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$\delta\beta$-open Sets in Pythagorean Neutrosophic Nano Topological Spaces and Their Application via Machine Learning

Mohanarao Navuluri[1] and Shantha lakshmi K[2*]

[1]Department of Mathematics, Government College of Engineering, Bodinayakkanur- 625582, Theni, India; (Mohanarao Navuluri) mohanaraonavuluri@gmail.com

[2]Department of Mathematics, M.Kumarasamy College of Engineering, Karur - 639 113, India; (Shantha lakshmi K) kslakshmi20@gmail.com

[1,2]Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India

[*]Correspondence: (Shantha lakshmi K) kslakshmi20@gmail.com; Tel.: (+91 9488323427)

Abstract. This paper introduces a novel class of generalized open sets, namely Pythagorean neutrosophic nano δ-open sets and their variants (δ-pre, δ-semi, $\delta\alpha$, and $\delta\beta$), within Pythagorean neutrosophic nano topological spaces, extending classical topological concepts to handle uncertainty, indeterminacy, and vagueness simultaneously. The fundamental properties of these generalized open and closed sets are rigorously analyzed, and corresponding closure and interior operators are defined to provide essential tools for topological analysis under uncertainty. Furthermore, the hierarchical interrelations among these sets are established, demonstrating that δ-open $\Rightarrow$ $\delta\alpha$-open $\Rightarrow$ $\delta\mathcal{S}$-open / $\delta\mathcal{P}$-open $\Rightarrow$ $\delta\beta$-open, with counterexamples confirming that converses do not hold. Finally, a machine learning application is presented to demonstrate the practical utility of constructing Pythagorean neutrosophic nano topological spaces from real-world datasets for classification and decision-making under incomplete information.

Keywords: Pythagorean neutrosophic nano open set, Pythagorean neutrosophic nano δ open set, Pythagorean neutrosophic nano δ interior, Pythagorean neutrosophic nano δ closure.

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1. Introduction

Zadeh [21] first introduced the concept of fuzzy sets in 1965, marking a significant generalization of classical (crisp) set theory. In a fuzzy set, each element is assigned a membership degree in the interval $[0, 1]$, thereby allowing partial belongingness to a set. This pioneering idea laid the foundation for numerous extensions and applications in uncertain and imprecise environments.

Building on this, Atanassov [3] proposed the notion of intuitionistic fuzzy sets in 1983 as a further generalization of Zadeh's fuzzy sets. In this framework, each element is characterized by both a membership degree and a non-membership degree, whose sum does not exceed one, thus providing a more refined structure for modeling uncertainty and hesitation.

The development of fuzzy topology began in 1968, when Chang [4] introduced the concept of fuzzy topological spaces, along with the fundamental notions of fuzzy open sets, fuzzy closed sets, and fuzzy continuity. Subsequently, Lowen [8] presented an alternative formulation of fuzzy topological spaces, enriching the theoretical foundations of the field. Later, Çoker [5] extended this study by defining intuitionistic fuzzy topological spaces and establishing several of their key properties.

To enhance the representational capacity of fuzzy sets, Yager [18–20] introduced Pythagorean fuzzy sets, a distinct class of fuzzy subsets in which the sum of the squares of the membership and non-membership degrees is at most one. This approach offers greater flexibility than intuitionistic fuzzy sets in handling uncertainty. Inspired by this, Olgun [8] developed the framework of Pythagorean fuzzy topological spaces, following the foundational structure established by Chang in fuzzy topology.

Parallel to these developments, Lellis Thivagar [6] introduced a novel topological framework known as nano topology in 2013, inspired by Pawlak's rough set theory. In this framework, nano topological spaces are constructed using the lower and upper approximations and boundary regions of subsets of a universe, determined via an equivalence relation. The open elements in such spaces are called nano open sets, and their complements are nano closed sets. The term "nano" signifies the study of very small or fine structures, so nano topology can be viewed as the topology of minute or granular spaces. The key notions underlying this theory are approximation and indiscernibility relations.

Further advancements in this area include the study of nano δ-open sets in nano topological spaces by Pankajam and Kavitha [10]. Recently, Lellis Thivagar et al. [7] extended the framework by proposing the concept of neutrosophic nano topology, which integrates the indeterminacy aspect of neutrosophic logic into nano topological structures. Additionally, Shiventhiradevi Sathaananthan et al. [13, 14] introduced and examined the properties of Z-closed sets in double fuzzy topological spaces in 2020. Vadivel et al. [16, 17] introduced new operators using Pythagorean fuzzy open sets, continuous maps, and irresolute maps in Pythagorean fuzzy topological spaces.

In the same vein, Padma et al. [9] investigated M-open sets in nano topological spaces, while Pankajam and Kavitha [10] explored δ-open sets within the same framework. Furthermore, the notion of neutrosophic nano topological spaces was later formalized and studied in detail by several researchers [15], thereby bridging the concepts of neutrosophy and nano topology to handle imprecise, indeterminate, and inconsistent information in a unified setting.

Research Gap: To the best of our knowledge, no comprehensive investigation has been conducted on the stronger and weaker forms of Pythagorean fuzzy nano open sets—such as Pythagorean fuzzy nano δ-open, δ-semi open, δ-pre open, $\delta\alpha$-open, and $\delta\beta$-open sets—within the framework of Pythagorean fuzzy nano topological spaces. These classes of sets remain unexplored in the existing Pythagorean fuzzy literature.

Contribution: The introduction of Pythagorean neutrosophic nano δ-open sets and their variants (δ-pre, δ-semi, $\delta\alpha$, and $\delta\beta$) represents a significant advancement in generalized topology that bridges multiple mathematical frameworks—classical topology, fuzzy sets, neutrosophic sets, rough sets, and nano topology—into a unified structure capable of handling complex real-world uncertainties. This research is theoretically significant because the Pythagorean constraint ($\mu^2 + \lambda^2 \leq 1$) offers greater flexibility than intuitionistic fuzzy sets in modeling membership and non-membership degrees, while the neutrosophic component explicitly captures indeterminacy, which is prevalent in practical scenarios where information is incomplete, vague, or conflicting. The systematic study of the hierarchy among these generalized open sets ($\delta \to \delta\alpha \to \delta\mathcal{S}/\delta\mathcal{P} \to \delta\beta$), along with their corresponding closure and interior operators, provides researchers with a refined toolkit for analyzing topological structures at multiple levels of granularity, enabling more nuanced approximations than standard open sets allow.

Importance of Neutrosophic in the Work : From a practical standpoint, this framework has profound implications for machine learning, data mining, medical diagnosis, image processing, decision-making systems, and pattern recognition, where uncertainty quantification is critical. The nano topological structure based on equivalence relations naturally connects to granular computing and rough set theory, allowing the identification of certain knowledge (lower approximation), possible knowledge (upper approximation), and uncertain regions (boundary). The δ-variants provide intermediate levels of approximation that enable soft boundaries, hierarchical clustering, and three-way decisions. The application to machine learning demonstrated in the paper illustrates how Pythagorean neutrosophic nano topological spaces can be constructed from real datasets to perform feature selection, classification with confidence scores, and uncertainty-aware predictions, thereby showcasing the practical utility of these theoretical developments and opening numerous future research directions in artificial intelligence, soft computing, information systems, and interdisciplinary applications where modeling uncertainty with mathematical rigor is essential.

2. Preliminaries

Definition 2.1. [18–20] Let U be a universal set. Then, a Pythagorean fuzzy set $\ominus$, which is a set of ordered pairs over U, is defined by the following: $\ominus = \{< \iota, \mu_{\ominus}(\iota), \lambda_{\ominus}(\iota) > | \iota \in U\}$

or $\ominus = \left\{\left\langle \frac{\mu_{\ominus}(\iota),\lambda_{\ominus}(\iota)}{x} \right\rangle |\iota \in U\right\}$, where the functions $\mu_{\ominus}(\iota) : U \to [0,1]$ and $\lambda_{\ominus}(\iota) : U \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $\iota \in U$ to $\ominus$, which is a subset of U, and for every $\iota \in U$, $0 \leq (\mu_{\ominus}(\iota))^2 + (\lambda_{\ominus}(\iota))^2 \leq 1$. Supposing $(\mu_{\ominus}(\iota))^2 + (\lambda_{\ominus}(\iota))^2 \leq 1$, then there is a degree of indeterminacy of $\iota \in U$ to $\ominus$ defined by $\pi_{\ominus}(\iota) = \sqrt{1 - [(\mu_{\ominus}(\iota))^2 + (\lambda_{\ominus}(\iota))^2]}$ and $\pi_{\ominus}(\iota) \in [0,1]$. In what follows, $(\mu_{\ominus}(\iota))^2 + (\lambda_{\ominus}(\iota))^2 + (\pi_{\ominus}(\iota))^2 = 1$. Otherwise, $\pi_{\ominus}(\iota) = 0$ whenever $(\mu_{\ominus}(\iota))^2 + (\lambda_{\ominus}(\iota))^2 = 1$. We denote the set of all PFS's over U by $pfs(X)$.

Definition 2.2. [20] Let $\ominus$ and $\mathfrak{Q}$ be pfs's of the forms $\ominus = \{< a, \mu_{\ominus}(a), \lambda_{\ominus}(a) > |a \in U\}$ and $\mathfrak{Q} = \{< a, \mu_{\mathfrak{Q}}(a), \lambda_{\mathfrak{Q}}(a) > |a \in U\}$. Then

(i) $\ominus \subseteq \mathfrak{Q}$ if and only if $\mu_{\ominus}(a) \leq \mu_{\mathfrak{Q}}(a)$ and $\lambda_{\ominus}(a) \geq \lambda_{\mathfrak{Q}}(a)$ for all $a \in U$.

(ii) $\ominus = \mathfrak{Q}$ if and only if $\ominus \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \subseteq \ominus$.

(iii) $\bar{\ominus} = \{< a, \lambda_{\ominus}(a), \mu_{\ominus}(a) > |a \in U\}$.

(iv) $\ominus \cap \mathfrak{Q} = \{< a, \mu_{\ominus}(a) \wedge \mu_{\mathfrak{Q}}(a), \lambda_{\ominus}(a) \vee \lambda_{\mathfrak{Q}}(a) > |a \in U\}$.

(v) $\ominus \cup \mathfrak{Q} = \{< a, \mu_{\ominus}(a) \vee \mu_{\mathfrak{Q}}(a), \lambda_{\ominus}(a) \wedge \lambda_{\mathfrak{Q}}(a) > |a \in U\}$.

(vi) $0_P = \{< a, 0, 1 > |a \in U\}$ and $1_P = \{< a, 1, 0 > |a \in U\}$.

(vii) $\bar{1_P} = 0_P$ and $\bar{0_P} = 1_P$.

Definition 2.3. [1,2] Let U be a non-empty set and R be an equivalence relation on U. Let $\ominus$ be a Pythagorean fuzzy set in U with the membership function $\mu_{\ominus}(\iota)$ and non membership function $\lambda_{\ominus}(\iota)$, $\forall\ \iota \in U$. The Pythagorean fuzzy nano lower, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary approximation of $\ominus$ in (U, R) denoted by $\underline{\mathcal{PFN}}(\ominus), \overline{\mathcal{PFN}}(\ominus)$ and $B_{\mathcal{PFN}}(\ominus)$ are respectively defined as follows:

(i) $\underline{\mathcal{PFN}}(\ominus) = \left\{\langle \iota, \mu_{\underline{R}(\ominus)}(\iota), \lambda_{\overline{R}(\ominus)}(\iota)\rangle / y \in [\iota]_R, \iota \in U\right\}$

(ii) $\overline{\mathcal{PFN}}(F) = \left\{\langle \iota, \mu_{\overline{R}(\ominus)}(\iota), \lambda_{\underline{R}(\ominus)}(\iota)\rangle / y \in [\iota]_R, \iota \in U\right\}$

(iii) $B_{\mathcal{PFN}}(F) = \overline{\mathcal{PFN}}(F) - \underline{\mathcal{PFN}}(F)$

where $\mu_{\underline{R}(\ominus)}(\iota) = \bigwedge_{y \in [\iota]_R} \mu_{\ominus}(y)$
$\lambda_{\underline{R}(\ominus)}(\iota) = \bigwedge_{y \in [\iota]_R} \lambda_{\ominus}(y)$,
$\mu_{\overline{R}(\ominus)}(\iota) = \bigvee_{y \in [\iota]_R} \mu_{\ominus}(y)$,
$\lambda_{\overline{R}(\ominus)}(\iota) = \bigvee_{y \in [\iota]_R} \lambda_{\ominus}(y)$.

Definition 2.4. [1, 2] Let U be an universe of discourse, R be an equivalence relation on U and $\ominus$ be a Pythagorean fuzzy set in U and if the collection $\tau_{\mathcal{R}}(\ominus) = \left\{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(\ominus), \overline{\mathcal{PFN}}(\ominus), B_{\mathcal{PFN}}(\ominus)\right\}$ forms a topology then it is said to be a Pythagorean fuzzy nano topology. We call $(U, \tau_{\mathcal{R}}(\ominus))$ (or simply U) as the Pythagorean fuzzy nano topological space. The elements of $\tau_{\mathcal{R}}(\ominus)$ are called Pythagorean fuzzy nano open (briefly, $\mathcal{PFN}o$) sets.

Remark 2.5. [1,2] $[\tau_{\mathcal{R}}(\ominus)]^c$ is called the dual Pythagorean fuzzy nano topology of $\tau_{\mathcal{R}}(\ominus)$. Elements of $[\tau_{\mathcal{R}}(\ominus)]^c$ are called Pythagorean fuzzy nano closed (briefly, $\mathcal{PF}\mathfrak{N}c$) sets. Thus, we note that a Pythagorean fuzzy set G of U is Pythagorean fuzzy nano closed in $\tau_{\mathcal{R}}(\ominus)$ if and only if $1_P - G$ is Pythagorean fuzzy nano open in $\tau_{\mathcal{R}}(\ominus)$.

Definition 2.6. [12] Let X be a universe of discourse with a generic element in U denoted by x, the neutrosophic set is an object having the form $\ominus = \{\langle \iota, \mu_\ominus(\iota), \sigma_\ominus(\iota), \lambda_\ominus(\iota)\rangle : \iota \in U\}$, where the functions $\mu : U \to [0,1]$ denote the degree of membership function, $\sigma : U \to [0,1]$ denote the degree of indeterminacy function and $\lambda : U \to [0,1]$ denote the degree of non-membership function respectively of each element $\iota \in U$ to the set S and $0 \leq \mu_\ominus(\iota) + \sigma_\ominus(\iota) + \lambda_\ominus(\iota) \leq 3$ for each $\iota \in U$.

Definition 2.7. [11] Let U be a non-empty set (Universe) A Pythagorean neutrosophic set (briefly, PNS) T and F as dependent neutrosophic components $\ominus$ on U is an object of the form $\ominus = \{\langle \iota, \mu_\ominus(\iota), \sigma_\ominus(\iota), \lambda_\ominus(\iota)\rangle | \iota \in U\}$,
where $\mu_\ominus(\iota)$, $\sigma_\ominus(\iota)$, $\lambda_\ominus(\iota)$ are the truth, indeterminacy and false respectively such that $\mu, \sigma, \lambda \in [0,1]$. Here when μ and λ are dependent components, then for all x in U; (i) $0 \leq \mu^2 + \lambda^2 \leq 1$, (ii) $0 \leq \mu^2 + \sigma^2 + \lambda^2 \leq 2$.

Definition 2.8. [11] Let U be a nonempty set and the Pythagorean neutrosophic sets $\ominus$ and $\mathfrak{Q}$ in the form $\ominus = \{\langle x : \mu_\ominus(\iota), \sigma_\ominus(\iota), \lambda_\ominus(\iota)\rangle,\ \iota \in U\}, \mathfrak{Q} = \{\langle x : \mu_{\mathfrak{Q}}(\iota), \sigma_{\mathfrak{Q}}(\iota), \lambda_{\mathfrak{Q}}(\iota)\rangle, \iota \in U\}$. Then the following statements hold:

(i) $0_N = \{\langle \iota, 0, 0, 1\rangle : \iota \in U\}$ and $1_N = \{\langle \iota, 1, 1, 0\rangle : \iota \in U\}$.
(ii) $\ominus \subseteq \mathfrak{Q}$ iff $\mu_\ominus(\iota) \leq \mu_{\mathfrak{Q}}(\iota), \sigma_\ominus(\iota) \leq \sigma_{\mathfrak{Q}}(\iota), \lambda_\ominus(\iota) \geq \lambda_{\mathfrak{Q}}(\iota)$ for all $\iota \in U\}$.
(iii) $\ominus = \mathfrak{Q}$ iff $\ominus \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \subseteq \ominus$
(iv) $\ominus^c = \{\langle \iota, \lambda_\ominus(\iota), 1 - \sigma_\ominus(\iota), \mu_\ominus(\iota)\rangle : \iota \in U\}$
(v) $\ominus \cap \mathfrak{Q} = \{x, \mu_\ominus(\iota) \wedge \mu_{\mathfrak{Q}}(\iota), \sigma_\ominus(\iota) \wedge \sigma_{\mathfrak{Q}}(\iota), \lambda_\ominus(\iota) \vee \lambda_{\mathfrak{Q}}(\iota)$ for all $\iota \in U\}$.
(vi) $\ominus \cup \mathcal{T} = \{x, \mu_\ominus(\iota) \vee \mu_{\mathfrak{Q}}(\iota), \sigma_\ominus(\iota) \vee \sigma_{\mathfrak{Q}}(\iota), \lambda_\ominus(\iota) \wedge \lambda_{\mathfrak{Q}}(\iota)$ for all $\iota \in U\}$.

3. Pythagorean neutrosophic nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-open sets

Definition 3.1. Let U be a non-empty set and R be an equivalence relation on U. Let A be a Pythagorean neutrosophic set in U with the membership function μ_A, the indeterminacy function σ_A and the non-membership function λ_A. The Pythagorean neutrosophic nano lower, Pythagorean neutrosophic nano upper approximation and Pythagorean neutrosophic nano boundary approximation of A in (U, R) denoted by $\underline{\mathcal{P}y\mathcal{N}eu}\mathfrak{N}(A), \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A)$ and $B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A)$ are respectively defined as follows:

(i) $\underline{\mathcal{P}y\mathcal{N}eu}\mathfrak{N}(A) = \left\{\langle \iota, \mu_{\underline{R}(A)}(\iota), \sigma_{\underline{R}(A)}(\iota), \lambda_{\underline{R}(A)}(\iota)\rangle / y \in [\iota]_R, \iota \in U\right\}$
(ii) $\overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A) = \left\{\langle \iota, \mu_{\overline{R}(A)}(\iota), \sigma_{\overline{R}(A)}(\iota), \lambda_{\overline{R}(A)}(\iota)\rangle / y \in [\iota]_R, \iota \in U\right\}$

(iii) $B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A) = \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A) - \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A)$

where $\mu_{\underline{R}(A)}(\iota) = \bigwedge_{y\in[\iota]_R} \mu_A(y), \sigma_{\underline{R}(A)}(\iota) = \bigwedge_{y\in[\iota]_R} \sigma_A(y), \lambda_{\underline{R}(A)}(\iota) = \bigvee_{y\in[\iota]_R} \lambda_A(y)$.
$\mu_{\overline{R}(A)}(\iota) = \bigvee_{y\in[\iota]_R} \mu_A(y), \sigma_{\overline{R}(A)}(\iota) = \bigvee_{y\in[\iota]_R} \sigma_A(y), \lambda_{\overline{R}(A)}(\iota) = \bigwedge_{y\in[\iota]_R} \lambda_A(y)$.

Definition 3.2. Let U be an universe, R be an equivalence relation on U and $\ominus$ be a neutrosophic set in U and if the collection $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A) = \{0_N, 1_N, \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A), \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A), B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(A)\}$ forms a topology then it is said to be a Pythagorean neutrosophic nano topology. We call $\left(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)\right)$ as the Pythagorean neutrosophic nano topological space (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$). The elements of $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)$ are called Pythagorean neutrosophic nano open (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}o$) sets.

Remark 3.3. $[\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)]^c$ is called the dual Pythagorean neutrosophic nano topology of $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)$. Elements of $[\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)]^c$ are called Pythagorean neutrosophic nano closed (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$) sets. Thus, we note that a Pythagorean neutrosophic set $N(G)$ of U is Pythagorean neutrosophic nano closed in $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)$ if and only if $U - N(G)$ is Pythagorean neutrosophic nano open in $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A)$.

Definition 3.4. Let $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ with respect to A where A is a neutrosophic subset of U. Let S be a neutrosophic subset of U. Then neutrosophic nano

(i) interior of S (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(S)$) is defined by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(S) = \bigcup\{I : I \subseteq S \;\&\; I$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}o$ set in $U\}$,

(ii) closure of S (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(S)$) is defined by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(S) = \bigcap\{A : S \subseteq A \;\&\; A$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ set in $U\}$.

Definition 3.5. Let $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(A))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ with respect to A where A is a neutrosophic subset of U. Then a neutrosophic subset S in U is said to be neutrosophic nano regular open (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ro$) set if $S = \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(S))$ and neutrosophic nano regular closed (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}rc$) set if $S = \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(S))$.

Definition 3.6. Let $(U, \tau_{\mathcal{N}}(\mathcal{A}))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ with respect to A where A is a PNS of U. Let S be a PNS of U. Then

(i) Pythagorean neutrosophic nano δ interior of S (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)$) is defined by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S) = \cup\{I : I \subseteq S \;\&\; I$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ro$ set in $U\}$,

(ii) Pythagorean neutrosophic nano δ closure of S (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)$) is defined by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S) = \cap\{A : S \subseteq A \;\&\; A$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}rc$ set in $U\}$.

Definition 3.7. Let $(U, \tau_{\mathcal{N}}(\mathcal{A}))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ with respect to A where A is a PNS of U. Then a PNS S in U is said to be Pythagorean:

(i) neutrosophic nano δ-open set (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$) if $S = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)$,

(ii) neutrosophic nano $\delta\mathcal{P}$-open set (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$) if $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))$,

(iii) neutrosophic nano $\delta\mathcal{S}$-open set (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$) if $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))$,

(iv) neutrosophic nano $\delta\alpha$ or a-open set (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$ or $\mathcal{P}y\mathcal{N}eu\mathfrak{N}aos$) if $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)))$,

(v) neutrosophic nano $\delta\beta$ or e^* -open set (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ or $\mathcal{P}y\mathcal{N}eu\mathfrak{N}e^*os$) if $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)))$.

The complement of a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$ & $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$) is called a Pythagorean neutrosophic nano δ (resp. $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) closed set (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cs$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cs$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$)) in U.

The family of all $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$) of U is denoted by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta OS(U)$, (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta CS(U)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}OS(U)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}CS(U)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}OS(U)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}CS(U), \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha OS(U), \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha CS(U)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta OS(U)$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta CS(U)$).

Definition 3.8. Let $(U, \tau_{\mathcal{N}}(\mathcal{A}))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ with respect to A where A is a PNS of U. Let S be a PNS of U. Then Pythagorean neutrosophic nano

(i) δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) interior of S (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}int(S)$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}int(S)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha int(S)$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$)) is defined by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}int(S)$
(resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}int(S), \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha int(S)$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)) = \cup\{I : I \subseteq S$ & I is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o, \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o$ & $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$) set in $U\}$,

(ii) δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) closure of S (briefly, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(S)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cl(S)$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$)) is defined by $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(S), \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cl(S)$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)) = \cap\{A : S \subseteq A$ & A is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}c$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}c, \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha c$ & $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$) set in $U\}$.

Theorem 3.9. Let $(U, \tau_{\mathcal{N}}(\mathcal{A}))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$. Then,

(i) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$) set,

(ii) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}c$) set,

(iii) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}c$) set,

(iv) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$) set,

(v) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$) set,

(vi) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}c$) set,

(vii) Every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha c$) set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}c$) set.

Proof.

(i) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ in U, then $S = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(S)$. Therefore, S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}os$.

(ii) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ in U, then S is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}os$ by (i). So, $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))$. Therefore, S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$.

(iii) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ in U, then S is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}os$ by (i). So, $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))$. Therefore, S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$.

(iv) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$ in U, then

$$S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))).$$

Therefore, S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

(v) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$ in U, then

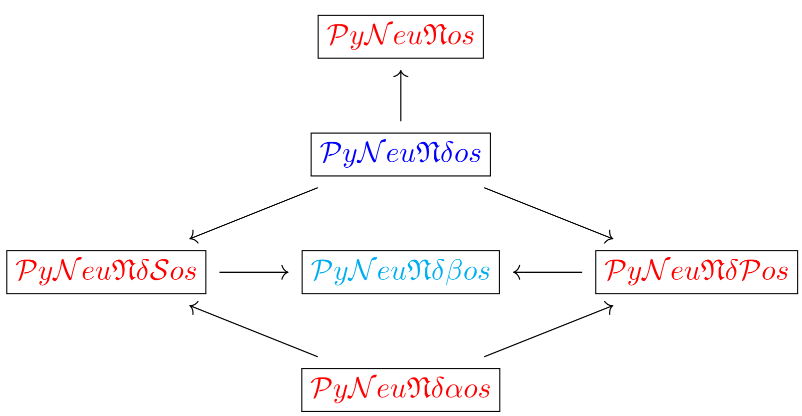
$$S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))).$$

Therefore, S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

(vi) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$ in U, then $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)))$. So $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))$. Therefore, S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$.

(vii) If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$ in U, then $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)))$. So, $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))$. Therefore S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$. It is also true for their respective $\mathcal{P}y\mathcal{N}eu\mathfrak{N}$ closed sets.

Remark 3.10. The diagram shows $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$'s in $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$.



The converse of the Theorem 3.9 need not to be true. The following examples show it.

Example 3.11. Assume $U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Let $\ominus = \left\{\left\langle\frac{s_1}{0.6,0.8,0.4}\right\rangle, \left\langle\frac{s_2}{0.4,0.6,0.8}\right\rangle, \left\langle\frac{s_3}{0.5,0.7,0.75}\right\rangle, \left\langle\frac{s_4}{0.7,0.9,0.55}\right\rangle\right\}$ be a Pythagorean neutrosophic set of U.

$$\underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\ominus) = \left\{\left\langle\frac{s_1, s_4}{0.6, 0.8, 0.55}\right\rangle, \left\langle\frac{s_2}{0.4, 0.6, 0.8}\right\rangle, \left\langle\frac{s_3}{0.5, 0.7, 0.75}\right\rangle\right\},$$
$$\overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\ominus) = \left\{\left\langle\frac{s_1, s_4}{0.7, 0.9, 0.4}\right\rangle, \left\langle\frac{s_2}{0.4, 0.6, 0.8}\right\rangle, \left\langle\frac{s_3}{0.5, 0.7, 0.75}\right\rangle\right\},$$
$$B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\ominus) = \left\{\left\langle\frac{s_1, s_4}{0.55, 0.2, 0.6}\right\rangle, \left\langle\frac{s_2}{0.4, 0.4, 0.8}\right\rangle, \left\langle\frac{s_3}{0.5, 0.3, 0.75}\right\rangle\right\}.$$

Thus $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\ominus) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\ominus), \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\ominus), B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\ominus)\}$. Then

(i) $\left\{\left\langle\frac{s_1,s_4}{0.7,0.9,0.4}\right\rangle, \left\langle\frac{s_2}{0.4,0.6,0.8}\right\rangle, \left\langle\frac{s_3}{0.5,0.7,0.75}\right\rangle\right\}$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\beta o$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$) set but not $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o$) set.

(ii) $\left\{\left\langle\frac{s_1,s_4}{0.55,0.2,0.6}\right\rangle, \left\langle\frac{s_2}{0.8,0.4,0.4}\right\rangle, \left\langle\frac{s_3}{0.75,0.3,0.5}\right\rangle\right\}$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o$ set but not $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$) set.

(iii) $\left\{\left\langle\frac{s_1,s_4}{0.55,0.2,0.65}\right\rangle, \left\langle\frac{s_2}{0.55,0.4,0.75}\right\rangle, \left\langle\frac{s_3}{0.6,0.4,0.5}\right\rangle\right\}$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ set but not $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$ set.

Theorem 3.12. Let $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$ be an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ and $\{S_\alpha|\alpha \in I\}$ be a family of $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$) sets in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$. Then $\bigcup\{S_\alpha|\alpha \in I\}$ (resp. $\bigcap\{S_\alpha|\alpha \in I\}$) is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ (resp. $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$).

Proof. Let $\{S_\alpha|\alpha \in I\}$ ba a family of $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ sets in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$. Then for each α, $S_\alpha \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S_\alpha)))$. Since $S_\alpha \subseteq \bigcup S_\alpha$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S_\alpha))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(\bigcup S_\alpha)))$ and also $S_\alpha \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(\bigcup S_\alpha)))$. Hence $\bigcup S_\alpha \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(\bigcup S_\alpha)))$ which shows that $\bigcup S_\alpha$ is an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$. The other case is similar.

Remark 3.13. The Theorem 3.12 is also true for $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o$.

Remark 3.14. The intersection of two $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$'s need not be $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

Example 3.15. In Example 3.11, let $A = \left\{\left\langle\frac{s_1,s_4}{0.4,0.1,0.6}\right\rangle, \left\langle\frac{s_2}{0.5,0.4,0.6}\right\rangle, \left\langle\frac{s_3}{0.6,0.3,0.4}\right\rangle\right\}$ and $B = \left\{\left\langle\frac{s_1,s_4}{0.5,0.3,0.7}\right\rangle, \left\langle\frac{s_2}{0.5,0.6,0.6}\right\rangle, \left\langle\frac{s_3}{0.7,0.5,0.5}\right\rangle\right\}$ are $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ sets but $A \cap B = \left\{\left\langle\frac{s_1,s_4}{0.4,0.1,0.7}\right\rangle, \left\langle\frac{s_2}{0.5,0.4,0.7}\right\rangle, \left\langle\frac{s_3}{0.6,0.3,0.5}\right\rangle\right\}$ is not $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$ set.

Theorem 3.16. Let K be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$ then $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$ does not contain any non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$.

Proof. Let K be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$ and S be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ subset of $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$. That is, $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$ implies $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \wedge (1_{\mathcal{P}} - K)$. That is $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$ and $S \subseteq (1_{\mathcal{P}} - K)$ which implies $K \subseteq (1_{\mathcal{P}} - S)$ where $1_{\mathcal{P}} - S$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}o$ set. Since K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \subseteq (1_{\mathcal{P}} - S)$. That is $S \subseteq (1_{\mathcal{P}} - \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K))$. Thus $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \wedge (1_{\mathcal{P}} - \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)) = 0_P$. Hence $S = 0_P$. Therefore $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$ does not contain any non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$.

Remark 3.17. The converse of the Theorem 3.16 need not be true as seen from the following example.

Example 3.18. In Example 3.11, let $K = \left\{\left\langle \frac{s_1, s_4}{0.7,0.9,0.4} \right\rangle, \left\langle \frac{s_2}{0.4,0.6,0.8} \right\rangle, \left\langle \frac{s_3}{0.5,0.7,0.75} \right\rangle\right\}$ be any subset of U then $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K = 1_P - K$ which is a non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cs$ in U, but K is not a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ in U.

Theorem 3.19. Let K be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}$ subset in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$ then K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ if and only if $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$.

Proof. Let K be a $\mathcal{P}y\mathcal{N}eu$ subset in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$. Assume that K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ then we have $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) = K$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K = 0_P$ which is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cs$.

Conversely, assume that $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$ be $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cs$ and K is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set in U. Now $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ subset of itself. Therefore by Theorem 3.16, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) - K = 0_P$. That is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) = K$, implies K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$.

Theorem 3.20. If K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$ and $K \subseteq L \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$, then L is also $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ in $(U, \tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{A}))$.

Proof. Let K be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set in U and $K \subseteq L \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$. Let $L \subseteq O$ where O be $\mathcal{P}y\mathcal{N}eu\mathfrak{N}c$ set in U. Since $K \subseteq L$, implies $K \subseteq O$ and K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$, implies $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \subseteq O$. By hypothesis $L \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$, implies $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(L) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \subseteq O$, which implies $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(L) \subseteq O$. Therefore L is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ in U.

Theorem 3.21. If a subset K of U is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set, then $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\{x_r\}) \cap K \neq 0_P$ for each $x_r \in \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$.

Proof. Suppose K is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set and $x_r \in \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$. If possible $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\{x_r\}) \cap K = 0_P$. Then $K \subseteq 1_{\mathcal{P}} - \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\{x_r\})$ and $1_{\mathcal{P}} - \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\{x_r\})$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}o$ set containing K. Since K is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set, implies $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \subseteq 1_{\mathcal{P}} - \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\{x_r\})$ which is a contradiction to $x_r \in \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K)$. Therefore, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\{x_r\}) \cap K \neq 0_P$.

Theorem 3.22. *If $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta O(U,A) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta C(U,A)$, then $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta C(U,A) = P(U)$ is the power set of U.*

Proof. Suppose $K \subseteq O$, where O is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ in U. Since every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$ set is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$, O is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$. By hypothesis, O is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$. Hence, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(K) \subseteq O$. Therefore, K is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ set. Since, K is arbitrary, by Theorem 3.21, every subset of U is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$. Thus $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta C(U,A) = P(U)$.

Remark 3.23. The Theorems 3.12, 3.16, 3.19, 3.20, 3.21 and 3.22 are also true for $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os(U)$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os(U)$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o(U)$.

Proposition 3.24. If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ and T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$, then $S \cap T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

Proof. $S \cap T \subseteq S \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(S \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S \cap T)))$. Therefore $S \cap T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

Remark 3.25. The Proposition 3.24 is also true if T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$.

Proposition 3.26. *If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$ and T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$. Then $S \cap T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$.*

Proof. $S \cap T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)) \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))) \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))) \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S \cap T))) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\ int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S \cap T))$. Therefore, $S \cap T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$.

Corollary 3.27. If S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cs$ and T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cs$. Then $S \cup T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cs$.

Proposition 3.28. If S is a $\mathcal{P}y\mathcal{N}eus$ of U and T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$ of U such that $T \subseteq S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T))$. Then S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

Proof. Since T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$, $T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T))$. Now $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\ \delta int(T)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\ int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)))) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)))$. Hence $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)))$. Therefore S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

Proposition 3.29. If each T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ which is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cs$ is also a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$.

Proof. Let T be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cs$. Then, $T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)))$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)) \subseteq T$. Therefore, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)) \subseteq T$ and so, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T))$.

Hence, $T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T)))$. Therefore, T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$.

Proposition 3.30. If T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$. Then T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cs$.

Proof. Since T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$. Then, $1_P - T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cs$ and so by Proposition 3.29, $1 - P - T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$. Therefore T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cs$.

Proposition 3.31. If each T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ which is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cs$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cs$.

Proof. Let T be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cs$. Then, $T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)))$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T))) \subseteq T$. Therefore, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)) \subseteq T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)))$. So, $T = \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(T)))$. Therefore, T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cs$.

Corollary 3.32. If each S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$, which is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$, is also a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$.

Theorem 3.33. Let $(U, \tau_R(\mathcal{P}))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$, S be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$ and T be a PNS such that $T \subseteq S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(T)$. Then T is also an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$.

Proof. $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(T)$, implies that $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(T)$ and so $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl\ (\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T))$ and so $T \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(T))$. Therefore T is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$.

Remark 3.34. Theorem 3.33 is also true for $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$.

Theorem 3.35. Let $(U, \tau_R(\mathcal{P}))$ be an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ and let S and T be PNS's. Then the following hold.

(i) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(0_P) = 0_P$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P) = 1_P$.
(ii) S is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ iff $S = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$.
(iii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$ is the greatest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing S.
(iv) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$.
(v) $S \subseteq T$ implies that $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T)$.
(vi) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S \cap T) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T)$.

(vii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S \cup T) \supseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T)$.

(viii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \subseteq S$.

(ix) $(1_P - \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(1_P - S)$.

(x) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$ is the greatest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ contained in S.

Proof.

(i) The proofs are directly from Definition 3.7.

(ii) Suppose S is any $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ of U. Then, the greatest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing in S is itself. Therefore $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) = S$.

(iii) If G is any $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ contained in S, then $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$. Hence, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$ is the greatest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing S.

(iv) By (iii) the greatest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$ is itself. Hence, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$.

(v) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T) = \cup\{G|G$ is an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ros$ and $G \subseteq T\} \supseteq \cup\{G|G$ is an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ros$ and $G \subseteq S\} = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$. Then $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T)$.

(vi) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S \cap T) = \cup\{C|C$ is an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ros$ and $C \subseteq S \cap T\} = (\cup\{C|C$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ros$ and $C \subseteq S\}) \cap (\cup\{C|C$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ros$ and $C \subseteq T\}) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T)$.

(vii) $S \subseteq S \cup T$ or $T \subseteq S \cup T$. Hence $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S \cup T)$ or $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S \cup T)$. Therefore, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S \cup T) \supseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(T)$.

(viii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) = \cup\{G|G$ is an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ros$ and $G \subseteq S\}$. Thus, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \subseteq S$.

(ix) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S)$ is the greatest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing S. The complement is the smallest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ contained in $1_P - S$. Therefore, $1_P - \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(1_P - S)$.

Theorem 3.36. Let $(U, \tau_R(\mathcal{P}))$ be an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ and let S and T be PNS's. Then the following hold.

(i) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(0_P) = 0_P$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(1_P) = 1_P$.

(ii) S is an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ iff $S = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$.

(iii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$ is the smallest $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ contained in S.

(iv) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$.

(v) $S \subseteq T$ implies that $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(T)$.

(vi) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S \cap T) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(T)$.

(vii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S \cup T) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(T)$.

(viii) $S \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$.

(ix) $(1_P - \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)) = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P - S)$.

(x) $\iota \in \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$ iff $S \cap T \neq 0_P$ for every $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ T containing x.

Proof. (x) Suppose $\iota \in \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$. Let T be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing x. If $S \cap T = 0_P$, then $1_P - T$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ containing S and so $x \notin \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$, a contradiction. Therefore, $S \cap T \neq 0_P$. If $x \notin \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$, then there exists a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ C containing S such that $x \notin C$. Then $D = 1_P - C$ is a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ containing x such that $S \cap D = 0_P$, a contradiction. Therefore, $\iota \in \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$. The other cases are follows from Theorem 3.35.

Theorem 3.37. Let $(U, \tau_R(\mathcal{P}))$ be an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ and S be a PNS. Then the following hold.

(i) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}int(S) \subseteq S \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))$.

(ii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S) \supseteq S \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))$.

(iii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}int(S) \subseteq S \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S))$.

(iv) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}cl(S) \supseteq S \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S))$.

(v) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha int(S) \subseteq S \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)))$.

(vi) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha cl(S) \supseteq S \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)))$.

(vii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(S) \subseteq S \cap \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S)))$.

(viii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) \supseteq S \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)))$.

Proof. (ii) Since $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S)$ is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cs$, we have $\mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S))) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S))$. Thus $S \cup \mathcal{P}y\mathcal{N}eu\mathfrak{N}cl(\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(S)) \subseteq \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}cl(S)$. The other cases are similar.

Theorem 3.38. Let $(U, \tau_R(\mathcal{P}))$ be a $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$ and let S be a PNS. Then the following are equivalent.

(i) S is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta$ dense.

(ii) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) = 1_P$.

(iii) If T is any $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ such that $S \subset T$, then $T = 1_P$.

(iv) Every non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ has a non-empty intersection with S.

(v) $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P - S) = 0_P$.

Proof. (i) $\Rightarrow$ (ii): Suppose $x \notin \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S)$. Then there exists an $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ G containing x such that $G \cap S \neq 0_P$. Since G is a non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$, there is a non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}os$ H such that $H \subseteq G$ and so $H \cap S = 0_P$, a contradiction. Therefore, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta cl(S) = U$.

(ii) $\Rightarrow$ (iii): If T is any $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$ such that $S \subset T$, then $1_P = \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) \subset \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(T) = T$ which implies that $T = U$.

(iii) ⇒ (iv): If G is a non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta os$ such that $G \cap S = 0_P$, then $S \subset 1_P - G$ and $1_P - G$ is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cs$. By (iii), it follows that $G = 0_P$, a contradiction.

(iv) ⇒ (v): Suppose that $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P - S) \neq 0_P$. Then $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P - S)$ is a non-empty $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$ such that $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta int(1_P - S) \cap S \neq 0_P$, a contradiction to the hypothesis.

(v) ⇒ (i): $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P - S) \neq 0_P$ implies that $1_P - \mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta int(1_P - S) = 1_P$. Thus $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) = 1_P$. Hence $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta cl(S) = 1_P$ which shows that S is $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta$-dense.

Remark 3.39. The Theorems 3.35, 3.36 and 3.38 are also true for $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}os$ and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha os$ of their respective interior and closure operators.

4. **Application**

The process of teaching a machine learning algorithm to identify patterns, anticipate outcomes, or carry out particular tasks by providing it with data is known as AI model training. To guarantee the model operates efficiently and dependably, this procedure includes a number of crucial actions, resources, and factors.

AI Model Training

Step:1 Clearly state the problem your model must address, such as sentiment analysis or image classification.

Step:2 Get relevant, high-quality data. Make sure to properly classify the data for supervised learning. To enhance model performance, preprocess the data by cleaning, standardizing, and enriching it.

Step:3 To guarantee an objective assessment, separate the data set into training, validation, and test sets.

Step:4 Model is changing its architecture by enhancing design procedures, automating tasks, and boosting sustainability and efficiency.

Step:5 Evaluation and optimization happen together, improving the model by adjusting based on performance checks during training.

Step:6 Fine-tuning adjusts a model for better performance by tweaking hyper parameters and often using pre-trained models for specific tasks.

Step:7 The learned model is implemented in actual operational environments and its performance is continually monitored to address issues such as model drift, thereby ensuring sustained efficacy and accuracy.

Suppose we have a AI model which is to be trained for real life problems. Actually the data set was classified into three groups namely D_1,D_2 and D_3. Hundred datas in each groups are used to train the model every day. The number of data used for training, fine tuned the model and no modification in the model by the grouped data is denoted by the membership value, indeterminacy value and the non membership value of the Pythagorean neutrosophic set respectively.Now their upper, the lower and boundary approximations are given below.

Let $A = \{D_1, D_2, D_3\}$ be the universe of discourse. Let $A/R = \{\{D_1, D_2\}, \{D_3\}\}$ be an equivalence relation R on A and Pythagorean neutrosophic set on A is denoted as follows $\ominus = \left\{\left\langle\frac{D_1}{0.6,0.5,0.5}\right\rangle, \left\langle\frac{D_2}{0.7,0.5,0.3}\right\rangle, \left\langle\frac{D_3}{0.6,0.3,0.5}\right\rangle\right\}$, be a Pythagorean neutrosophic set on A then

$$\underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1, D_2}{0.6, 0.5, 0.5}\right\rangle, \left\langle\frac{D_3}{0.6, 0.6, 0.5}\right\rangle\right\},$$
$$\overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1, D_2}{0.6, 0.5, 0.5}\right\rangle, \left\langle\frac{D_3}{0.6, 0.6, 0.5}\right\rangle\right\},$$
$$B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1, D_2}{0.5, 0.5, 0.6}\right\rangle, \left\langle\frac{D_3}{0.5, 0.4, 0.6}\right\rangle\right\}.$$

$\underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) \cup \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1,D_2}{0.7,0.5,0.3}\right\rangle, \left\langle\frac{D_3}{0.6,0.6,0.5}\right\rangle\right\} = \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$\underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) \cup B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1,D_2}{0.6,0.5,0.5}\right\rangle, \left\langle\frac{D_3}{0.6,0.6,0.5}\right\rangle\right\} = \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$\overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) \cup B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1,D_2}{0.7,0.5,0.3}\right\rangle, \left\langle\frac{D_3}{0.6,0.6,0.5}\right\rangle\right\} = \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$\underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) \cap \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1,D_2}{0.6,0.5,0.5}\right\rangle, \left\langle\frac{D_3}{0.6,0.6,0.5}\right\rangle\right\} = \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$\underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) \cap B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1,D_2}{0.5,0.5,0.6}\right\rangle, \left\langle\frac{D_3}{0.5,0.4,0.6}\right\rangle\right\} = B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$\overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) \cap B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \left\{\left\langle\frac{D_1,D_2}{0.5,0.5,0.6}\right\rangle, \left\langle\frac{D_3}{0.5,0.4,0.6}\right\rangle\right\} = B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$0_{\mathcal{P}} \cup \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$, $0_{\mathcal{P}} \cup \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$, $0_{\mathcal{P}} \cup B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$1_{\mathcal{P}} \cup \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = 1_{\mathcal{P}}$, $1_{\mathcal{P}} \cup \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = 1_{\mathcal{P}}$, $1_{\mathcal{P}} \cup B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = 1_{\mathcal{P}}$,

$1_{\mathcal{P}} \cap \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$, $1_{\mathcal{P}} \cap \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$, $1_{\mathcal{P}} \cap B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})$,

$0_{\mathcal{P}} \cap \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = 0_{\mathcal{P}}$, $0_{\mathcal{P}} \cap \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = 0_{\mathcal{P}}$, $0_{\mathcal{P}} \cap B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}) = 0_{\mathcal{P}}$.

Therefore, $\tau_{\mathcal{R}}^{\mathcal{P}y\mathcal{N}eu}(\mathcal{P}) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}), \overline{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P}), B_{\mathcal{P}y\mathcal{N}eu\mathfrak{N}}(\mathcal{P})\}$ forms a topology.

5. **Conclusions**

In this paper, we introduce and investigate several new types of strongly and weakly open (respectively closed) sets, namely $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta o$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta c$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}o$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{S}c$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}o$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\mathcal{P}c$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha o$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\alpha c$, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta o$, and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta\beta c$ sets, along with their corresponding interior and closure operators. Furthermore, we explore several fundamental properties of these sets within $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$. The study is further extended to define $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta$-continuous, $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta$-open, and $\mathcal{P}y\mathcal{N}eu\mathfrak{N}\delta$-closed mappings, including their stronger and weaker variants in $\mathcal{P}y\mathcal{N}eu\mathfrak{N}ts$. Finally, we provide an application illustrating how a Pythagorean neutrosophic nano topological space can be utilized in a machine learning framework.

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References

1. D. Ajay and J. Joseline Charisma, *Pythagorean nano topological space*, International Journal of Recent Technology and Engineering **2020**, 8, 3415-3419.
2. D. Ajay and J. Joseline Charisma, *On weak forms of Pythagorean nano open sets*, Advances in Mathematics: Scientific Journal **2020**, 9, 5953-5963.
3. K. T. Atanassov, *Intuitionistic fuzzy sets*, VII ITKR's Session, Sofia (deposed in Central Sci.-Technical Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian) **1983**.
4. C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **1968**, 24, 182-190.
5. D. Coker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy sets and systems **1997**, 88, 81-89.
6. M. Lellis Thivagar and C. Richard, *On nano forms of weekly open sets*, International journal of mathematics and statistics invention **2013**, 1, 31-37.
7. M. Lellis Thivagar, S. Jafari, V. Sutha Devi and V. Antonysamy, *A novel approach to nano topology via neutrosophic sets*, Neutrosophic Sets and Systems **2018**, 20, 86-94.
8. Murat Olgun, Mehmet Unver and Seyhmus Yardimci , *Pythagorean fuzzy topological spaces*, Complex & Intelligent Systems **2019**, 177-183, https://doi.org/10.1007/s40747-019-0095-2.
9. A. Padma, M. Saraswathi, A. Vadivel and G. Saravanakumar, *New Notions of Nano M-open Sets*, Malaya Journal of Matematik **2019**, S1, 656–660.
10. V. Pankajam and K. Kavitha, *δ-open sets and δ-nano continuity in δ-nano topological spaces*, International Journal of Innovative Science and Research Technology **2017**, 2, 110-118.
11. R. Radha, A. S. A. Mary, R. Prema and S. Broumi, *Neutrosophic Pythagorean sets with dependent neutrosophic Pythagorean components and its improved correlation coefficients*, Neutrosophic Sets Syst., **2021**, 46, 77-86.
12. A. A. Salama and S. A. Alblowi, *Generalized neutrosophic set and generalized neutrosophic topological spaces*, Journal of Computer Sci. Engineering **2012**, 2 , 129-132.
13. Shiventhiradevi Sathaananthan, S. Tamilselvan, A. Vadivel and G. Saravanakumar, *Fuzzy $\mathcal{Z}$ closed sets in double fuzzy topological spaces,* AIP Conf Proceeding **2020**, 2277, 090001.
14. Shiventhiradevi Sathaananthan, A. Vadivel, S. Tamilselvan and G. Saravanakumar, *Generalized fuzzy $\mathcal{Z}$ closed sets in double fuzzy topological spaces,* Adv. Math: Sci. J., **2020**, 9, 2107-2112.
15. A. Vadivel, C. John Sundar, K. Kirubadevi and S. Tamilselvan, *More on Neutrosophic Nano Open Sets*, International Journal of Neutrosophic Science **2022**, 18, 204-222.
16. A. Vadivel, G. Gavaskar and G. Saravanakumar , *New operators using Pythagorean fuzzy open sets*, Nanotechnology Perceptions **2024**, 20, 1616-1635.
17. A. Vadivel, G. Gavaskar and C. John Sundar, *More on Continuous and Irresolute Maps in Pythagorean Fuzzy Topological Spaces*, TWMS J. App. & Eng. Math. **2025**, 15, 2697-2708.
18. R. R. Yager , *Pythagorean membership grades in multicriteria decision making*, In: Technical report *MII*-3301. Machine Intelligence Institute, Iona College, New Rochelle **2013**.
19. R. R. Yager, *Pythagorean fuzzy subsets*, In: Proceedings of the joint *IFSA* world congress *NAFIPS* annual meeting **2013**, 57-61.

20. R. R. Yager (2014), *Pythagorean membership grades in multicriteria decision making*, *IEEE* Trans Fuzzy Syst **2013**, 22, 958-965.
21. L. A. Zadeh, *Fuzzy sets*, Information and control **1965**, 8, 338-353.

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