall\ \mathfrak{z} > 0$.

Also, if $\mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{O}_{\mathfrak{nf}1} \in [0,2] : \mathfrak{O}_{\mathfrak{nf}1} = \mathfrak{T}(\mathfrak{O}_{\mathfrak{nf}_o})$, $\exists\ \mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{O}_{\mathfrak{nf}1} \in \mathfrak{J}_0(\mathfrak{z}) : \mathfrak{B}(\mathfrak{O}_{\mathfrak{nf}1}, \mathfrak{T}(\mathfrak{O}_{\mathfrak{nf}0}), \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$, $\mathfrak{D}(\mathfrak{O}_{\mathfrak{nf}1}, \mathfrak{T}(\mathfrak{O}_{\mathfrak{nf}0}), \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z})$ and, $\alpha(\mathfrak{O}_{\mathfrak{nf}0}, \mathfrak{O}_{\mathfrak{nf}1}, \mathfrak{z}) \geq \mathfrak{z}$.

Now suppose,

$$\begin{cases} \alpha(\mathfrak{O}_{\mathfrak{nf}}, m, \mathfrak{z}) \geq \mathfrak{z}, \\ \mathfrak{B}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{cases}$$

Then,

$$\begin{cases} \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m} \in [0,2], \\ \mathfrak{B}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{B}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{B}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{u}, \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}), \\ \mathfrak{D}(\mathfrak{v}, \mathfrak{T}\mathfrak{m}, \mathfrak{z}) = \mathfrak{D}(\mathfrak{J}, \mathfrak{K}, \mathfrak{z}). \end{cases}$$

Therefore, $\mathfrak{u} = \mathfrak{T}\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{v} = \mathfrak{T}\mathfrak{m}$ and $\mathfrak{u}, \mathfrak{v} \in [0,2) \subset [0,2]$. That is, $\alpha(\mathfrak{u}, \mathfrak{v}, \mathfrak{z}) \geq \mathfrak{z}$.

Hence $\mathfrak{T}$ is an α-P_A mapping.

Furthermore,

$$\frac{1}{\mathfrak{B}(\mathfrak{u}, \mathfrak{v}, \mathfrak{z})} - 1 = 0 \leq \psi(\mathcal{B}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) - \mathcal{F}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z})),$$
$$\frac{1}{\mathfrak{D}(\mathfrak{u}, \mathfrak{v}, \mathfrak{z})} - 1 = 0 \leq \psi(\mathcal{D}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}) - \mathcal{H}^{\mathfrak{T}}(\mathfrak{O}_{\mathfrak{nf}}, \mathfrak{m}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z})).$$

Hence $\mathfrak{T}$ is also an α-ψ-P_{CV} mapping of Ciric type.

Also if $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} : \alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}_{\mathfrak{n}+1}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{n} \in \mathbb{W}, \mathfrak{z} \in (0, \infty)$, where $\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to \mathfrak{O}_{\mathfrak{nf}}$ as $\mathfrak{n} \to \infty$, then $\{\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}\} \subseteq [0,2]$.

Now from the function T, it s also clear that $\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}} \to 0$ as $\mathfrak{n} \to \infty$.

$\therefore \mathfrak{O}_{\mathfrak{nf}} = 0 \in [0,2]$; which gives, $\alpha(\mathfrak{O}_{\mathfrak{nf}_\mathfrak{n}}, \mathfrak{O}_{\mathfrak{nf}}, \mathfrak{z}) \geq \mathfrak{z}\ \forall\ \mathfrak{n} \in \mathbb{W}, \mathfrak{z} \in (0, \infty)$.

$\implies$ All the constraints of Theorem 3.2 are fulfilled and, $\mathfrak{p} = 0$ is the unique BPP, for $\mathfrak{T}$.

But since, $\mathfrak{T}$ is a self mapping, $\mathfrak{p} = 0$ gets converted to fixed point w.r.t all the conditions satisfied.

$\therefore$ It can be said that, $\mathfrak{T}(0) = 0$ is a solution of our rainfall-windspeed function if all the

conditions of Theorem 3.2 are to be satisfied.
Hence the rainfall and windspeed on a particular day given by the funtion $\mathfrak{T}$ satisfying all the conditions of Theorem 3.2 is 0.

Or, neither there is any chance of wind/storm, nor any chance of rainfall on that particular day.

6. Conclusion

The BPP results are discussed under the the purview of NFMS for a new type of rational contraction. The triangular property for NFMS is used to establish the results for the first time. As well as, the concept of t-uniformly continuous mapping has also been used in Theorem 3.1. Also $\alpha - P_A$ mapping and $\alpha - \psi$ proximal contractive mappings of Ciric type has been used.

To find the applications, the BPP results are converted to fixed point contraction. For the said purpose, some new concepts such as α-admissible and $\alpha - \psi$ contractive mappings has been defined under the realm of NFMS.

In future, the deduced results can be generalized by using the concepts of soft sets and rough sets as neutrosophic soft fuzzy metric space and neutrosophic rough metric space. The established results are useful to solve initial value and boudary value problems, as well as in real life, the reckoned results may be used in decision making problems.

As in this research, the climate forecasting is done for rainfall, wind, storm or cyclone using this phenomena. In future, more generalized real life examples and applications can be established for decision making problems.

Funding: This research received no external funding.
Acknowledgments: Contributors are thankful to the honorable reviewers for improvement of the article.
Conflicts of Interest: Contributors affirm that there are no known competing interests happening.

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Received: Oct 12, 2025. Accepted: March 16, 2026