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Application of New Neutrosophic Open Set via Neutrosophic Topological Spaces

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Abstract. We introduce a new class of open sets in neutrosophic topological spaces, called neutrosophic $\mathfrak{I}$-open (NI-open) sets, together with their dual NI-closed sets, defined via neutrosophic δ-interior and δ-closure operators. We establish basic stability (arbitrary unions of NI-open sets and arbitrary intersections of NI-closed sets), give characterizations in terms of the associated interior/closure operators, and prove a decomposition formula expressing every NI-open set as the union of its $N\delta S$-interior and $N\delta\beta$-interior. We further locate NI-openness within the existing landscape by showing that $N\delta S$-open, $N\delta P$-open, N-eopen, and Ne^*-open sets are, in general, contained in NI-open sets, while the converses may fail; illustrative examples are provided. Finally, we present a neutrosophic multi-criteria assessment for supplier selection that combines the neutrosophic score with its negative counterpart to rank alternatives under ambiguity and indeterminacy, including a worked example with per-strategy rankings and an assignment rule.

Keywords: Supplier selection, Neutrosophic scoring function, Neutrosophic β-open sets.

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1. Introduction

Since Zadeh's seminal proposal of fuzzy sets [44] and Chang's extension to fuzzy topological spaces [7], a rich line of research has explored graded notions of membership in topology. Atanassov's intuitionistic fuzzy sets [6] introduced simultaneous membership and non-membership degrees, with topological developments by Coker [8]. In parallel, Smarandache initiated neutrosophy and neutrosophic sets [27, 28], later specialized to single-valued neutrosophic sets by Wang *et al.* [43] and investigated topologically by Salama and collaborators [23, 24]. Classical generalized openness has also evolved: e-open sets were studied by Ekici [9] and extended to fuzzy and intuitionistic fuzzy settings [25]; Saha's δ-open sets [21] were recently adapted to the neutrosophic framework [14, 36, 42]. Beyond foundations, neutrosophic models have seen broad adoption in multi-criteria decision making and information processing, including applications to ranking, medical diagnosis, and materials selection [15, 18, 19, 22, 30, 32, 33, 39]. Recent advacnces also address nano/neutrosophic open-set structures and classes of (ir)resolute mappings in neutrosophic crisp and related spaces [31, 35, 37, 38, 40, 41], as well as generalized closedness and weak separation axioms in neutrosophic environments [1, 2, 10–12].

Recent contributions further complement this line of research: the interplay between soft and neutrosophic sets has been analyzed in [16]; computational tools for neutrosophic topologies in Python have been developed in [17]; algebraic automata in the neutrosophic setting were introduced in [13]; and refined notions of neutrosophic crisp generalized closed functions were proposed in [3].

Motivated by these developments, we introduce neutrosophic $\mathfrak{I}$-open sets (NI-open) in neutrosophic topological spaces, together with their dual NI-closed sets. A set E is NI-open if $E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))$, and NI-closed if $\mathrm{Nint}(\mathrm{N}\delta\mathrm{cl}(E)) \cap \mathrm{Nint}(\mathrm{N}\delta\mathrm{cl}(\mathrm{N}\delta\mathrm{int}(E))) \subseteq E$. We establish stability (arbitrary unions of NI-open sets and arbitrary intersections of NI-closed sets), characterizations via $N\mathfrak{I}\mathrm{int}$ and $N\mathfrak{I}\mathrm{cl}$ (namely, E is NI-closed iff $E = N\mathfrak{I}\mathrm{cl}(E)$ and NI-open iff $E = N\mathfrak{I}\mathrm{int}(E)$), and a decomposition formula $E = \mathrm{N}\delta\mathrm{Sint}(E) \cup \mathrm{N}\delta\beta\mathrm{int}(E)$ for every NI-open set. We also locate NI-openness among related classes by proving the inclusions $\mathrm{N}\delta S$-os, $\mathrm{N}\delta P$-os, Ne-os, Ne^*-os $\subseteq$ NI-os, with counterexamples to the converses. Finally, we illustrate practical relevance through a neutrosophic multi-criteria supplier-selection study using the neutrosophic score and its negative counterpart to deliver robust rankings under ambiguity and indeterminacy.

This paper is structured as follows. Section 2 recalls the background on neutrosophic sets and topological operators. Section 3 introduces NI-open/NI-closed sets, develops their stability and characterization results, establishes the decomposition formula, clarifies inclusion relations, and presents illustrative examples. Section 4 applies neutrosophic scoring to a supplier-selection problem, including a worked numerical example, ranking tables, and an assignment rule. Section 5 concludes with a summary and directions for future work.

2. Background and Definitions

In this section we recall the basic notions of neutrosophic sets and neutrosophic topological spaces used throughout the paper; see, e.g., [9, 23, 29, 31].

Definition 2.1 (Neutrosophic set [23]). Let Ψ be a nonempty set. A neutrosophic set (NS) $\mathcal{M}$ on Ψ is a collection

$$\mathcal{M} = \{\langle \eta, \mu_{\mathcal{M}}(\eta), \sigma_{\mathcal{M}}(\eta), \nu_{\mathcal{M}}(\eta)\rangle : \eta \in \Psi\},$$

where the membership, indeterminacy and nonmembership functions $\mu_{\mathcal{M}}, \sigma_{\mathcal{M}}, \nu_{\mathcal{M}} \colon \Psi \to [0,1]$ satisfy

$$0 \le \mu_{\mathcal{M}}(\eta) + \sigma_{\mathcal{M}}(\eta) + \nu_{\mathcal{M}}(\eta) \le 3 \quad \text{for all } \eta \in \Psi.$$

Definition 2.2 (Basic operations [23]). Let $\mathcal{M}, \Omega$ be NSs on Ψ. Then:

(i) $0_N = \langle \eta, 0, 0, 1\rangle$ and $1_N = \langle \eta, 1, 1, 0\rangle$ (for all $\eta \in \Psi$).

(ii) $\mathcal{M} \subseteq \Omega$ iff $\mu_{\mathcal{M}} \le \mu_\Omega$, $\sigma_{\mathcal{M}} \le \sigma_\Omega$ and $\nu_{\mathcal{M}} \ge \nu_\Omega$ pointwise on Ψ.

(iii) $\mathcal{M} = \Omega$ iff $\mathcal{M} \subseteq \Omega$ and $\Omega \subseteq \mathcal{M}$.

(iv) The complement $\mathcal{M}^c = 1_N - \mathcal{M}$ is given by

$$\mathcal{M}^c = \{\langle \eta,\, \nu_{\mathcal{M}}(\eta),\, 1 - \sigma_{\mathcal{M}}(\eta),\, \mu_{\mathcal{M}}(\eta)\rangle : \eta \in \Psi\}.$$

(v) The union and intersection are defined by

$$\mathcal{M} \cup \Omega = \{\langle \eta,\, \max(\mu_{\mathcal{M}}, \mu_\Omega),\, \max(\sigma_{\mathcal{M}}, \sigma_\Omega),\, \min(\nu_{\mathcal{M}}, \nu_\Omega)\rangle : \eta \in \Psi\},$$

$$\mathcal{M} \cap \Omega = \{\langle \eta,\, \min(\mu_{\mathcal{M}}, \mu_\Omega),\, \min(\sigma_{\mathcal{M}}, \sigma_\Omega),\, \max(\nu_{\mathcal{M}}, \nu_\Omega)\rangle : \eta \in \Psi\}.$$

Definition 2.3 (Neutrosophic topology [23]). A family Φ_N of NSs on Ψ is a neutrosophic topology (NT) if:

(i) $0_N, 1_N \in \Phi_N$;

(ii) $\Omega_\kappa \cap \Omega_\omega \in \Phi_N$ for all $\Omega_\kappa, \Omega_\omega \in \Phi_N$;

(iii) $\bigcup_{\lambda \in \Lambda} \Omega_\lambda \in \Phi_N$ for any subfamily $\{\Omega_\lambda\}_{\lambda \in \Lambda} \subseteq \Phi_N$.

Then (Ψ, Φ_N) is a neutrosophic topological space (NTS). Members of Φ_N are neutrosophic open sets (N-os). A NS is neutrosophic closed (N-cs) if its complement is N-os.

Definition 2.4 (Interior and closure). Let (Ψ, Φ_N) be an NTS and E a NS on Ψ. Define

$$\mathrm{Nint}(E) = \bigcup\{U : U \subseteq E,\ U \in \Phi_N\}, \qquad \mathrm{Ncl}(E) = \bigcap\{C : E \subseteq C,\ C^c \in \Phi_N\}.$$

Similarly, the δ-interior and δ-closure are denoted by $\mathrm{N}\delta\mathrm{int}(E)$ and $\mathrm{N}\delta\mathrm{cl}(E)$.

Definition 2.5 (Semi/regular/β open sets [31]). Let (Ψ, Φ_N) be an NTS and $\mathcal{M}$ a NS. Then

$$\mathcal{M} \text{ is N-sos if } \mathcal{M} \subseteq \mathrm{Nint}(\mathrm{Ncl}(\mathcal{M})), \quad \mathcal{M} \text{ is N-ros if } \mathcal{M} = \mathrm{Ncl}(\mathrm{Nint}(\mathcal{M})),$$

$$\mathcal{M} \text{ is N-}\beta\text{os if } \mathcal{M} \subseteq \mathrm{Ncl}\big(\mathrm{Nint}(\mathrm{Ncl}(\mathcal{M}))\big).$$

Their complements are, respectively, N-scs and N-βcs.

Definition 2.6 (δ-, δ-semi-, and e^*-open sets [9])**.** A NS $\mathcal{M}$ is:

$$\text{N}\delta\text{-os if } \mathcal{M} = \text{N}\delta\text{int}(\mathcal{M}), \qquad \text{N}\delta\text{S-os if } \mathcal{M} \subseteq \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M})),$$

$$\text{N}e^*\text{-os if } \mathcal{M} \subseteq \text{Ncl}\big(\text{Nint}(\text{N}\delta\text{cl}(\mathcal{M}))\big).$$

Complements are denoted by Nδ-cs, NδS-cs, and Ne^*-cs, respectively.

Definition 2.7 (e-open / e-closed sets [9])**.** A NS $\mathcal{M}$ is

$$\text{N-eos if } \mathcal{M} \subseteq \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M})) \ \cup \ \text{Nint}(\text{N}\delta\text{cl}(\mathcal{M})),$$

$$\text{N-ecs if } \ \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M})) \ \cap \ \text{Nint}(\text{N}\delta\text{cl}(\mathcal{M})) \subseteq \mathcal{M}.$$

We denote by N-eos(Ψ) (resp. N-ecs(Ψ)) the family of all N-eos (resp. N-ecs) on Ψ.

Theorem 2.8 (Basic inclusions [9])**.** *Let* (Ψ, Φ_N) *be an NT and* $\mathcal{M} \subseteq \Psi$*. Then:*

(1) $\mathcal{M} \cup \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M})) \subseteq \text{N}\delta\text{Pcl}(\mathcal{M})$.
(2) $\text{N}\delta\text{Pint}(\mathcal{M}) \subseteq \mathcal{M} \cap \text{Nint}(\text{N}\delta\text{cl}(\mathcal{M}))$.
(3) $\mathcal{M} \cup \text{Nint}(\text{N}\delta\text{cl}(\mathcal{M})) \subseteq \text{N}\delta\text{Scl}(\mathcal{M})$.
(4) $\text{N}\delta\text{Sint}(\mathcal{M}) \subseteq \mathcal{M} \cap \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M}))$.
(5) $\mathcal{M} \cup \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M})) \subseteq \text{N}\delta\beta\text{cl}(\mathcal{M})$.
(6) $\text{N}\delta\beta\text{int}(\mathcal{M}) \subseteq \mathcal{M} \cap \text{Ncl}(\text{N}\delta\text{int}(\mathcal{M}))$.

Definition 2.9 (Neutrosophic scoring functions [29])**.** For $(\mu, \sigma, \nu) \in [0,1]^3$, the neutrosophic score and the negative score are defined by

$$s(\mu, \sigma, \nu) = \frac{2 + \mu - \sigma - \nu}{3}, \qquad s^-(\mu, \sigma, \nu) = \frac{1 - \mu + \sigma + \nu}{3}.$$

3. Characteristics of Neutrosophic $\mathfrak{I}$-Open Sets

In this section we introduce neutrosophic $\mathfrak{I}$-open sets and describe their basic properties in an NTS (Ψ, Φ_N).

Definition 3.1 (Neutrosophic $\mathfrak{I}$-open / $\mathfrak{I}$-closed)**.** A set E is called *neutrosophic* $\mathfrak{I}$*-open* (briefly, $N\mathfrak{I}$-os) if

$$E \subseteq \text{Ncl}(\text{N}\delta\text{int}(E)) \ \cup \ \text{Ncl}\big(\text{N}\delta\text{int}(\text{N}\delta\text{cl}(E))\big).$$

Dually, E is *neutrosophic* $\mathfrak{I}$*-closed* (briefly, $N\mathfrak{I}$-cs) if

$$\text{Nint}(\text{N}\delta\text{cl}(E)) \ \cap \ \text{Nint}\big(\text{N}\delta\text{cl}(\text{N}\delta\text{int}(E))\big) \subseteq E.$$

We denote by N$\mathfrak{I}$-os(Ψ) (resp. N$\mathfrak{I}$-cs(Ψ)) the family of all $N\mathfrak{I}$-open (resp. $N\mathfrak{I}$-closed) sets in Ψ.

Example 3.2. Let $\Psi = \{e, d, c\}$ and $\Phi_N = \{0_N, 1_N, \mathcal{P}\}$, where

$$\mathcal{P} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.5}, \tfrac{\mu_d}{0.7}, \tfrac{\mu_c}{0.6}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.5}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.6}, \tfrac{v_d}{0.7}, \tfrac{v_c}{0.9}\right) \right\rangle.$$

Consider

$$E = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.3}, \tfrac{\mu_d}{0.3}, \tfrac{\mu_c}{0.1}\right),\ \left(\tfrac{\sigma_e}{0.37}, \tfrac{\sigma_d}{0.41}, \tfrac{\sigma_c}{0.26}\right),\ \left(\tfrac{v_e}{0.61}, \tfrac{v_d}{0.5}, \tfrac{v_c}{0.61}\right) \right\rangle.$$

Then (by construction of the chosen topology) $\mathrm{N}\delta\mathrm{int}(E) = 0_N$ and $\mathrm{N}\delta\mathrm{cl}(E) = 1_N$, hence

$$\mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) = 0_N, \qquad \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big) = 1_N.$$

It follows that

$$\mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big) = 1_N,$$

and therefore $E \subseteq 1_N$, i.e., E is an $N\mathfrak{I}$-os.

Now set

$$\mathcal{O} = \langle (e, 0.7, 0.2, 0.2),\ (d, 0.7, 0.3, 0.3),\ (c, 0.8, 0.4, 0.4) \rangle.$$

We have $\mathrm{N}\delta\mathrm{int}(\mathcal{O}) = \mathcal{P}$ and $\mathrm{N}\delta\mathrm{cl}(\mathcal{O}) = 1_N$, hence

$$\mathrm{Nint}(\mathrm{N}\delta\mathrm{cl}(\mathcal{O})) = 1_N, \qquad \mathrm{Nint}\big(\mathrm{N}\delta\mathrm{cl}(\mathrm{N}\delta\mathrm{int}(\mathcal{O}))\big) = \mathcal{P}.$$

Thus

$$\mathrm{Nint}(\mathrm{N}\delta\mathrm{cl}(\mathcal{O})) \cap \mathrm{Nint}\big(\mathrm{N}\delta\mathrm{cl}(\mathrm{N}\delta\mathrm{int}(\mathcal{O}))\big) = \mathcal{P} \subseteq \mathcal{O},$$

so $\mathcal{O}$ is an $N\mathfrak{I}$-cs.

Theorem 3.3. *Let (Ψ, Φ_N) be an NTS. Then:*

(1) *The union of an arbitrary family of $N\mathfrak{I}$-os is an $N\mathfrak{I}$-os.*

(2) *The intersection of an arbitrary family of $N\mathfrak{I}$-cs is an $N\mathfrak{I}$-cs.*

Proof. (1) Let $\{\mathcal{G}_b\}_{b \in B} \subseteq \mathrm{N}\mathfrak{I}\text{-os}(\Psi)$. By definition,

$$\mathcal{G}_b \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathcal{G}_b)) \cup \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(\mathcal{G}_b))\big) \quad \forall b.$$

Using the monotonicity of Ncl, Nδint, Nδcl and the distributivity of unions:

$$\begin{aligned}\bigcup_b \mathcal{G}_b &\subseteq \bigcup_b \Big(\mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathcal{G}_b)) \cup \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(\mathcal{G}_b))\big) \Big) \\ &\subseteq \mathrm{Ncl}\Big(\mathrm{N}\delta\mathrm{int}\big(\textstyle\bigcup_b \mathcal{G}_b\big) \Big) \ \cup \ \mathrm{Ncl}\Big(\mathrm{N}\delta\mathrm{int}\big(\mathrm{N}\delta\mathrm{cl}(\textstyle\bigcup_b \mathcal{G}_b) \big) \Big),\end{aligned}$$

hence $\bigcup_b \mathcal{G}_b$ is $N\mathfrak{I}$-os.

(2) The statement follows dually by passing to complements: the intersection of $N\mathfrak{I}$-cs is $N\mathfrak{I}$-cs. □

Remark 3.4. The intersection of two $N\mathfrak{I}$-os need not be an $N\mathfrak{I}$-os; see Example 3.5 below.

Example 3.5. Let $\Psi = \{e, d, c\}$ and define the NSs $\mathcal{P}, \mathrm{E}, \mathcal{N}$ on Ψ by

$$\mathcal{P} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.2}, \tfrac{\mu_d}{0.3}, \tfrac{\mu_c}{0.2}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.4}\right),\ \left(\tfrac{v_e}{0.8}, \tfrac{v_d}{0.7}, \tfrac{v_c}{0.8}\right) \right\rangle,$$
$$\mathrm{E} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.4}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.6}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.4}\right),\ \left(\tfrac{v_e}{0.6}, \tfrac{v_d}{0.4}, \tfrac{v_c}{0.6}\right) \right\rangle,$$
$$\mathcal{N} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.1}, \tfrac{\mu_d}{0.2}, \tfrac{\mu_c}{0.3}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.4}\right),\ \left(\tfrac{v_e}{0.9}, \tfrac{v_d}{0.8}, \tfrac{v_c}{0.7}\right) \right\rangle.$$

Assume $\Phi_N = \{0_N, 1_N, \mathcal{P}, \mathrm{E}, \mathcal{N}\}$. Then $\mathcal{P}, \mathrm{E} \in \mathrm{N}\mathfrak{I}\text{-os}(\Psi)$ but $\mathcal{P} \cap \mathrm{E}$ is not $N\mathfrak{I}$-os.

Definition 3.6 ($N\mathfrak{I}$-closure and $N\mathfrak{I}$-interior)**.** For $E \subseteq \Psi$ set

$$N\mathfrak{I}\mathrm{cl}(E) = \bigcap\{\mathfrak{D} : E \subseteq \mathfrak{D},\ \mathfrak{D} \text{ is } N\mathfrak{I}\text{-cs in } \Psi\},$$
$$N\mathfrak{I}\mathrm{int}(E) = \bigcup\{\mathcal{V} : \mathcal{V} \subseteq E,\ \mathcal{V} \text{ is } N\mathfrak{I}\text{-os in } \Psi\}.$$

Remark 3.7. For every $E \subseteq \Psi$ the following hold:

- $N\mathfrak{I}\mathrm{cl}(0_N) = 0_N$, $N\mathfrak{I}\mathrm{cl}(1_N) = 1_N$; $N\mathfrak{I}\mathrm{cl}(E)$ is $N\mathfrak{I}$-cs and $N\mathfrak{I}\mathrm{cl}(N\mathfrak{I}\mathrm{cl}(E)) = N\mathfrak{I}\mathrm{cl}(E)$.
- $N\mathfrak{I}\mathrm{int}(0_N) = 0_N$, $N\mathfrak{I}\mathrm{int}(1_N) = 1_N$; $N\mathfrak{I}\mathrm{int}(E)$ is $N\mathfrak{I}$-os and $N\mathfrak{I}\mathrm{int}(N\mathfrak{I}\mathrm{int}(E)) = N\mathfrak{I}\mathrm{int}(E)$.

Theorem 3.8. *If* (Ψ, Φ_N) *is an NTS and* E *is* $N\mathfrak{I}$*-os, then*

$$E = \mathrm{N}\delta\mathrm{Sint}(E)\ \cup\ \mathrm{N}\delta\beta\mathrm{int}(E).$$

Proof. If E is $N\mathfrak{I}$-os, by Definition 3.1

$$E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big).$$

By Theorem 2.8 (items (4) and (6)) we have

$$\mathrm{N}\delta\mathrm{Sint}(E) \cup \mathrm{N}\delta\beta\mathrm{int}(E) = \big(E \cap \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E))\big)\ \cup\ \big(E \cap \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))\big) = E.$$

Conversely, if $E = \mathrm{N}\delta\mathrm{Sint}(E) \cup \mathrm{N}\delta\beta\mathrm{int}(E)$, again by Theorem 2.8:

$$E = \big(E \cap \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E))\big)\ \cup\ \big(E \cap \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))\big)$$
$$= E \cap \Big(\mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))\Big) \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))),$$

hence E is $N\mathfrak{I}$-os. □

Theorem 3.9. *Let* (Ψ, Φ_N) *be an NTS and* $E \subseteq \Psi$*. Then* E *is* $N\mathfrak{I}$*-cs (resp.* $N\mathfrak{I}$*-os) if and only if*

$$E = N\mathfrak{I}\mathrm{cl}(E) \quad (\textit{resp. } E = N\mathfrak{I}\mathrm{int}(E)).$$

Proof. If $E = N\mathfrak{I}\mathrm{cl}(E) = \bigcap\{\mathcal{K} : E \subseteq \mathcal{K},\ \mathcal{K} \text{ is } N\mathfrak{I}\text{-cs}\}$, then E belongs to every $N\mathfrak{I}$-cs that contains it, hence E is $N\mathfrak{I}$-cs. Conversely, if E is $N\mathfrak{I}$-cs, then $E \supseteq \bigcap\{\mathcal{K} : E \subseteq \mathcal{K},\ \mathcal{K} \text{ is } N\mathfrak{I}\text{-cs}\}$

and also $E \subseteq$ that intersection, so $E = N\mathfrak{I}\mathrm{cl}(E)$. The dual argument yields the equivalence for $N\mathfrak{I}\mathrm{int}(E)$. □

Theorem 3.10. *Let (Ψ, Φ_N) be an NTS and $E \subseteq \Psi$. Then:*

(1) $\mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \ \cap \ \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big) \ \subseteq \ N\mathfrak{I}\mathrm{cl}(E)$.

(2) $N\mathfrak{I}\mathrm{int}(E) \ \subseteq \ \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \ \cup \ \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big)$.

Proof. (1) Since $E \subseteq N\mathfrak{I}\mathrm{cl}(E)$, by monotonicity of $\mathrm{N}\delta\mathrm{int}, \mathrm{N}\delta\mathrm{cl}, \mathrm{Ncl}$ we get

$$\mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E))\cap\mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big) \subseteq \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(N\mathfrak{I}\mathrm{cl}(E))\big)\cap\mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(N\mathfrak{I}\mathrm{cl}(E)))\big) \subseteq N\mathfrak{I}\mathrm{cl}(E),$$

where the last inclusion follows from Definition 3.1 applied to the $N\mathfrak{I}$-cs $N\mathfrak{I}\mathrm{cl}(E)$.

(2) By Definition 3.6, $N\mathfrak{I}\mathrm{int}(E)$ is $N\mathfrak{I}$-os and $N\mathfrak{I}\mathrm{int}(E) \subseteq E$. Using again Definition 3.1:

$$N\mathfrak{I}\mathrm{int}(E) \subseteq \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(N\mathfrak{I}\mathrm{int}(E))\big) \cup \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(N\mathfrak{I}\mathrm{int}(E)))\big) \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}\big(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E))\big).$$

□

Theorem 3.11. *Let (Ψ, Φ_N) be an NTS and $E \subseteq \Psi$. Then:*

(1) $N\mathfrak{I}\mathrm{cl}(1_N - E) = 1_N - N\mathfrak{I}\mathrm{int}(E)$.

(2) $N\mathfrak{I}\mathrm{int}(1_N - E) = 1_N - N\mathfrak{I}\mathrm{cl}(E)$.

Proof. (1) By the definition of $N\mathfrak{I}\mathrm{int}$,

$$N\mathfrak{I}\mathrm{int}(E) = \bigcup\{U : U \subseteq E,\ U \text{ is } N\mathfrak{I}\text{-os}\} = 1_N - \bigcap\{U^c : U \subseteq E,\ U \text{ is } N\mathfrak{I}\text{-os}\}.$$

Since U is $N\mathfrak{I}$-os iff U^c is $N\mathfrak{I}$-cs and $U \subseteq E \iff U^c \supseteq 1_N - E$, it follows that

$$N\mathfrak{I}\mathrm{int}(E) = 1_N - \bigcap\{\mathcal{N} : \mathcal{N} \supseteq 1_N - E,\ \mathcal{N} \text{ is } N\mathfrak{I}\text{-cs}\} = 1_N - N\mathfrak{I}\mathrm{cl}(1_N - E).$$

(2) is the dual statement. □

Proposition 3.12. *Let (Ψ, Φ_N) be an NTS. Then:*

(1) *Every $N\delta S$-os (resp. $N\delta S$-cs) is $N\mathfrak{I}$-os (resp. $N\mathfrak{I}$-cs).*

(2) *Every $N\delta P$-os (resp. $N\delta P$-cs) is $N\mathfrak{I}$-os (resp. $N\mathfrak{I}$-cs).*

(3) *Every N-eos (resp. N-ecs) is Ne*-os (resp. Ne*-cs).*

(4) *Every N-eos (resp. N-ecs) is $N\mathfrak{I}$-os (resp. $N\mathfrak{I}$-cs).*

(5) *Every Ne*-os (resp. Ne*-cs) is $N\mathfrak{I}$-os (resp. $N\mathfrak{I}$-cs).*

Proof. (1) If E is $N\delta S$-os, then by Definition 2.6 $E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E))$, hence $E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))$, i.e., E is $N\mathfrak{I}$-os. The "cs" case is dual.

(2) If E is $N\delta P$-os, by definition $E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))$, thus again E is $N\mathfrak{I}$-os; dual for "cs".

(3) This implication is known in the literature (see, e.g., [9]): N-eos $\Rightarrow$ Ne^*-os; similarly for "cs".

(4) If E is N-eos, then $E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Nint}(\mathrm{N}\delta\mathrm{cl}(E))$. Since $\mathrm{Nint}(\cdot) \subseteq \mathrm{Ncl}(\cdot)$, we get $E \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(E)) \cup \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))$, hence E is $N\mathfrak{I}$-os; dual for "cs".

(5) If E is Ne^*-os, by Definition 2.6 $E \subseteq \mathrm{Ncl}(\mathrm{Nint}(\mathrm{N}\delta\mathrm{cl}(E))) \subseteq \mathrm{Ncl}(\mathrm{N}\delta\mathrm{int}(\mathrm{N}\delta\mathrm{cl}(E)))$, so E is $N\mathfrak{I}$-os; dual for "cs". □

Remark 3.13. The converses of Proposition 3.12 do not hold in general.

Example 3.14. Let $\Psi = \{e, d, c\}$ and consider the NSs $\mathcal{P}, E, \mathcal{N}$ given by

$$\mathcal{P} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.3}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.5}\right),\ \left(\tfrac{\sigma_e}{0.4}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.4}\right),\ \left(\tfrac{v_e}{0.9}, \tfrac{v_d}{0.8}, \tfrac{v_c}{0.6}\right) \right\rangle,$$

$$E = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.2}, \tfrac{\mu_d}{0.3}, \tfrac{\mu_c}{0.5}\right),\ \left(\tfrac{\sigma_e}{0.4}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.4}\right),\ \left(\tfrac{v_e}{0.8}, \tfrac{v_d}{0.8}, \tfrac{v_c}{0.5}\right) \right\rangle,$$

$$\mathcal{N} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.3}, \tfrac{\mu_d}{0.5}, \tfrac{\mu_c}{0.6}\right),\ \left(\tfrac{\sigma_e}{0.4}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.4}\right),\ \left(\tfrac{v_e}{0.9}, \tfrac{v_d}{0.7}, \tfrac{v_c}{0.7}\right) \right\rangle.$$

With $\Phi_N = \{0_N, 1_N, \mathcal{P}, E, \mathcal{N}\}$ one checks (by direct computation) that $\mathcal{N}$ is $N\mathfrak{I}$-os but not $N\delta S$-os.

Example 3.15. Let $\Psi = \{e, d, c\}$ and define the NSs $\mathcal{P}, \mathrm{E}, \mathcal{N}, \mathcal{K}$ and $\mathcal{M}$ on Ψ by

$$\mathcal{P} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.3}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.4}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.8}, \tfrac{v_d}{0.4}, \tfrac{v_c}{0.5}\right) \right\rangle,$$

$$\mathrm{E} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.8}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.5}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.3}, \tfrac{v_d}{0.4}, \tfrac{v_c}{0.4}\right) \right\rangle,$$

$$\mathcal{N} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.5}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.5}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.8}, \tfrac{v_d}{0.7}, \tfrac{v_c}{0.7}\right) \right\rangle,$$

$$\mathcal{K} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.8}, \tfrac{\mu_d}{0.7}, \tfrac{\mu_c}{0.7}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.5}, \tfrac{v_d}{0.4}, \tfrac{v_c}{0.5}\right) \right\rangle,$$

$$\mathcal{M} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.5}, \tfrac{\mu_d}{0.5}, \tfrac{\mu_c}{0.4}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.4}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.5}, \tfrac{v_d}{0.5}, \tfrac{v_c}{0.4}\right) \right\rangle.$$

Let $\Phi_N = \{0_N, 1_N, \mathcal{P}, \mathrm{E}, \mathcal{N}, \mathcal{K}, \mathcal{M}\}$. Then, by direct computation, $\mathcal{M}$ *is an $N\mathfrak{I}$-os but not an $N\delta P$-os.*

Example 3.16. Let $\Psi = \{e, d, c\}$ and define the NSs $\mathcal{P}, \mathrm{E}, \mathcal{N}$ and $\mathcal{K}$ on Ψ by

$$\mathcal{P} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.4}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.4}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.5}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.6}, \tfrac{v_d}{0.6}, \tfrac{v_c}{0.6}\right) \right\rangle,$$

$$\mathrm{E} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.8}, \tfrac{\mu_d}{0.4}, \tfrac{\mu_c}{0.5}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.5}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.4}, \tfrac{v_d}{0.4}, \tfrac{v_c}{0.4}\right) \right\rangle,$$

$$\mathcal{N} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.6}, \tfrac{\mu_d}{0.3}, \tfrac{\mu_c}{0.8}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.5}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.4}, \tfrac{v_d}{0.4}, \tfrac{v_c}{0.2}\right) \right\rangle,$$

$$\mathcal{K} = \left\langle \Psi,\ \left(\tfrac{\mu_e}{0.7}, \tfrac{\mu_d}{0.6}, \tfrac{\mu_c}{0.7}\right),\ \left(\tfrac{\sigma_e}{0.5}, \tfrac{\sigma_d}{0.5}, \tfrac{\sigma_c}{0.5}\right),\ \left(\tfrac{v_e}{0.3}, \tfrac{v_d}{0.8}, \tfrac{v_c}{0.6}\right) \right\rangle.$$

Let $\Phi_N = \{0_N, 1_N, \mathcal{P}, \mathrm{E}, \mathcal{N}, \mathcal{K}\}$. Then, by direct computation, $\mathcal{K}$ *is an* $N\mathfrak{I}$*-os but not an* N*-eos; moreover,* $\mathcal{K}$ *is not* Ne^**-os.*

Remark 3.17. The relationships among several notions in NTS are depicted in Fig. 1.

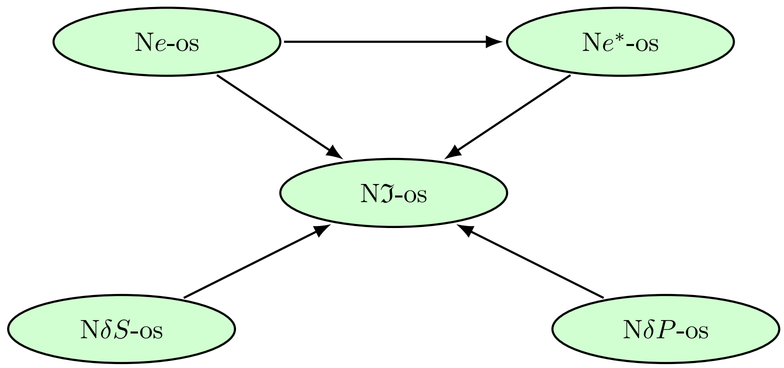

FIGURE 1. Inclusion relations among classes of neutrosophic open sets in NTS (arrows indicate $\subseteq$). See Proposition 3.12.

4. A Neutrosophic Multi-Criteria Assessment for Choosing Suppliers

Choosing suppliers is a critical component of modern supply chain management. As global supply chains grow in complexity, firms must evaluate multiple vendors under ambiguous and sometimes conflicting criteria (quality, delivery reliability, and cost effectiveness). In this section we outline a neutrosophic multi-criteria decision process that leverages the *neutrosophic positive score* and the *neutrosophic negative score* (see Definition 2.9; cf. [29]) to handle indeterminacy and support robust rankings.

4.1. *Problem Description*

We consider three candidate suppliers

$$S = \{A, B, C\},$$

three evaluation criteria

$$C_1 = \text{Product Quality}, \quad C_2 = \text{Delivery Reliability}, \quad C_3 = \text{Cost Efficiency},$$

and three procurement strategies

$$D_1,\ D_2,\ D_3,$$

each representing a different emphasis on the criteria (e.g., quality-driven, reliability-driven, cost-driven).

4.2. *Neutrosophic Decision Model*

For each strategy D_k we adopt a weight vector

$$\mathbf{w}^{(k)} = (w_1^{(k)}, w_2^{(k)}, w_3^{(k)}), \qquad w_j^{(k)} \in [0,1],\ \sum_{j=1}^{3} w_j^{(k)} = 1.$$

Judgments are encoded in a neutrosophic decision matrix

$$\mathcal{E}^{(k)} = \big[(\mu_{ij}^{(k)}, \sigma_{ij}^{(k)}, \nu_{ij}^{(k)})\big]_{i=1,\dots,3}^{j=1,\dots,3},$$

where $(\mu,\sigma,\nu) \in [0,1]^3$ respectively denote truth (membership), indeterminacy, and falsity (non-membership) degrees for supplier i on criterion j under strategy D_k.
Normalisation (optional). If raw measurements are not already on $[0,1]$, apply a monotone normalisation per criterion so that higher μ means "better" (for cost-type indicators, invert the scale).

4.3. *Scoring, Aggregation, and Ranking*

We use the *positive* and *negative* neutrosophic scores (Definition 2.9)

$$s(\mu,\sigma,\nu) = \frac{2+\mu-\sigma-\nu}{3}, \qquad s^{-}(\mu,\sigma,\nu) = \frac{1-\mu+\sigma+\nu}{3},$$

to compute, for each D_k,

$$S_i^{(k)} = \sum_{j=1}^{3} w_j^{(k)}\, s\big(\mu_{ij}^{(k)}, \sigma_{ij}^{(k)}, \nu_{ij}^{(k)}\big), \qquad N_i^{(k)} = \sum_{j=1}^{3} w_j^{(k)}\, s^{-}\big(\mu_{ij}^{(k)}, \sigma_{ij}^{(k)}, \nu_{ij}^{(k)}\big).$$

Two equivalent ranking rules are typical:

- **Bi-objective outranking:** i is preferred to r if $S_i^{(k)} > S_r^{(k)}$ and $N_i^{(k)} < N_r^{(k)}$ (with tie-breakers if needed).
- **Composite index:** for $\alpha \in [0,1]$,

$$F_i^{(k)} = \alpha\, S_i^{(k)} + (1-\alpha)\big(1 - N_i^{(k)}\big),$$

and rank by decreasing $F_i^{(k)}$ (a neutral choice is $\alpha = \frac{1}{2}$).

4.4. *Algorithm*

The following algorithm operationalizes the neutrosophic multi-criteria framework.

(1) Fix S, the criteria set $\{C_1, C_2, C_3\}$, and strategies D_1, D_2, D_3.
(2) For each D_k, set weights $\mathbf{w}^{(k)}$ with $\sum_j w_j^{(k)} = 1$.
(3) Elicit/construct $\mathcal{E}^{(k)}$ with neutrosophic triplets $(\mu,\sigma,\nu) \in [0,1]^3$.
(4) (If needed) normalise raw indicators to $[0,1]$ so that higher μ is better.
(5) Compute $S_i^{(k)}$ and $N_i^{(k)}$ via the scores s and s^{-}.

(6) Rank suppliers by the bi-objective rule or by the composite index $F_i^{(k)}$.

(7) Select the top-ranked supplier for each strategy D_k; report $(S_i^{(k)}, N_i^{(k)}, F_i^{(k)})$.

4.5. *Template Decision Matrix*

For quick use, here is a minimal table to be filled with (μ, σ, ν) values on $[0, 1]$:

	C_1 (Quality)	C_2 (Reliability)	C_3 (Cost Efficiency)
A	$(\mu_{11}, \sigma_{11}, \nu_{11})$	$(\mu_{12}, \sigma_{12}, \nu_{12})$	$(\mu_{13}, \sigma_{13}, \nu_{13})$
B	$(\mu_{21}, \sigma_{21}, \nu_{21})$	$(\mu_{22}, \sigma_{22}, \nu_{22})$	$(\mu_{23}, \sigma_{23}, \nu_{23})$
C	$(\mu_{31}, \sigma_{31}, \nu_{31})$	$(\mu_{32}, \sigma_{32}, \nu_{32})$	$(\mu_{33}, \sigma_{33}, \nu_{33})$

(For each strategy D_k, use a weight row $\mathbf{w}^{(k)}$.)

Remark 4.1. The composite-index approach yields a total preorder; the bi-objective rule induces a partial order that may return ties or incomparabilities, which is often informative in practice. Sensitivity analysis over α and $\mathbf{w}^{(k)}$ is recommended.

Step 1: Neutrosophic Supplier Decision Matrix

The Single-Valued Neutrosophic Values (SVNVs) for each supplier under the three criteria are reported in Table 1. Each entry is a triplet (μ, σ, ν), where:

- μ is the truth-membership (degree of satisfaction),
- σ is the indeterminacy-membership,
- ν is the falsity-membership.

Table 1. SVNVs for each supplier across the three criteria.

Criteria	Supplier A	Supplier B	Supplier C
Product Quality	(0.76, 0.52, 0.25)	(0.68, 0.56, 0.32)	(0.87, 0.49, 0.17)
Delivery Reliability	(0.84, 0.48, 0.20)	(0.75, 0.54, 0.27)	(0.89, 0.50, 0.14)
Cost Efficiency	(0.76, 0.48, 0.24)	(0.68, 0.56, 0.29)	(0.84, 0.48, 0.18)

Step 2: Neutrosophic Purchasing Strategy Decision Matrix

Table 2 lists the SVNVs of the three procurement strategies with respect to the same criteria.

Step 3: Neutrosophic Topology Construction

We now form neutrosophic structures for suppliers (τ_i) and strategies (σ_i).

TABLE 2. Neutrosophic values for quality, reliability, and cost efficiency under each purchasing strategy.

Strategy	Product Quality	Delivery Reliability	Cost Efficiency
$\mathcal{D}_1$	(0.82, 0.49, 0.20)	(0.85, 0.48, 0.19)	(0.78, 0.52, 0.23)
$\mathcal{D}_2$	(0.74, 0.54, 0.25)	(0.78, 0.48, 0.22)	(0.76, 0.52, 0.24)
$\mathcal{D}_3$	(0.78, 0.48, 0.20)	(0.83, 0.49, 0.18)	(0.82, 0.46, 0.19)

(i) Supplier topology τ^*.

$$\tau^* = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3,$$

where

$$\mathcal{M}_1 = \{(0.79, 0.51, 0.21), (0.82, 0.48, 0.19), (0.75, 0.49, 0.23)\},$$

$$\mathcal{M}_2 = \{(0.86, 0.49, 0.16), (0.89, 0.50, 0.13), (0.83, 0.47, 0.17)\},$$

$$\mathcal{M}_3 = \{(0.67, 0.56, 0.31), (0.73, 0.53, 0.26), (0.70, 0.55, 0.29)\}.$$

(ii) Strategy topology σ^*.

$$\sigma^* = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3,$$

where

$$\mathcal{N}_1 = \{(0.81, 0.49, 0.19), (0.84, 0.47, 0.18), (0.80, 0.50, 0.22)\},$$

$$\mathcal{N}_2 = \{(0.73, 0.53, 0.26), (0.77, 0.49, 0.21), (0.75, 0.52, 0.24)\},$$

$$\mathcal{N}_3 = \{(0.78, 0.48, 0.20), (0.82, 0.49, 0.17), (0.83, 0.46, 0.19)\}.$$

Step 4: Neutrosophic Score Function

For a neutrosophic triplet (μ, σ, ν) we use the (positive) neutrosophic score

$$\mathrm{NSF}(\mu, \sigma, \nu) = \frac{\mu + (1 - \nu) - \sigma}{2}.$$

(Equivalently, one may use the positive/negative scores of Definition 2.9; these are affine transforms of the same linear form and thus preserve rankings.)

Applying the above to the aggregated sets gives the following illustrative values:

Suppliers: $\mathrm{NSF}(\tau_1) \approx 0.707$, $\mathrm{NSF}(\tau_2) \approx 0.657$, $\mathrm{NSF}(\tau_3) \approx 0.760$;

Strategies: $\mathrm{NSF}(\sigma_1) \approx 0.724$, $\mathrm{NSF}(\sigma_2) \approx 0.675$, $\mathrm{NSF}(\sigma_3) \approx 0.745$.

Step 4.1: Worked Numerical Example (Weights, Scores, Rankings)

We illustrate the procedure using the supplier matrix of Step 1. For each strategy D_k we choose the following weight vectors (criteria order: C_1 = Quality, C_2 = Reliability, C_3 = Cost):

$$\mathbf{w}^{(1)} = (0.50,\ 0.30,\ 0.20), \quad \mathbf{w}^{(2)} = (0.30,\ 0.50,\ 0.20), \quad \mathbf{w}^{(3)} = (0.25,\ 0.25,\ 0.50).$$

We use the positive and negative neutrosophic scores of Definition 2.9:

$$s(\mu,\sigma,\nu)=\frac{2+\mu-\sigma-\nu}{3}, \qquad s^{-}(\mu,\sigma,\nu)=\frac{1-\mu+\sigma+\nu}{3}.$$

For supplier i under strategy D_k, the aggregated scores are

$$S_i^{(k)}=\sum_{j=1}^{3} w_j^{(k)}\, s(\mu_{ij},\sigma_{ij},\nu_{ij}), \qquad N_i^{(k)}=\sum_{j=1}^{3} w_j^{(k)}\, s^{-}(\mu_{ij},\sigma_{ij},\nu_{ij}).$$

Remark. Since $s(\cdot)+s^{-}(\cdot)=1$ pointwise, it follows that $N_i^{(k)}=1-S_i^{(k)}$. Hence ranking by decreasing S is equivalent to ranking by increasing N.

(i) Aggregated positive scores $S_i^{(k)}$.

Supplier	D_1	D_2	D_3
A	0.684	0.695	0.686
B	0.616	0.625	0.617
C	0.739	0.741	0.735

(ii) Aggregated negative scores $N_i^{(k)}$ (for completeness).

Supplier	D_1	D_2	D_3
A	0.316	0.305	0.314
B	0.384	0.375	0.383
C	0.261	0.259	0.265

(iii) Resulting rankings. By positive scores (maximize S; equivalently, minimize N):

$$D_1: \quad C \succ A \succ B,$$

$$D_2: \quad C \succ A \succ B,$$

$$D_3: \quad C \succ A \succ B.$$

Therefore, under all three strategies the top-ranked supplier is C. If a one-to-one assignment of strategies to different suppliers is required (e.g., capacity or diversification constraints), one may apply secondary tie-breakers (e.g., lexicographic on the most weighted criterion, or a composite business rule).

Step 4.2: Final Ranking Summary and One-to-One Assignment (Optional)

We summarise the per-strategy rankings by positive scores $S_i^{(k)}$ and derive a one-to-one assignment (each strategy uses a different supplier). Rule: assign the supplier with the largest margin (Top–1 − Top–2) first; then proceed with remaining strategies and remaining suppliers.

Strategy	Top–1 (score)	Top–2 (score)	Margin	Chosen (1–1)
$\mathcal{D}_1$	C (0.739)	A (0.684)	0.055	C
$\mathcal{D}_2$	C (0.741)	A (0.695)	0.046	A *(since C is used)*
$\mathcal{D}_3$	C (0.735)	A (0.686)	0.049	B *(remaining)*

Resulting one-to-one assignment:

$$\mathcal{D}_1 \to C, \qquad \mathcal{D}_2 \to A, \qquad \mathcal{D}_3 \to B.$$

Remark 4.2. Because $N_i^{(k)} = 1 - S_i^{(k)}$ for the positive/negative score pair, any composite index $F_i^{(k)} = \frac{1}{2}\big(S_i^{(k)} + 1 - N_i^{(k)}\big)$ coincides with $S_i^{(k)}$ and yields the same ranking.

Step 4.3: Final Ranking Table (S, N, F)

Table 3 reports, for each supplier and strategy, the aggregated positive score $S_i^{(k)}$, the aggregated negative score $N_i^{(k)}$, and the composite index $F_i^{(k)} = \frac{1}{2}\big(S_i^{(k)} + 1 - N_i^{(k)}\big)$. *Note:* since $S + N = 1$ for the score pair in Definition 2.9, here $F_i^{(k)} \equiv S_i^{(k)}$.

TABLE 3. Final per-strategy rankings with $S_i^{(k)}$, $N_i^{(k)}$, and $F_i^{(k)}$.

	$\mathcal{D}_1$				$\mathcal{D}_2$				$\mathcal{D}_3$			
Supplier	S	N	F	Rank	S	N	F	Rank	S	N	F	Rank
A	0.684	0.316	0.684	2	0.695	0.305	0.695	2	0.686	0.314	0.686	2
B	0.616	0.384	0.616	3	0.625	0.375	0.625	3	0.617	0.383	0.617	3
C	**0.739**	**0.261**	**0.739**	**1**	**0.741**	**0.259**	**0.741**	**1**	**0.735**	**0.265**	**0.735**	**1**

Optional overall view (averaging S across strategies):

$$\overline{S}_A = 0.688, \quad \overline{S}_B = 0.619, \quad \overline{S}_C = 0.738 \quad \Rightarrow \quad \text{Overall ranking: } C \succ A \succ B.$$

Step 4.4: Overall Summary Across Strategies

We aggregate performances by averaging the positive score S across the three strategies ($\overline{S} = \frac{1}{3}\sum_{k=1}^{3} S^{(k)}$). Since $N = 1 - S$, we report $\overline{N} = 1 - \overline{S}$. The composite index F coincides numerically with S for the adopted score pair.

TABLE 4. Overall supplier performance (averaged across strategies).

Supplier	$\overline{S}$	$\overline{N}$	F	Overall Rank
A	0.688	0.312	0.688	2
B	0.619	0.381	0.619	3
C	**0.738**	**0.262**	**0.738**	**1**

Overall ordering: $C \succ A \succ B$.

Step 5: Final Choice

(i) Based on positive scores. Ordering:

$$\text{Suppliers: } \tau_3 \geq \tau_1 \geq \tau_2, \qquad \text{Strategies: } \sigma_3 \geq \sigma_1 \geq \sigma_2.$$

One possible assignment of "best supplier per strategy" is:

$$\mathcal{B} \text{ (Supplier B)} \to \mathcal{D}_1, \qquad \mathcal{A} \text{ (Supplier A)} \to \mathcal{D}_2, \qquad \mathcal{C} \text{ (Supplier C)} \to \mathcal{D}_3.$$

(ii) Based on negative scores. Ordering:

$$\text{Suppliers: } \tau_2 \geq \tau_1 \geq \tau_3, \qquad \text{Strategies: } \sigma_2 \geq \sigma_1 \geq \sigma_3.$$

Possible allocations:

$$\mathcal{D}_1 \to \mathcal{C}, \qquad \mathcal{D}_2 \to \mathcal{A}, \qquad \mathcal{D}_3 \to \mathcal{B}.$$

(iii) Takeaways. The dual scoring view offers decision-makers two complementary perspectives "maximizing satisfaction" vs. "minimizing risk" while the neutrosophic framework captures both uncertainty and inconsistency in assessments, supporting more reliable, data-driven supplier choices.

5. **Conclusions and future work**

Within neutrosophic topological spaces (NTS), we introduce and study $N\mathfrak{I}$-open and $N\mathfrak{I}$-closed sets, establishing basic stability properties namely, that arbitrary unions of $N\mathfrak{I}$-open sets and arbitrary intersections of $N\mathfrak{I}$-closed sets remain within the same classes. We provide sharp characterizations showing that a set E is $N\mathfrak{I}$-closed precisely when $E = N\mathfrak{I}\text{cl}(E)$, and $N\mathfrak{I}$-open precisely when $E = N\mathfrak{I}\text{int}(E)$. A structural decomposition is proved as well: every $N\mathfrak{I}$-open set E satisfies the formula $E = \text{N}\delta\text{Sint}(E) \cup \text{N}\delta\beta\text{int}(E)$. Furthermore, we clarify the position of $N\mathfrak{I}$-openness among related notions by proving inclusion relations that place $N\delta S$-open sets, $N\delta P$-open sets, N-eopen sets, and Ne^*-open sets inside the class of $N\mathfrak{I}$-open sets, while also exhibiting counterexamples to the converses. Finally, we illustrate the practical impact of these ideas through a supplier-selection study, where neutrosophic positive and negative scoring yield robust rankings in the presence of ambiguity and indeterminacy.

Several directions naturally follow from our results. From a topological viewpoint, one may develop bases and subbases for the $\mathfrak{I}$-topology and investigate the separation axioms and compactness concepts induced by $N\mathfrak{I}$-openness; in parallel, it is of interest to analyze continuity, irresoluteness, and homeomorphisms defined via preimages of $N\mathfrak{I}$-open sets. On the decision-theoretic side, the framework can be extended to interval, bipolar, and soft neutrosophic settings and compared systematically with rough and fuzzy soft models. Algorithmic aspects merit attention as well, including the design of composite indices, sensitivity analysis with respect to weights and scoring parameters, and robustness under inconsistent assessments. Finally, alternative ranking functionals such as accuracy and certainty alongside score and dynamic scenarios with time-varying data streams offer promising ground for further development.

Overall, the $N\mathfrak{I}$-open concept adds a flexible layer to the neutrosophic topological toolkit and supports practical multi-criteria decisions by explicitly encoding uncertainty and inconsistency in evaluations.

6. Conflict of Interest

The authors of this paper declare that they have no conflicts of interest.

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