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Dealing with missingness in uncertain data via neutrosophic exponential imputation: Climate data insights

Anoop Kumar[1], Vishal Kumar[*,1] and Vrijesh Tripathi[2]

[1]Department of Statistics, Central University of Haryana, Mahendergarh, Haryana, India, 123031.

[2]Depatment of Mathematics and Statistics, The University of The West Indies St. Augustine, Trinidad and Tobago.

[*]Correspondence: vishalk.stats@gmail.com

Abstract. Missing data is a common problem across numerous real-world datasets, particularly in domains such as climate research where data is frequently inadequate because of environmental conditions, defective sensors or human mistakes. Due to the inherent uncertainty in these datasets, traditional imputation approaches produce inefficient estimates. Neutrosophy offers a strong framework for managing uncertainty because of its ability to express truth, falsehood, and indeterminacy concurrently. This work presents some neutrosophic exponential imputations and the corresponding neutrosophic exponential estimators of the population mean for handling missing values in uncertain data, with a focus on climate data. The proposed neutrosophic exponential imputations and the corresponding estimators are exemplified through a comprehensive simulation study and a real climate dataset. The findings indicate that the proposed neutrosophic exponential imputations and the corresponding estimators perform better at maintaining data integrity and increasing estimation accuracy compared to the adapted ones.

Keywords: Neutrosophic sets; Uncertainty; Climate data; Exponential imputation approaches; Missing data.

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1. **Introduction**

In survey sampling, one typical problem is missing data, which occurs when respondents don't answer some or all of the survey items. A number of factors, including participant non-response, incorrect data input, and equipment failure, may cause this problem. Missing data can cause biased estimates and decrease statistical accuracy, which makes it challenging to extrapolate reliable inferences from the findings of survey. Effectively handling missing data is essential to guaranteeing the validity and precision of the conclusions. The impact of missing data can be lessened by the well-known imputation methods. Managing missing data appropriately is crucial to preserving the integrity of survey-based research.

In survey sampling, resolving the missing data problem is essential to ensure the reliability and validity of statistical findings. Imputation is one of the popular methods that estimates missing values from observable data. Three key ideas, including parameter distribution (PD), observed at random (OAR), and missing at random (MAR), were invented by [1] in his notable work. [2] distinguished between missing completely at random (MCAR) and MAR. Using reliable auxiliary data is also essential to improve the effectiveness of the suggested imputation approaches. Several imputation approaches that make use of auxiliary information have been presented, all based on the MCAR technique. [3] estimated the population mean utilizing modified robust extreme ranked set sampling. [4] estimated the population mean in the presence of missing data under SRS with known correlation coefficient using some imputation approaches. In order to estimate the population mean, [5] created the optimal approach for imputation of missing data. In survey sampling, [6] designed several novel techniques for ratio exponential type imputation. The imputation of missing values using raw moments was investigated by [7]. [8] examined the optimality of imputation frameworks for the estimation of population mean using higher order moment of an auxiliary variable, while [9] developed an optimal imputation framework of the missing data utilizing multi-auxiliary information. Under SRS, [10] estimated the population mean when there were missing data. [11] carried out a comparative analysis of several imputation schemes. The unique mean imputation methodology was created by [12] and demonstrated it through abalone data. Using a simulation method, [13] created a generalised class of factor type exponential imputation frameworks for the population mean. In the case of correlated measurement errors, [14] provided various optimal imputation frameworks with actual data examples. An optimal random non-response framework for evaluating mean on present occasion was proposed by [15].

Neutrosophic statistics provides an extensive framework than the classical and interval statistics by explicitly incorporating uncertainty and indeterminacy into data analysis. In classical statistics, observations are assumed to be precise and completely known, for instance, computing the average income of a population consisting of the exact survey responses. Interval statistics extends this by representing uncertain or imprecise data within ranges, such as recording the monthly income of a household as lying between 25,000–30,000 instead of a single fixed amount. However, both approaches fail to fully capture situations where data not only varies within a range but also contains elements of indeterminacy, inconsistency, or incomplete truth. Introduced by [16], neutrosophic statistics addresses this by representing data as a triplet (T, I, F), where T represents truth (certainty), I indeterminacy, and F falsity. For example, in medical diagnosis, test result of a patient may be 70% true (indicating presence of a disease), 20% indeterminate (borderline or unclear), and 10% false (indicating absence), giving richer insights than a crisp positive/negative or even an interval of probabilities. Similarly, in stock market prediction, where data are often noisy and conflicting, neutrosophic statistics gives a framework to model the uncertain and indeterminate parts of the information, providing a richer and more flexible analysis compared to interval or classical approaches. Thus, while classical statistics works best with precise data, and interval statistics with bounded imprecision, neutrosophic statistics is more suitable for real-world complex systems where contradictions, indeterminacy, and uncertainty coexist.

In the last few years, several researchers have extended the application of neutrosophic statistics and related indeterminacy-based methods to different domains, particularly in climatology, energy, and environmental data analysis. These studies not only show the flexibility of neutrosophic approaches in handling uncertain and indeterminate information but also highlight their practical relevance in real-world scientific problems where traditional approaches often fail to capture uncertainty adequately. [17] conducted a case study on mathematical modeling and fuzzy availability analysis of stainless steel utensil manufacturing unit under steady state. [18] explored short-term prediction of solar energy in Saudi Arabia employing automated-design fuzzy logic systems. [19] analyzed the fuzzy availability of washing system in paper industry by Markov modeling. [20] introduced a generalized estimator for population mean estimation under the neutrosophic ranked set sampling, applying it to solar energy data. [21] proposed a classification of trapezoidal bipolar neutrosophic numbers, a de-bipolarization technique, and its application to cloud service-based multi-criteria group decision-making (MCGDM) problems. In the domain of climatology and environmental studies, [22] examined radar circular data utilizing a novel Watson's goodness-of-fit test under complexity, while [23] introduced Cochran's test to assess normality of indeterminate temperature data in Jeddah city. Extending this work, [24], [25] and [26] proposed a neutrosophic statistical test for counts in climatology, developed a Chi-square test under indeterminacy for pulse count data, and introduced an autocorrelation test for metrology data in uncertain settings. Additionally, [27] presented a CUSUM control chart under uncertainty with applications in petroleum and meteorology, whereas [28] and [29] analyzed radar data in the presence of uncertainty and designed a sampling plan for testing average wind speed under a Weibull distribution within the neutrosophic framework.

Previous research on survey sampling has only looked at clear, exact, and definite data. Adaption of these traditional methods yields a single, unambiguous result that sometimes overestimates or underestimates the true population parameter. This limitation is crucial when working with neutrosophic data. When traditional methods are inadequate in these circumstances, neutrosophic statistics become significant. Data collection is costly for many study variables, particularly when the data is unclear. Given unclear data, estimating the unknown population parameter using standard approaches is expensive and risky. The survey literature contains very few studies where both the study and the auxiliary variables have a neutrosophic origin. [30] proposed the neutrosophic ratio type population mean estimation under simple random sampling (SRS). [31] developed a generalized neutrosophic sampling strategy for improved estimation of population mean. [32] created the neutrosophic factor-type exponential estimators using additional data. The neutrosophic robust ratio-type estimator for mean estimation was introduced by [33]. [34] suggested utilising robust estimations of the auxiliary variable to estimate the neutrosophic finite median. [35] constructed an almost unbiased estimator for population median using neutrosophic information. [36] developed the robust neutrosophic exponential estimators of population mean under uncertainty. [37] introduced an optimal neutrosophic framework for population mean estimation under SRS. [38] estimated population mean using generalized neutrosophic exponential estimators. [39] suggested a neutrosophic mean estimation method for both sensitive and non-sensitive variables utilizing robust Hartley-Ross-type estimators. [40] developed the estimation of the population mean with the help of two neutrosophic auxiliary variables under imprecise information. [41] further contributed by proposing neutrosophic advancements in Horvitz-Thompson-type estimators. Similarly, [42] suggested a generalized class of estimators for the population mean estimation under the neutrosophic framework. [43] proposed a comprehensive mean estimation approach based on a regression-cum-exponential type estimator by showing its applicability on neutrosophic data. These studies are based on mean estimation under neutrosophic framework when the data are precise and certain. Moreover, several classical imputation methods based on MCAR setup have been put out in the literature to deal with the missingness in certain data, but they are typically unable to handle the uncertainty and indeterminacy that accompany missing data. Hence, in this study, we set the following objectives:

(i). To adapt some prominent neutrosophic imputation approaches for handling missingness in uncertain data using neutrosophic logic.

(ii). To propose some neutrosophic exponential imputation approaches to handle missing data under uncertainty.

(iii). To theoretically compare the performance of the proposed neutrosophic imputation approaches with the adapted neutrosophic imputation approaches.

(iv). To assess the performance of the proposed approaches by using simulation study based on hypothetically generated neutrosophic symmetric and asymmetric populations.

(v). To validate the adapted and proposed approaches with real-world applications using climate data.

The novelty of this paper lies in its integration of neutrosophic theory with the imputation techniques to address missing values under uncertainty. Unlike traditional imputation approaches that assume complete precision, this work develops the neutrosophic exponential imputation methods which incorporate truth, indeterminacy, and falsity components to more accurately represent incomplete and uncertain data. The paper further contributes by deriving new neutrosophic estimators for the population mean under SRS and demonstrating their efficiency through simulation and real climate data analysis. The use of climate datasets highlights its practical significance, as missingness in such data is common due to sensor errors, extreme conditions, and inconsistent reporting. Thus, the paper introduces a novel methodological framework that enhances the reliability of imputation in uncertain environments and provides more robust insights for environmental monitoring and decision-making.

1.1. *Notations*

In statistics, various forms of neutrosophic observations, including quantitative neutrosophic data, are encountered where a value may lie within an imprecisely known interval $[l,\ m]$. Neutrosophic numbers provide several ways to represent such uncertain intervals. In this study, the neutrosophic interval values are represented as $W_N = W_L + W_U I_N$, where $I_N \in [I_L,\ I_U]$ captures the indeterminacy component. Consequently, the neutrosophic data are expressed in the interval form $W_N \in [l,\ m]$, where l and m denote the lower and upper bounds of the neutrosophic value, respectively.

Remark 1.1. For a more comprehensive understanding of the notations and their interpretations within the neutrosophic framework, reader is encouraged to refer to [44], which provides the foundational principles and formal definitions used in neutrosophic set theory and its statistical applications.

Consider a finite population $(U = U_1, U_2, ..., U_N)$ of size N from which a neutrosophic random sample of size $n_N \in [n_L,\ n_U]$ is drawn. Let $y_N(i) \in [y_L,\ y_U]$ and $x_N(i) \in [x_L,\ x_U]$ be the data on i^{th} sample observation of the neutrosophic variable of interest and auxiliary variable, respectively. Let r_N be the number of neutrosophic responding units from the extracted n_N units. Let R_N and $\bar{R}_N$ be the set of neutrosophic responding and non-responding units, respectively. The value $y_{N(i)}$ is deduced for every unit $i \in R_N$, while, for the units $i \in \bar{R}_N$, the values are missing and must be imputed to complete the sample data. Suppose the imputation is carried out utilizing neutrosophic auxiliary information $x_{N(i)}$, the value of x_N for i^{th} unit is available and positive for all $i \in s_N$, ensuring that the data $x_{s_N} = x_{iN}$ are available. Let $\bar{Y}_N \in [\bar{Y}_L,\ \bar{Y}_U]$ and $\bar{X}_N \in [\bar{X}_L,\ \bar{X}_U]$ be the population means corresponding to the sample means $\bar{y}_N \in [\bar{y}_L,\ \bar{y}_U]$ and $\bar{x}_N \in [\bar{x}_L,\ \bar{x}_U]$ of the neutrosophic variable of interest y_N and the neutrosophic auxiliary variable x_N, respectively. Let $C_{y_N} \in [C_{y_L},\ C_{y_U}]$ and $C_{x_N} \in [C_{x_L},\ C_{x_U}]$ be the neutrosophic coefficients of variation of the neutrosophic variables y_N and x_N, respectively. The neutrosophic correlation coefficient between neutrosophic variables y_N and x_N is ρ_{xy_N}. The neutrosophic skewness and neutrosophic kurtosis coefficients are denoted by $\beta_1(x_N) \in [\beta_1(x_L),\ \beta_1(x_U)]$ and $\beta_2(x_N) \in [\beta_2(x_L),\ \beta_2(x_U)]$, respectively.

To establish the $Bias_N \in [Bias_L,\ Bias_U]$ and mean square error (MSE) $MSE_N \in [MSE_L,\ MSE_U]$ of the resultant neutrosophic estimators, we take the neutrosophic error terms $e_{0N} \in [e_{0L},\ e_{0U}]$, $e_{1N} \in [e_{1L},\ e_{1U}]$, and $e_{2N} \in [e_{2L},\ e_{2U}]$, together with the following expectations:

$$\left.\begin{aligned}
e_{0N} &= \frac{(\bar{y}_{r_N} - \bar{Y}_N)}{\bar{Y}_N},\\
e_{1N} &= \frac{(\bar{x}_{r_N} - \bar{X}_N)}{\bar{X}_N},\\
e_{2N} &= \frac{(\bar{x}_{n_N} - \bar{X}_N)}{\bar{X}_N},\\
E(e_{0N}^2) &= f_{r_N} C_{y_N}^2,\\
E(e_{1N}^2) &= f_{r_N} C_{x_N}^2,\\
E(e_{2N}^2) &= f_{n_N} C_{x_N}^2,\\
E(e_{0N} e_{1N}) &= f_{r_N} \rho_{xy_N} C_{x_N} C_{y_N},\\
E(e_{0N} e_{2N}) &= f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},\\
E(e_{1N} e_{2N}) &= f_{n_N} C_{x_N}^2,
\end{aligned}\right\} \tag{1}$$

where $f_{n_N} = (n_N^{-1} - N^{-1})$, $f_{r_N} = (r_N^{-1} - N^{-1})$, $f_{rn_N} = (r_N^{-1} - n_N^{-1})$, $C_{y_N} = S_{y_N}/\bar{Y}_N$, $C_{x_N} = S_{x_N}/\bar{X}_N$, and $\rho_{xy_N} = S_{xy_N}/S_{x_N} S_{y_N}$.

This article will be divided into various sections. Some popular adapted neutrosophic imputation approaches and the accompanying neutrosophic estimators are considered in section 2. section 3 covers the suggested neutrosophic exponential imputation approaches, the resulting neutrosophic estimators, and their related characteristics. A comparison study of the suggested resultant estimators and the adapted estimators is done in section 4. A thorough simulation study using symmetric and asymmetric hypothetical neutrosophic populations is carried out in section 5. The adapted and proposed resultant neutrosophic estimators are analysed in section 6 using a real world climate dataset. The conclusion of the study is provided in section 7.

2. Adapted neutrosophic imputation approaches

The literature on neutrosophic imputation approach and their associated estimators remains very limited. To evaluate the efficiency of the neutrosophic imputation approaches and the resulting estimators, we adapt and extend some fundamental imputation techniques and their associated estimators under neutrosophic framework along with their characteristics.

2.1. *Neutrosophic mean imputation approach*

The neutrosophic mean imputation approach is a novel method for handling missing or indeterminate data within the neutrosophic framework. Unlike classical mean imputation that replaces missing values with a single deterministic average, neutrosophic mean imputation represents the imputed values as intervals or sets to reflect the uncertainty inherent in the data. Specifically, each missing value is replaced by a neutrosophic mean, often expressed as an interval such as $[\bar{y}_L, \bar{y}_U]$. This approach provides a more realistic and flexible treatment of incomplete data by acknowledging and modeling the imprecision rather than ignoring it. Following [45], when a sample unit 'i' is missing and needs imputation, the neutrosophic mean imputation approach is as follows:

$$y_{.i_{mN}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \bar{y}_{r_N} & \text{for } i \in \bar{R}_N \end{cases}$$

The resultant neutrosophic mean estimator is given as

$$T_{mN} = \bar{y}_{r_N}.$$

Theorem 2.1. *The variance of the resultant neutrosophic mean estimator* $T_{mN} = \bar{y}_{r_N}$ *is provided by*

$$V(T_{mN}) = f_{r_N} \bar{Y}_N^2 C_{y_N}^2.$$

Proof. Appendix A provides an outline of the proof. □

When more neutrosophic auxiliary data is available, the neutrosophic imputation approach is broken down into three categories, which are detailed below.

Category I: When $\bar{X}_N$ is available, consider $\bar{x}_{n_N}$.

Category II: When $\bar{X}_N$ is available, consider $\bar{x}_{r_N}$.

Category III: When $\bar{X}_N$ is not available, consider $\bar{x}_{n_N}$ and $\bar{x}_{r_N}$.

2.2. *Neutrosophic ratio imputation approaches*

When neutrosophic study and neutrosophic auxiliary variables are positively correlated, then the traditional neutrosophic ratio imputation approaches produce efficient results. Following [46], we define the neutrosophic ratio imputation approaches under categories I-III as
Category I

$$y_{.i_{r_{1N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category II

$$y_{.i_{r_{2N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category III

$$y_{.i_{r_{3N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N}\left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

The resultant neutrosophic ratio estimators under categories I-III are given by

$$T_{r_{1N}} = \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right),$$
$$T_{r_{2N}} = \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right),$$
$$\text{and } T_{r_{3N}} = \bar{y}_{r_N}\left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right).$$

Theorem 2.2. *Using Taylor series expansion, the first order approximated bias and MSE of the neutrosophic ratio estimators are given by*

$$Bias(T_{r_{1N}}) \approx f_{n_N}\bar{Y}_N(C^2_{x_N} - \rho_{xy_N}C_{x_N}C_{y_N}),$$
$$Bias(T_{r_{2N}}) \approx f_{r_N}\bar{Y}_N(C^2_{x_N} - \rho_{xy_N}C_{x_N}C_{y_N}),$$
$$Bias(T_{r_{3N}}) \approx f_{rn_N}\bar{Y}_N(C^2_{x_N} - \rho_{xy_N}C_{x_N}C_{y_N}),$$
$$MSE(T_{r_{1N}}) \approx \bar{Y}^2_N(f_{r_N}C^2_{y_N} + f_{n_N}C^2_{x_N} - 2f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}),$$
$$MSE(T_{r_{2N}}) \approx f_{r_N}\bar{Y}^2_N(C^2_{y_N} + C^2_{x_N} - 2\rho_{xy_N}C_{x_N}C_{y_N}),$$
$$\textit{and } MSE(T_{r_{3N}}) \approx f_{n_N}\bar{Y}^2_N C^2_{y_N} + f_{rn_N}\bar{Y}^2_N(C^2_{y_N} + C^2_{x_N} - 2\rho_{xy_N}C_{x_N}C_{y_N}).$$

Proof. Appendix B provides an outline of the proof. □

2.3. *Neutrosophic regression imputation approaches*

Neutrosophic regression imputation approaches extend traditional regression imputation by incorporating the principles of neutrosophic logic, which allows modeling of truth (T), indeterminacy (I), and falsity (F) simultaneously. In such approaches, missing values in a dataset are imputed using a regression model where the dependent and/or independent variables are represented as neutrosophic numbers, capturing both uncertainty and vagueness in the data. These methods are particularly useful when the underlying data contains ambiguity or partial information, and they offer a more flexible framework than classical imputation under uncertainty. Following [47], the neutrosophic regression imputation approaches are described under categories I-III as

Category I

$$y_{.i_{lr_{1N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \bar{y}_{r_N} + \frac{n_N\beta_{1N}}{n_N - r_N}(\bar{X}_N - \bar{x}_{n_N}) & \text{for } i \in \bar{R}_N \end{cases}$$

Category II

$$y_{.i_{lr_{2N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \bar{y}_{r_N} + \frac{n_N\beta_{2N}}{n_N - r_N}(\bar{X}_N - \bar{x}_{r_N}) & \text{for } i \in \bar{R}_N \end{cases}$$

Category III

$$y_{.i_{lr_{3N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \bar{y}_{r_N} + \frac{n_N\beta_{3N}}{n_N - r_N}(\bar{x}_{n_N} - \bar{x}_{r_N}) & \text{for } i \in \bar{R}_N \end{cases}$$

The resultant neutrosophic regression estimators are given as

$$\begin{aligned} T_{lr_{1N}} &= \bar{y}_{r_N} + \beta_{1N}(\bar{X}_N - \bar{x}_{n_N}), \\ T_{lr_{2N}} &= \bar{y}_{r_N} + \beta_{2N}(\bar{X}_N - \bar{x}_{r_N}), \\ \text{and } T_{lr_{3N}} &= \bar{y}_{r_N} + \beta_{3N}(\bar{x}_{n_N} - \bar{x}_{r_N}). \end{aligned}$$

where $\beta_{jN},\ j = 1, 2, 3$ are neutrosophic regression coefficients.

Theorem 2.3. *The bias and minimum MSE of the resultant neutrosophic regression estimators are given by*

$$\begin{aligned} Bias(T_{lr_{jN}}) &= 0,\ j = 1, 2, 3, \\ min.MSE(T_{lr_{1N}}) &= \bar{Y}_N^2 C_{y_N}^2 (f_{r_N} - f_{n_N}\rho_{xy_N}^2), \\ min.MSE(T_{lr_{2N}}) &= \bar{Y}_N^2 f_{r_N} C_{y_N}^2 (1 - \rho_{xy_N}^2), \\ \textit{and } min.MSE(T_{lr_{3N}}) &= f_{n_N}\bar{Y}_N^2 C_{y_N}^2 + f_{rn_N} C_{y_N}^2 \bar{Y}_N^2 (1 - \rho_{xy_N}^2). \end{aligned}$$

Proof. Appendix C provides an outline of the proof. □

2.4. *Neutrosophic exponential ratio imputation approaches*

The neutrosophic exponential ratio imputation approach is a novel method that extends traditional exponential ratio imputation techniques into the neutrosophic framework to effectively handle missing and indeterminate data. In this approach, both the study variable and auxiliary information are represented using neutrosophic numbers to capture uncertainty, imprecision, and indeterminacy. The imputation formula incorporates a neutrosophic exponential function of the ratio between known auxiliary values and their neutrosophic means, enhancing robustness in imputation when classical assumptions fail. Following [48], the neutrosophic exponential ratio imputation approaches are provided under categories I-III as
Category I

$$y_{.i_{bt_{1N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category II

$$y_{.i_{bt_{2N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N} \exp\left(\frac{\bar{X}_N - \bar{x}_{r_N}}{\bar{X}_N + \bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category III

$$y_{.i_{bt_{3N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N} \exp\left(\frac{\bar{x}_{n_N} - \bar{x}_{r_N}}{\bar{x}_{n_N} + \bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

The resultant neutrosophic exponential ratio estimators under categories I-III are

$$T_{bt_{1N}} = \bar{y}_{r_N} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right),$$
$$T_{bt_{2N}} = \bar{y}_{r_N} \exp\left(\frac{\bar{X}_N - \bar{x}_{r_N}}{\bar{X}_N + \bar{x}_{r_N}}\right),$$
$$\text{and } T_{bt_{3N}} = \bar{y}_{r_N} \exp\left(\frac{\bar{x}_{n_N} - \bar{x}_{r_N}}{\bar{x}_{n_N} + \bar{x}_{r_N}}\right).$$

Theorem 2.4. *Using Taylor series expansion, the bias and MSE of the resultant neutrosophic exponential ratio estimators are given by*

$$Bias(T_{bt_{1N}}) \approx f_{n_N}\bar{Y}_N\left(\frac{3}{8}C_{x_N}^2 - \frac{1}{2}\rho_{xy_N}C_{x_N}C_{y_N}\right),$$
$$Bias(T_{bt_{2N}}) \approx f_{r_N}\bar{Y}_N\left(\frac{3}{8}C_{x_N}^2 - \frac{1}{2}\rho_{xy_N}C_{x_N}C_{y_N}\right),$$
$$Bias(T_{bt_{3N}}) \approx f_{rn_N}\bar{Y}_N\left(\frac{3}{8}C_{x_N}^2 - \frac{1}{2}\rho_{xy_N}C_{x_N}C_{y_N}\right),$$
$$MSE(T_{bt_{1N}}) \approx \bar{Y}_N^2\left(f_{r_N}C_{y_N}^2 + \frac{1}{4}f_{n_N}C_{x_N}^2 - f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\right),$$
$$MSE(T_{bt_{2N}}) \approx f_{r_N}\bar{Y}_N^2\left(C_{y_N}^2 + \frac{1}{4}C_{x_N}^2 - \rho_{xy_N}C_{x_N}C_{y_N}\right),$$
$$and\ MSE(T_{bt_{3N}}) \approx \bar{Y}_N^2\left(f_{r_N}C_{y_N}^2 + \frac{1}{4}f_{rn_N}C_{x_N}^2 - f_{rn_N}\rho_{xy_N}C_{x_N}C_{y_N}\right).$$

Proof. Appendix D provides an outline of the proof. □

2.5. *Neutrosophic power ratio-cum exponential ratio imputation approaches*

The neutrosophic power ratio-cum exponential ratio imputation approaches represent an advanced class of imputation techniques designed to handle missing and indeterminate data under the neutrosophic statistical framework. These methods extend classical ratio and exponential ratio imputation by incorporating power transformation and neutrosophic components into the estimation process. By fusing the robustness of exponential ratio-type adjustments with the flexibility of power transformations, these approaches aim to improve estimation efficiency in the presence of both random nonresponse and indeterminate observations, often encountered in complex or uncertain data environments. Following [49], we adapt the neutrosophic power ratio-cum exponential ratio imputation approaches under categories I-III as:

Category I

$$y_{.i_{p_{1N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{\kappa_{1N}}\exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right) - r_N\bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category II

$$y_{.i_{p_{2N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right)^{\kappa_{2N}}\exp\left(\frac{\bar{X}_N - \bar{x}_{r_N}}{\bar{X}_N + \bar{x}_{r_N}}\right) - r_N\bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category III

$$y_{.i_{p_{3N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \bar{y}_{r_N}\left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right)^{\kappa_{3N}} \exp\left(\frac{\bar{x}_{n_N}-\bar{x}_{r_N}}{\bar{x}_{n_N}+\bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

The resultant neutrosophic power ratio-cum exponential ratio estimators under categories I-III are provided as

$$T_{p_{1N}} = \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{\kappa_{1N}} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right),$$

$$T_{p_{2N}} = \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right)^{\kappa_{2N}} \exp\left(\frac{\bar{X}_N - \bar{x}_{r_N}}{\bar{X}_N + \bar{x}_{r_N}}\right),$$

$$\text{and } T_{p_{3N}} = \bar{y}_{r_N}\left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right)^{\kappa_{3N}} \exp\left(\frac{\bar{x}_{n_N} - \bar{x}_{r_N}}{\bar{x}_{n_N} + \bar{x}_{r_N}}\right).$$

where $\kappa_{jN},\ j = 1, 2, 3$ are optimizing scalars.

Theorem 2.5. *Using Taylor series expansion, the bias and minimum MSE of the neutrosophic power ratio-cum exponential ratio estimators are provided by*

$$Bias(T_{p_{1N}}) = f_{n_N}\bar{Y}_N\left\{\left(\frac{\kappa_{1N}(\kappa_{1N}+2)}{2} + \frac{3}{8}\right)C_{x_N}^2 - \left(\kappa_{1N} + \frac{1}{2}\right)\rho_{xy_N}C_{x_N}C_{y_N}\right\},$$

$$Bias(T_{p_{2N}}) = f_{r_N}\bar{Y}_N\left\{\left(\frac{\kappa_{2N}(\kappa_{2N}+2)}{2} + \frac{3}{8}\right)C_{x_N}^2 - \left(\kappa_{2N} + \frac{1}{2}\right)\rho_{xy_N}C_{x_N}C_{y_N}\right\},$$

$$Bias(T_{p_{3N}}) = f_{rn_N}\bar{Y}_N\left\{\left(\frac{\kappa_{3N}(\kappa_{3N}+2)}{2} + \frac{3}{8}\right)C_{x_N}^2 - \left(\kappa_{3N} + \frac{1}{2}\right)\rho_{xy_N}C_{x_N}C_{y_N}\right\},$$

$$min.MSE(T_{p_{1N}}) = \bar{Y}_N^2 C_{y_N}^2 (f_{r_N} - f_{n_N}\rho_{xy_N}^2),$$

$$min.MSE(T_{p_{2N}}) = f_{r_N}\bar{Y}_N^2 C_{y_N}^2 (1 - \rho_{xy_N}^2),$$

$$and\ min.MSE(T_{p_{3N}}) = f_{n_N}\bar{Y}_N^2 C_{y_N}^2 + f_{rn_N}\bar{Y}_N^2 C_{y_N}^2 (1 - \rho_{xy_N}^2).$$

Proof. Appendix E provides an outline of the proof. □

3. Proposed neutrosophic imputation frameworks

The need for proposing imputation methods and the resulting estimators in this study arises from the fact that missing values in real-world datasets are not only frequent but also often uncertain due to sensor failures, measurement errors, or inconsistent reporting. Traditional imputation frameworks and classical statistical methods typically assume precise and complete data, which may lead to biased estimates and unreliable predictions when applied to such uncertain datasets. Neutrosophic exponential imputation and the corresponding estimator becomes important here, as it can incorporate the truth, indeterminacy, and falsity components, thereby addressing both incompleteness and uncertainty simultaneously. This is crucial in studies where accurate data modeling under uncertainty directly impacts environmental monitoring, prediction of extreme events, and policy decisions. Therefore, proposing such an imputation methods enhance the reliability of uncertain data analysis, making the results more robust and practically relevant for sustainable planning and decision-making. Motivated by this need, we propose some neutrosophic exponential imputation approaches and corresponding resultant estimators for the population mean under SRS. These imputation approaches are provided under categories I-III as follows:

Category I

$$y_{.i_{k_{1N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \Delta_{1N} \bar{y}_{r_N} \left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{\zeta_{1N}} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

$$y_{.i_{k_{4N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \left\{\Delta_{4N} \bar{y}_{r_N} + \zeta_{4N} \bar{y}_{r_N} \left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{g}\right\} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category II

$$y_{.i_{k_{2N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \Delta_{2N} \bar{y}_{r_N} \left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right)^{\zeta_{2N}} \exp\left(\frac{\bar{X}_N - \bar{x}_{r_N}}{\bar{X}_N + \bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

$$y_{.i_{k_{5N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \left\{\Delta_{5N} \bar{y}_{r_N} + \zeta_{5N} \bar{y}_{r_N} \left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right)^{g}\right\} \exp\left(\frac{\bar{X}_N - \bar{x}_{r_N}}{\bar{X}_N + \bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

Category III

$$y_{.i_{k_{3N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \Delta_{3N} \bar{y}_{r_N} \left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right)^{\zeta_{3N}} \exp\left(\frac{\bar{x}_{n_N} - \bar{x}_{r_N}}{\bar{x}_{n_N} + \bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

$$y_{.i_{k_{6N}}} = \begin{cases} y_{i_N} & \text{for } i \in R_N \\ \frac{1}{n_N - r_N}\left[n_N \left\{\Delta_{6N} \bar{y}_{r_N} + \zeta_{6N} \bar{y}_{r_N} \left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right)^{g}\right\} \exp\left(\frac{\bar{x}_{n_N} - \bar{x}_{r_N}}{\bar{x}_{n_N} + \bar{x}_{r_N}}\right) - r_N \bar{y}_{r_N}\right] & \text{for } i \in \bar{R}_N \end{cases}$$

The resultant proposed neutrosophic exponential estimators under categories I-III are provided as

$$T_{k_{1N}} = \Delta_{1N}\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{\zeta_{1N}}\exp\left(\frac{\bar{X}_N-\bar{x}_{n_N}}{\bar{X}_N+\bar{x}_{n_N}}\right),$$
$$T_{k_{2N}} = \Delta_{2N}\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right)^{\zeta_{2N}}\exp\left(\frac{\bar{X}_N-\bar{x}_{r_N}}{\bar{X}_N+\bar{x}_{r_N}}\right),$$
$$T_{k_{3N}} = \Delta_{3N}\bar{y}_{r_N}\left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right)^{\zeta_{3N}}\exp\left(\frac{\bar{x}_{n_N}-\bar{x}_{r_N}}{\bar{x}_{n_N}+\bar{x}_{r_N}}\right),$$
$$T_{k_{4N}} = \left\{\Delta_{4N}\bar{y}_{r_N}+\zeta_{4N}\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{g}\right\}\exp\left(\frac{\bar{X}_N-\bar{x}_{n_N}}{\bar{X}_N+\bar{x}_{n_N}}\right),$$
$$T_{k_{5N}} = \left\{\Delta_{5N}\bar{y}_{r_N}+\zeta_{5N}\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{r_N}}\right)^{g}\right\}\exp\left(\frac{\bar{X}_N-\bar{x}_{r_N}}{\bar{X}_N+\bar{x}_{r_N}}\right),$$
$$\text{and } T_{k_{6N}} = \left\{\Delta_{6N}\bar{y}_{r_N}+\zeta_{6N}\bar{y}_{r_N}\left(\frac{\bar{x}_{n_N}}{\bar{x}_{r_N}}\right)^{g}\right\}\exp\left(\frac{\bar{x}_{n_N}-\bar{x}_{r_N}}{\bar{x}_{n_N}+\bar{x}_{r_N}}\right).$$

where $\Delta_{jN},\ j=1,2,...,6$ and ζ_{jN} are optimizing scalars that are determined later. Also, g takes values 1 and -1 to generate different estimators.

Theorem 3.1. *Using Taylor series expansion, the first order approximated bias and minimum MSE of the resultant neutrosophic exponential estimators $T_{k_{jN}},\ j=1,2,3$ are provided by*

$$Bias(T_{k_{jN}}) \approx \bar{Y}_N(\Delta_{jN}G_{jN}-1) \tag{2}$$
$$\text{and } min.MSE(T_{k_{jN}}) \approx \bar{Y}_N^2\left(1-\frac{G_{jN}^2}{F_{jN}}\right). \tag{3}$$

where

$$F_{1N} = 1+f_{r_N}C_{y_N}^2+(2\zeta_{1N}^2+3\zeta_{1N}+1)f_{n_N}C_{x_N}^2-2(2\zeta_{1N}+1)f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N},$$
$$G_{1N} = 1+\left(\frac{\zeta_{1N}(\zeta_{1N}+1)}{2}+\frac{\zeta_{1N}}{2}+\frac{3}{8}\right)f_{n_N}C_{x_N}^2-\left(\zeta_{1N}+\frac{1}{2}\right)f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N},$$
$$F_{2N} = 1+f_{r_N}C_{y_N}^2+(2\zeta_{2N}^2+3\zeta_{2N}+1)f_{r_N}C_{x_N}^2-2(2\zeta_{2N}+1)f_{r_N}\rho_{xy_N}C_{x_N}C_{y_N},$$
$$G_{2N} = 1+\left(\frac{\zeta_{2N}(\zeta_{2N}+1)}{2}+\frac{\zeta_{2N}}{2}+\frac{3}{8}\right)f_{r_N}C_{x_N}^2-\left(\zeta_{2N}+\frac{1}{2}\right)f_{r_N}\rho_{xy_N}C_{x_N}C_{y_N},$$
$$F_{3N} = 1+f_{r_N}C_{y_N}^2+(2\zeta_{3N}^2+3\zeta_{3N}+1)f_{rn_N}C_{x_N}^2-2(2\zeta_{3N}+1)f_{rn_N}\rho_{xy_N}C_{x_N}C_{y_N},$$
$$\text{and } G_{3N} = 1+\left(\frac{\zeta_{3N}(\zeta_{3N}+1)}{2}+\frac{\zeta_{3N}}{2}+\frac{3}{8}\right)f_{rn_N}C_{x_N}^2-\left(\zeta_{3N}+\frac{1}{2}\right)f_{rn_N}\rho_{xy_N}C_{x_N}C_{y_N}.$$

Proof. Appendix F provides an outline of the proof. □

Theorem 3.2. *Using Taylor series expansion, the first order approximated bias and minimum MSE of the resultant neutrosophic exponential estimators* $T_{k_{jN}}$*, j=4, 5, 6 are given by*

$$Bias(T_{k_{jN}}) \approx \bar{Y}_N(\Delta_{j_N} L_{j_N} + \zeta_{jN} M_{j_N} - 1) \tag{4}$$

$$\text{and } min.MSE(T_{k_{jN}}) \approx \bar{Y}_N^2 \left(1 - \frac{I_{jN} M_{jN}^2 + J_{jN} L_{jN}^2 - 2K_{jN} L_{jN} M_{jN}}{I_{jN} J_{jN} - K_{jN}^2}\right). \tag{5}$$

where

$$I_{4N} = 1 + f_{r_N} C_{y_N}^2 + f_{n_N} C_{x_N}^2 - 2 f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$J_{4N} = 1 + f_{r_N} C_{y_N}^2 + (2g^2 + 3g + 1) f_{n_N} C_{x_N}^2 - (4g + 2) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$K_{4N} = 1 + f_{r_N} C_{y_N}^2 + \left(\frac{g(g+1)}{2} + g + 1\right) f_{n_N} C_{x_N}^2 - 2(g+1) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$L_{4N} = 1 + \frac{3}{8} f_{n_N} C_{x_N}^2 - \frac{1}{2} f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$M_{4N} = 1 + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8}\right) f_{n_N} C_{x_N}^2 - \left(g + \frac{1}{2}\right) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$I_{5N} = 1 + f_{r_N} C_{y_N}^2 + f_{r_N} C_{x_N}^2 - 2 f_{r_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$J_{5N} = 1 + f_{r_N} C_{y_N}^2 + (2g^2 + 3g + 1) f_{r_N} C_{x_N}^2 - (4g + 2) f_{r_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$K_{5N} = 1 + f_{r_N} C_{y_N}^2 + \left(\frac{g(g+1)}{2} + g + 1\right) f_{r_N} C_{x_N}^2 - 2(g+1) f_{r_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$L_{5N} = 1 + \frac{3}{8} f_{r_N} C_{x_N}^2 - \frac{1}{2} f_{r_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$M_{5N} = 1 + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8}\right) f_{r_N} C_{x_N}^2 - \left(g + \frac{1}{2}\right) f_{r_N} \rho_{xy_N} C_{x_N} C_{y_N},$$

$$I_{6N} = 1 + f_{r_N} C_{y_N}^2 + f_{rn_N} C_{x_N}^2 - 2 f_{rn_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$J_{6N} = 1 + f_{r_N} C_{y_N}^2 + (2g^2 + 3g + 1) f_{rn_N} C_{x_N}^2 - (4g + 2) f_{rn_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$K_{6N} = 1 + f_{r_N} C_{y_N}^2 + \left(\frac{g(g+1)}{2} + g + 1\right) f_{rn_N} C_{x_N}^2 - 2(g+1) f_{rn_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$L_{6N} = 1 + \frac{3}{8} f_{rn_N} C_{x_N}^2 - \frac{1}{2} f_{rn_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$M_{6N} = 1 + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8}\right) f_{rn_N} C_{x_N}^2 - \left(g + \frac{1}{2}\right) f_{rn_N} \rho_{xy_N} C_{x_N} C_{y_N}.$$

Proof. Appendix G provides an outline of the proof. □

4. Mathematical comparisons

In survey sampling, mathematical comparison of estimators is critical to assuring the quality, reliability, and efficiency of statistical conclusions generated from sample data. Researchers can determine which estimator performs best under certain sampling situations by comparing them using MSE. This helps in choosing an estimator that minimises errors while producing more precise and valid findings. Thus, we accomplish a mathematical comparison by comparing the MSE of the suggested and adapted resultant neutrosophic estimators under the following lemmas.

Lemma 4.1. *The proposed neutrosophic exponential estimators $T_{k_{jN}},\ j=1,2,3$ dominate the neutrosophic mean estimator T_{mN} under categories I-III, if*

$$min.MSE(T_{k_{jN}}) < V(T_{mN}) \implies \frac{G_{jN}^2}{F_{jN}} > 1 - f_{r_N} C_{y_N}^2.$$

Lemma 4.2. *(i). The proposed neutrosophic exponential estimator $T_{k_{1N}}$ dominates the neutrosophic ratio estimator $T_{r_{1N}}$ under category I, if*

$$min.MSE(T_{k_{1N}}) < MSE(T_{r_{1N}}) \implies \frac{G_{1N}^2}{F_{1N}} > 1 - f_{r_N} C_{y_N}^2 - f_{n_N}\left(C_{x_N}^2 - 2\rho_{xy_N} C_{x_N} C_{y_N}\right).$$

(ii). The proposed neutrosophic exponential estimator $T_{k_{2N}}$ dominates the neutrosophic ratio estimator $T_{r_{2N}}$ under category II, if

$$min.MSE(T_{k_{2N}}) < MSE(T_{r_{2N}}) \implies \frac{G_{2N}^2}{F_{2N}} > 1 - f_{r_N}(C_{y_N}^2 + C_{x_N}^2 - 2\rho_{xy_N} C_{x_N} C_{y_N}).$$

(iii). The proposed neutrosophic exponential estimator $T_{k_{3N}}$ dominates the neutrosophic ratio estimator $T_{r_{3N}}$ under category III, if

$$min.MSE(T_{k_{3N}}) < MSE(T_{r_{3N}}) \implies \frac{G_{3N}^2}{F_{3N}} > 1 - f_{r_N} C_{y_N}^2 - f_{rn_N}(C_{x_N}^2 - 2\rho_{xy_N} C_{x_N} C_{y_N}).$$

Lemma 4.3. *(i). The proposed neutrosophic exponential estimator $T_{k_{1N}}$ dominates the neutrosophic regression estimator $T_{lr_{1N}}$ and the neutrosophic power ratio-cum exponential ratio estimator $T_{p_{1N}}$ under category I, if*

$$\begin{aligned} &min.MSE(T_{k_{1N}}) < min.MSE(t_{1N}^*),\ \textit{where } t_{1N}^* = T_{lr_{1N}} \textit{ and } T_{p_{1N}} \\ &\implies \frac{G_{1N}^2}{F_{1N}} > 1 - f_{r_N} C_{y_N}^2 + f_{n_N} C_{y_N}^2 \rho_{xy_N}^2. \end{aligned}$$

(ii). The proposed neutrosophic exponential estimator $T_{k_{2N}}$ dominates the neutrosophic regression estimator $T_{lr_{2N}}$ and neutrosophic power ratio-cum exponential ratio estimator $T_{p_{2N}}$ under category II, if

$$\begin{aligned} &min.MSE(T_{k_{2N}}) < min.MSE(t_{2N}^*),\ \textit{where } t_{2N}^* = T_{lr_{2N}} \textit{ and } T_{p_{2N}} \\ &\implies \frac{G_{2N}^2}{F_{2N}} > 1 - f_{r_N} C_{y_N}^2 (1 - \rho_{xy_N}^2). \end{aligned}$$

(iii). The proposed neutrosophic exponential estimator $T_{k_{3N}}$ dominates the neutrosophic regression estimator $T_{lr_{3N}}$ and neutrosophic power ratio-cum exponential ratio estimator $T_{p_{3N}}$

under category III, if

$$min.MSE(T_{k_{3N}}) < min.MSE(t^*_{3N}), \ \ where \ t^*_{3N} = T_{lr_{3N}} \ \ and \ T_{p_{3N}}$$
$$\implies \frac{G^2_{3N}}{F_{3N}} > 1 - f_{n_N} C^2_{y_N} - f_{rn_N} C^2_{y_N} (1 - \rho^2_{xy_N}).$$

Lemma 4.4. *(i). The proposed neutrosophic exponential estimator $T_{k_{1N}}$ dominates the neutrosophic exponential ratio estimator $T_{bt_{1N}}$ under category I, if*

$$min.MSE(T_{k_{1N}}) < MSE(T_{bt_{1N}}) \implies \frac{G^2_{1N}}{F_{1N}} > 1 - f_{r_N} C^2_{y_N} - f_{n_N} C_{x_N} \left(\frac{1}{4} C_{x_N} - \rho_{xy_N} C_{y_N} \right).$$

(ii). The proposed neutrosophic exponential estimator $T_{k_{2N}}$ dominates the neutrosophic exponential ratio estimator $T_{bt_{2N}}$ under category II, if

$$min.MSE(T_{k_{2N}}) < MSE(T_{bt_{2N}}) \implies \frac{G^2_{2N}}{F_{2N}} > 1 - f_{r_N} \left(C^2_{y_N} + \frac{1}{4} C^2_{x_N} - \rho_{xy_N} C_{x_N} C_{y_N} \right).$$

(iii). The proposed neutrosophic exponential estimator $T_{k_{3N}}$ dominates the neutrosophic exponential ratio estimator $T_{bt_{3N}}$ under category III, if

$$min.MSE(T_{k_{3N}}) < MSE(T_{bt_{3N}}) \implies \frac{G^2_{3N}}{F_{3N}} > 1 - f_{r_N} C^2_{y_N} - f_{rn_N} C_{x_N} \left(\frac{1}{4} C_{x_N} - \rho_{xy_N} C_{y_N} \right).$$

Lemma 4.5. *The proposed neutrosophic exponential estimator $T_{k_{jN}}, \ j = 4, 5, 6$ dominate the neutrosophic mean estimator T_{mN} under categories I-III, if*

$$min.MSE(T_{k_{jN}}) < V(T_{mN}) \implies \frac{I_{jN} M^2_{jN} + J_{jN} L^2_{jN} - 2K_{jN} L_{jN} M_{jN}}{I_{jN} J_{jN} - K^2_{jN}} > 1 - f_{r_N} C^2_{y_N}$$

Lemma 4.6. *(i). The proposed neutrosophic exponential estimator $T_{k_{4N}}$ dominates the neutrosophic ratio estimator $T_{r_{1N}}$ under category I, if*

$$min.MSE(T_{k_{4N}}) < MSE(T_{r_{1N}})$$
$$\implies \frac{I_{4N} M^2_{4N} + J_{4N} L^2_{4N} - 2K_{4N} L_{4N} M_{4N}}{I_{4N} J_{4N} - K^2_{4N}} > 1 - f_{r_N} C^2_{y_N} - f_{n_N} \left(C^2_{x_N} - 2\rho_{xy_N} C_{x_N} C_{y_N} \right).$$

(ii). The proposed neutrosophic exponential estimator $T_{k_{5N}}$ dominates the neutrosophic ratio estimator $T_{r_{2N}}$ under category II, if

$$min.MSE(T_{k_{5N}}) < MSE(T_{r_{2N}})$$
$$\implies \frac{I_{5N} M^2_{5N} + J_{5N} L^2_{5N} - 2K_{5N} L_{5N} M_{5N}}{I_{5N} J_{5N} - K^2_{5N}} > 1 - f_{r_N} (C^2_{y_N} + C^2_{x_N} - 2\rho_{xy_N} C_{x_N} C_{y_N}).$$

(iii). The proposed neutrosophic exponential estimator $T_{k_{6N}}$ dominates the neutrosophic ratio estimator $T_{r_{3N}}$ under category III, if

$$min.MSE(T_{k_{6N}}) < MSE(T_{r_{3N}})$$
$$\implies \frac{I_{6N} M^2_{6N} + J_{6N} L^2_{6N} - 2K_{6N} L_{6N} M_{6N}}{I_{6N} J_{6N} - K^2_{6N}} > 1 - f_{r_N} C^2_{y_N} - f_{rn_N} (C^2_{x_N} - 2\rho_{xy_N} C_{x_N} C_{y_N}).$$

Lemma 4.7. *(i). The proposed neutrosophic exponential estimator $T_{k_{4N}}$ dominates the neutrosophic regression estimator $T_{lr_{1N}}$ and neutrosophic power ratio-cum exponential ratio estimator $T_{p_{1N}}$ under category I, if*

$$min.MSE(T_{k_{4N}}) < min.MSE(T^*_{1N}), \text{ where } T^*_{1N} = T_{lr_{1N}} \text{ and } T_{p_{1N}}$$

$$\implies \frac{I_{4N}M_{4N}^2 + J_{4N}L_{4N}^2 - 2K_{4N}L_{4N}M_{4N}}{I_{4N}J_{4N} - K_{4N}^2} > 1 - f_{r_N}C_{y_N}^2 + f_{n_N}C_{y_N}^2\rho_{xy_N}^2.$$

(ii). The proposed neutrosophic exponential estimator $T_{k_{5N}}$ dominates the neutrosophic regression estimator $T_{lr_{2N}}$ and neutrosophic power ratio-cum exponential ratio estimator $T_{p_{2N}}$ under category II, if

$$min.MSE(T_{k_{5N}}) < min.MSE(T^*_{2N}), \text{ where } T^*_{2N} = T_{lr_{2N}} \text{ and } T_{p_{2N}}$$

$$\implies \frac{I_{5N}M_{5N}^2 + J_{5N}L_{5N}^2 - 2K_{5N}L_{5N}M_{5N}}{I_{5N}J_{5N} - K_{5N}^2} > 1 - f_{r_N}C_{y_N}^2(1 - \rho_{xy_N}^2).$$

(iii). The proposed neutrosophic exponential estimator $T_{k_{6N}}$ dominates the neutrosophic regression estimator $T_{lr_{3N}}$ and neutrosophic power ratio-cum exponential ratio estimator $T_{p_{3N}}$ under category III, if

$$min.MSE(T_{k_{6N}}) < MSE(T^*_{3N}), \text{ where } T^*_{3N} = T_{lr_{3N}} \text{ and } T_{pr_{3N}}$$

$$\implies \frac{I_{6N}M_{6N}^2 + J_{6N}L_{6N}^2 - 2K_{6N}L_{6N}M_{6N}}{I_{6N}J_{6N} - K_{6N}^2} > 1 - f_{n_N}C_{y_N}^2 - f_{rn_N}C_{y_N}^2(1 - \rho_{xy_N}^2).$$

Lemma 4.8. *(i). The proposed neutrosophic exponential estimator $T_{k_{4N}}$ dominates the neutrosophic exponential ratio estimator $T_{bt_{1N}}$ under category I, if*

$$min.MSE(T_{k_{4N}}) < min.MSE(T_{bt_{1N}})$$

$$\implies \frac{I_{4N}M_{4N}^2 + J_{4N}L_{4N}^2 - 2K_{4N}L_{4N}M_{4N}}{I_{4N}J_{4N} - K_{4N}^2} > 1 - f_{r_N}C_{y_N}^2 - f_{n_N}C_{x_N}\left(\frac{1}{4}C_{x_N} - \rho_{xy_N}C_{y_N}\right).$$

(ii). The proposed neutrosophic exponential estimator $T_{k_{5N}}$ dominates the neutrosophic exponential ratio estimator $T_{bt_{2N}}$ under category II, if

$$min.MSE(T_{k_{5N}}) < min.MSE(T_{bt_{2N}})$$

$$\implies \frac{I_{5N}M_{5N}^2 + J_{5N}L_{5N}^2 - 2K_{5N}L_{5N}M_{5N}}{I_{5N}J_{5N} - K_{5N}^2} > 1 - f_{r_N}\left(C_{y_N}^2 + \frac{1}{4}C_{x_N}^2 - \rho_{xy_N}C_{x_N}C_{y_N}\right).$$

(iii). The proposed neutrosophic exponential estimator $T_{k_{6N}}$ dominates the neutrosophic exponential ratio estimator $T_{bt_{3N}}$ under category III, if

$$min.MSE(T_{k_{6N}}) < min.MSE(T_{bt_{3N}})$$

$$\implies \frac{I_{6N}M_{6N}^2 + J_{6N}L_{6N}^2 - 2K_{6N}L_{6N}M_{6N}}{I_{6N}J_{6N} - K_{6N}^2} > 1 - f_{r_N}C_{y_N}^2 - f_{rn_N}C_{x_N}\left(\frac{1}{4}C_{x_N} - \rho_{xy_N}C_{y_N}\right).$$

5. **Simulation analysis**

Simulation analysis is important because it allows mathematical results to be tested under controlled settings. Following [50] and [51], a simulation analysis is performed on a symmetric population created by the neutrosophic normal distribution (NND) and an asymmetric population generated by the neutrosophic gamma distribution (NGD). The models used to create the neutrosophic data are mentioned below.

$$\left.\begin{aligned} y_N &= 12.8 + \sqrt{(1-\rho_{xy_N}^2)}y_N^* + \rho_{xy_N}\left(\frac{S_{y_N}}{S_{x_N}}\right)x_N^* \\ \text{and } x_N &= 8.4 + x_N^*. \end{aligned}\right\} \tag{6}$$

The neutrosophic variables x_N^* and y_N^* possess distinct neutrosophic distributions. The simulation procedure is described in the following points.

(i). Using the models in (6), we generate the following populations:

(a). Draw a normal (symmetric) population from NND of size $N_N = [1000,\ 1000]$ with $x_N^* \sim NND([8,\ 9],\ [22,\ 30])$, $y_N^* \sim NND([6,\ 5],\ [28,\ 32])$, and varying values of neutrosophic correlation coefficient $\rho_{xy_N} \in [(0.85,\ 0.9),\ (0.65,\ 0.7),\ (0.45,\ 0.5),\ (0.25,\ 0.3)]]$.

(b). Draw a gamma (asymmetric) population from NGD of size $N_N = [1000,\ 1000]$ with $x_N^* \sim NGD([0.018,\ 0.04],\ [0.008,\ 0.0052])$, $y_N^* \sim NGD([0.06,\ 0.07],\ [0.002,\ 0.0091])$, and varying values of neutrosophic correlation coefficient $\rho_{xy_N} \in [(0.85,\ 0.9),\ (0.65,\ 0.7),\ (0.45,\ 0.5),\ (0.25,\ 0.3)]$.

(ii). Select a neutrosophic sample of size $n_N = (150, 150)$ from the neutrosophic normal and gamma populations produced in the preceding step. It is assumed that $r_N = (140, 145)$ units give response out of the sampled n_N units drawn from the neutrosophic normal and gamma populations.

(iii). After selecting a sample of n_N from both populations, take $n_N - r_N$ units.

(iii). Determine the neutrosophic mean square error ($MSE_N \in [MSE_L, MSE_U]$) and neutrosophic percent relative efficiency ($PRE_N \in [PRE_L, PRE_U]$) of the resulting neutrosophic estimators for the samples and responding units taken in the previous step using the corresponding equations provided in (7) based on 10,000 replications.

$$\left.\begin{aligned} MSE(T^*) &= \frac{1}{10,000}\sum_{i=1}^{10,000}(T^* - \bar{Y}_N)^2 \\ \text{and } PRE &= \frac{V(T_{mN})}{MSE(T^*)} \times 100, \end{aligned}\right\} \tag{7}$$

where $T^* = T_{mN},\ T_{r_{jN}},\ j = 1, 2, 3,\ T_{lr_{jN}},\ T_{bt_{jN}},\ T_{p_{jN}},\ T_{k_{jN}},\ j = 1, 2, ..., 6.$

(iv). Tables 1-2 display the simulated $MSE_N \in (MSE_L, MSE_U)$, while Tables 3-4 display the simulated $PRE_N \in (PRE_L, PRE_U)$ for neutrosophic normal and neutrosophic gamma populations, respectively.

5.1. *Discussion of simulation findings*

A discussion of simulation findings is critical for understanding the results. It validates the efficiency conditions of the suggested estimators established through mathematical comparisons. The simulation results of Tables 1-4 are described below:

- Tables 1-2 show MSE_N findings for neutrosophic normal and neutrosophic gamma populations, respectively. These findings show that, in categories I-III, the suggested resultant neutrosophic exponential estimators $T_{k_{jN}},\ j = 1, 2, ..., 6$ achieve the lowest MSE_N for any value of ρ_{xy_N} in contrast to the adapted resultant neutrosophic mean estimator T_{mN}, neutrosophic ratio and regression estimators $T_{r_{jN}},\ j = 1, 2, 3$ and $T_{lr_{jN}}$, neutrosophic exponential ratio estimators $T_{bt_{jN}},\ j = 1, 2, 3$, and neutrosophic power ratio-cum exponential ratio estimators $T_{p_{jN}},\ j = 1, 2, 3$. These findings demonstrate improved performance and accuracy of the proposed neutrosophic estimators over the adapted ones, while processing uncertain data.

- Tables 3-4 show PRE_N findings for neutrosophic normal and neutrosophic gamma populations, respectively. These findings show that, in categories I-III, the suggested resultant neutrosophic exponential estimators $T_{k_{jN}},\ j = 1, 2, ..., 6$ achieve the highest PRE_N for any value of ρ_{xy_N} in contrast to the adapted resultant neutrosophic mean estimator T_{mN}, neutrosophic ratio and regression estimators $T_{r_{jN}},\ j = 1, 2, 3$ and $T_{lr_{jN}}$, neutrosophic exponential ratio estimators $T_{bt_{jN}},\ j = 1, 2, 3$, and neutrosophic power ratio-cum exponential ratio estimators $T_{p_{jN}},\ j = 1, 2, 3$. This indicates that the proposed estimators $T_{k_{jN}}$ are not only robust across different populations, but also demonstrate the highest efficiency for producing efficient estimations for various values of ρ_{xy_N}, providing enhanced reliability in neutrosophic population parameter estimations.

- Tables 1-2 demonstrate that, for any value of ρ_{xy_N}, the MSE_N of the resultant neutrosophic mean estimator T_{mN} becomes essentially stable. Moreover, under categories I-III, the MSE_N of the adapted resultant neutrosophic ratio and regression estimators $T_{r_{jN}},\ j = 1, 2, 3$ and $T_{lr_{jN}}$, neutrosophic exponential ratio estimators $T_{bt_{jN}},\ j = 1, 2, 3$, neutrosophic power ratio-cum exponential ratio estimators $T_{p_{jN}},\ j = 1, 2, 3$, and proposed neutrosophic exponential estimators $T_{k_{jN}},\ j = 1, 2, ..., 6$ steadily reduces as ρ_{xy_N} changes from minimum to maximum value.

- Tables 3-4 demonstrate that, for any value of ρ_{xy_N}, the PRE_N of the resultant neutrosophic mean estimator T_{mN} becomes essentially stable. Moreover, under categories I-III, the PRE_N of the adapted resultant neutrosophic ratio and regression estimators $T_{r_{jN}},\ j = 1, 2, 3$

and $T_{lr_{jN}}$, neutrosophic exponential ratio estimators $T_{bt_{jN}},\ j = 1, 2, 3$, neutrosophic power ratio-cum exponential ratio estimators $T_{p_{jN}},\ j = 1, 2, 3$, and proposed neutrosophic exponential estimators $T_{k_{jN}},\ j = 1, 2, ..., 6$ steadily increases as ρ_{xy_N} changes from minimum to maximum value.

• Furthermore, in both populations, the suggested neutrosophic estimators perform much better in category II than in categories I and III. This improved performance in category II indicates that the suggested estimators highlight the importance of using respondent auxiliary information over the full sample.

TABLE 1. $MSE_N \in [MSE_L,\ MSE_U]$ of resulting neutrosophic estimators using neutrosophic normal population

	ρ_{xy_N}			
Estimators	(0.25, 0.3)	(0.45, 0.5)	(0.65, 0.7)	(0.85, 0.9)
T_{mN}	(5.1151, 6.4601)	(5.2508, 6.6164)	(5.3290, 6.6876)	(5.2898, 6.5731)
Category I				
$T_{r_{1N}}$	(7.5733, 9.9094)	(6.1500, 7.8379)	(4.3311, 5.2280)	(2.1870, 2.1769)
$T_{lr_{1N}}$	(4.8202, 5.9013)	(4.2700, 5.0268)	(3.2520, 3.5384)	(1.7642, 1.4564)
$T_{bt_{1N}}$	(5.1317, 6.2948)	(4.3062, 5.0700)	(3.2925, 3.5948)	(2.1408, 1.9557)
$T_{p_{1N}}$	(4.8202, 5.9013)	(4.2700, 5.0268)	(3.2520, 3.5384)	(1.7642, 1.4564)
$T_{k_{1N}}$	(4.7658, 5.8110)	(4.2282, 4.9604)	(3.2272, 3.5023)	(1.7557, 1.4470)
$T_{k_{4N}}$	(4.7798, 5.8357)	(4.2317, 4.9662)	(3.2241, 3.4964)	(1.7510, 1.4391)
Category II				
$T_{r_{2N}}$	(7.7798, 10.0493)	(6.2255, 7.8875)	(4.2473, 5.1688)	(1.9263, 1.9986)
$T_{lr_{2N}}$	(4.7954, 5.8787)	(4.1875, 4.9623)	(3.0775, 3.4107)	(1.4679, 1.2488)
$T_{bt_{2N}}$	(5.1331, 6.2881)	(4.2268, 5.0073)	(3.1214, 3.4693)	(1.8762, 1.7684)
$T_{p_{2N}}$	(4.7954, 5.8787)	(4.1875, 4.9623)	(3.0775, 3.4107)	(1.4679, 1.2488)
$T_{k_{2N}}$	(4.7407, 5.7881)	(4.1462, 4.8964)	(3.0540, 3.3757)	(1.4611, 1.2407)
$T_{k_{5N}}$	(4.7559, 5.8138)	(4.1500, 4.9024)	(3.0507, 3.3697)	(1.4564, 1.2329)
Category III				
$T_{r_{3N}}$	(5.3217, 6.6000)	(5.3264, 6.6660)	(5.2452, 6.6284)	(5.0291, 6.3947)
$T_{lr_{3N}}$	(5.0904, 6.4374)	(5.1684, 6.5519)	(5.1545, 6.5599)	(4.9935, 6.3655)
$T_{bt_{3N}}$	(5.1165, 6.4534)	(5.1714, 6.5537)	(5.1579, 6.5622)	(5.0252, 6.3858)
$T_{p_{3N}}$	(5.0904, 6.4374)	(5.1684, 6.5519)	(5.1545, 6.5599)	(4.9935, 6.3655)
$T_{k_{3N}}$	(5.0388, 6.3518)	(5.1218, 6.4738)	(5.1123, 6.4882)	(4.9556, 6.2997)
$T_{k_{6N}}$	(5.0400, 6.3528)	(5.1221, 6.4741)	(5.1120, 6.4879)	(4.9548, 6.2990)

TABLE 2. $MSE_N \in [MSE_L,\ MSE_U]$ of resulting neutrosophic estimators using neutrosophic gamma population

	ρ_{xy_N}			
Estimators	(0.25, 0.3)	(0.45, 0.5)	(0.65, 0.7)	(0.85, 0.9)
T_{mN}	(58.1667, 3.7145)	(58.0808, 3.8343)	(58.0313, 3.8888)	(58.0561, 3.8011)
Category I				
$T_{r_{1N}}$	(62.4547, 14.1545)	(49.2016, 11.6136)	(35.4291, 8.4793)	(23.1447, 4.7567)
$T_{lr_{1N}}$	(54.8131, 3.3932)	(47.2312, 2.9131)	(35.4137, 2.0576)	(35.4762, 0.8422)
$T_{bt_{1N}}$	(55.0308, 5.2342)	(48.1243, 3.9030)	(41.3670, 2.4062)	(36.0303, 0.8589)
$T_{p_{1N}}$	(54.8131, 3.3932)	(47.2312, 2.9131)	(35.4137, 2.0576)	(19.3621, 0.8422)
$T_{k_{1N}}$	(53.0438, 3.3522)	(45.9602, 2.8732)	(34.7513, 2.0269)	(19.2488, 0.8268)
$T_{k_{4N}}$	(53.0900, 3.3918)	(45.8601, 2.9050)	(34.6468, 2.0440)	(19.2421, 0.8292)
Category II				
$T_{r_{2N}}$	(62.8151, 14.5780)	(48.4555, 11.9292)	(33.5298, 8.6655)	(20.2109, 4.7955)
$T_{lr_{2N}}$	(54.5313, 3.3802)	(46.3194, 2.8757)	(33.5131, 1.9833)	(16.1105, 0.7222)
$T_{bt_{2N}}$	(54.7673, 5.2958)	(47.2877, 3.9057)	(39.9667, 2.3460)	(34.1794, 0.7395)
$T_{p_{2N}}$	(54.5313, 3.3802)	(46.3194, 2.8757)	(33.5131, 1.9833)	(16.1105, 0.7222)
$T_{k_{2N}}$	(52.7664, 3.3387)	(45.0847, 2.8355)	(32.9207, 1.9527)	(16.0520, 0.7074)
$T_{k_{5N}}$	(52.8164, 3.3791)	(44.9776, 2.8682)	(32.8129, 1.9703)	(16.0467, 0.7098)
Category III				
$T_{r_{3N}}$	(58.5271, 4.1381)	(57.3347, 4.1499)	(56.1320, 4.0750)	(55.1224, 3.8398)
$T_{lr_{3N}}$	(57.8849, 3.7015)	(57.1691, 3.7969)	(56.1307, 3.8145)	(54.8045, 3.6810)
$T_{bt_{3N}}$	(57.9032, 3.7762)	(57.2441, 3.8371)	(56.6309, 3.8287)	(56.2052, 3.6817)
$T_{p_{3N}}$	(57.8849, 3.7015)	(57.1691, 3.7969)	(56.1307, 3.8145)	(54.8045, 3.6810)
$T_{k_{3N}}$	(56.0672, 3.6706)	(55.4702, 3.7652)	(54.4590, 3.7819)	(52.9741, 3.6475)
$T_{k_{6N}}$	(56.0712, 3.6729)	(55.4606, 3.7667)	(54.4456, 3.7828)	(52.9720, 3.6476)

6. **Real data insights**

Real data insights are important in survey sampling because they give an accurate depiction of population features, bridging the gap between theoretical frameworks and actual implementations. The findings enable researchers to validate and improve the imputation approaches and the resulting estimators in real-world complexity. Moreover, real data frequently has intrinsic complexities, such as missing values or biases that simulated data cannot fully portray. Thus, we have demonstrated the adapted and suggested neutrosophic imputation approaches using real-world uncertain climate data from Georgia, USA, of November month. It is easily accessible via the publicly available website `http://mrcc.purdueedu`. This data was recently utilized by [52] and [53] and is presented in the Appendix H of this study. The neutrosophic study variable $y_N \in [y_L,\ y_U]$ is relative humidity, while the neutrosophic auxiliary variable $x_N \in [x_L,\ x_U]$ is the dew point temperature (F).

Table 3. $PRE_N \in [PRE_L,\ PRE_U]$ of resulting neutrosophic estimators using neutrosophic normal population

	ρ_{xy_N}			
Estimators	(0.25, 0.3)	(0.45, 0.5)	(0.65, 0.7)	(0.85, 0.9)
T_{m_N}	(100, 100)	(100, 100)	(100, 100)	(100, 100)
Category I				
$T_{r_{1N}}$	(67.5420, 65.1920)	(85.3797, 84.4157)	(123.0394, 127.918)	(241.8727, 301.9370)
$T_{lr_{1N}}$	(106.1183, 109.468)	(122.9713, 131.6228)	(163.8667, 188.9990)	(299.8431, 451.3064)
$T_{bt_{1N}}$	(99.6776, 102.6259)	(121.9364, 130.5006)	(161.8523, 186.0366)	(247.0920, 336.0961)
$T_{p_{1N}}$	(106.1183, 109.4680)	(122.9713, 131.6228)	(163.8667, 188.9990)	(299.8431, 451.3064)
$T_{k_{1N}}$	(107.3300, 111.1698)	(124.1868, 133.3849)	(165.1270, 190.9510)	(301.2846, 454.2416)
$T_{k_{4N}}$	(107.0152, 110.6999)	(124.0828, 133.2292)	(165.2874, 191.2725)	(302.1004, 456.7329)
Category II				
$T_{r_{2N}}$	(65.7487, 64.2843)	(84.3434, 83.8853)	(125.4686, 129.3834)	(274.6121, 328.8793)
$T_{lr_{2N}}$	(106.6667, 109.8901)	(125.3918, 133.3333)	(173.1602, 196.0784)	(360.3604, 526.3158)
$T_{bt_{2N}}$	(99.6506, 102.7354)	(124.2264, 132.1355)	(170.7261, 192.7647)	(281.9420, 371.6970)
$T_{p_{2N}}$	(106.6667, 109.8901)	(125.3918, 133.3333)	(173.1602, 196.0784)	(360.3604, 526.3158)
$T_{k_{2N}}$	(107.8971, 111.6091)	(126.6428, 135.1288)	(174.4901, 198.1076)	(362.0353, 529.7710)
$T_{k_{5N}}$	(107.5534, 111.1171)	(126.5262, 134.9630)	(174.6792, 198.4620)	(363.2072, 533.1112)
Category III				
$T_{r_{3N}}$	(96.1184, 97.8798)	(98.5814, 99.2566)	(101.5987, 100.8933)	(105.1846, 102.7889)
$T_{lr_{3N}}$	(100.4869, 100.3521)	(101.5948, 100.9843)	(103.3861, 101.9475)	(105.9331, 103.2609)
$T_{bt_{3N}}$	(99.9728, 100.1039)	(101.5350, 100.9572)	(103.3179, 101.9120)	(105.2659, 102.9334)
$T_{p_{3N}}$	(100.4869, 100.3521)	(101.5948, 100.9843)	(103.3861, 101.9475)	(105.9331, 103.2609)
$T_{k_{3N}}$	(101.5153, 101.7049)	(102.5196, 102.2026)	(104.2391, 103.0730)	(106.7449, 104.3391)
$T_{k_{6N}}$	(101.4908, 101.6883)	(102.5132, 102.1986)	(104.2460, 103.0782)	(106.7612, 104.3512)

Table 5 displays the neutrosophic and classical parameters of the real data set. Using these parameters, we have calculated neutrosophic mean square error $MSE_N \in [MSE_L,\ MSE_U]$, neutrosophic percent relative efficiency $PRE_N \in [PRE_L,\ PRE_U]$, classical MSE, and classical PRE, for the resultant neutrosophic and classical estimators and presented the findings in Table 6. The PRE_N is computed using the following equation.

$$PRE_N = \frac{V(T_{mN})}{MSE(T^*)} \times 100. \tag{8}$$

The real data findings of Table 6 show that the MSE_N and PRE_N of the proposed neutrosophic estimators $T_{k_{jN}},\ j = 1, 2, ..., 6$ are minimum and maximum, respectively, compared to the MSE_N and PRE_N of the adapted neutrosophic estimators such as neutrosophic mean estimator T_{mN}, neutrosophic ratio and regression estimators $T_{r_{jN}},\ j = 1, 2, 3$ and $T_{lr_{jN}}$, neutrosophic exponential ratio estimators $T_{bt_{jN}}$, and neutrosophic power ratio-cum exponential

Table 4. $PRE_N \in [PRE_L,\ PRE_U]$ of resulting neutrosophic estimators using neutrosophic gamma population

	ρ_{xy_N}			
Estimators	(0.25, 0.3)	(0.45, 0.5)	(0.65, 0.7)	(0.85, 0.9)
T_{mN}	(100, 100)	(100, 100)	(100, 100)	(100, 100)
Category I				
$T_{r_{1N}}$	(93.1342, 26.2429)	(118.0465, 33.0157)	(163.7955, 45.8630)	(250.8397, 79.9100)
$T_{lr_{1N}}$	(106.1183, 109.4680)	(122.9713, 131.6228)	(163.8667, 188.9990)	(299.8431, 451.3064)
$T_{bt_{1N}}$	(105.6985, 70.9669)	(120.6890, 98.2402)	(140.2839, 161.6170)	(161.1314, 442.5376)
$T_{p_{1N}}$	(106.1183, 109.4680)	(122.9713, 131.6228)	(163.8667, 188.9990)	(299.8431, 451.3064)
$T_{k_{1N}}$	(109.6580, 110.8099)	(126.3720, 133.4493)	(166.9904, 191.8557)	(301.6083, 459.7218)
$T_{k_{4N}}$	(109.5626, 109.5162)	(126.6478, 131.9894)	(167.4940, 190.2557)	(301.7140, 458.4018)
Category II				
$T_{r_{2N}}$	(92.5999, 25.4805)	(119.8643, 32.1422)	(173.0740, 44.8774)	(287.2505, 79.2640)
$T_{lr_{2N}}$	(106.6667, 109.8901)	(125.3918, 133.3333)	(173.1602, 196.0784)	(360.3604, 526.3158)
$T_{bt_{2N}}$	(106.2071, 70.1408)	(122.8244, 98.1701)	(145.1991, 165.7605)	(169.8572, 513.9574)
$T_{p_{2N}}$	(106.6667, 109.8901)	(125.3918, 133.3333)	(173.1602, 196.0784)	(360.3604, 526.3158)
$T_{k_{2N}}$	(110.2345, 111.2579)	(128.8260, 135.2253)	(176.2762, 199.1466)	(361.6739, 537.3309)
$T_{k_{5N}}$	(110.1301, 109.9253)	(129.1327, 133.6795)	(176.8551, 197.3698)	(361.7936, 535.4886)
Category III				
$T_{r_{3N}}$	(99.3843, 89.7651)	(101.3014, 92.3952)	(103.3837, 95.4301)	(105.3222, 98.9903)
$T_{p_{3N}}$	(100.4869, 100.3521)	(101.5948, 100.9843)	(103.3861, 101.9475)	(105.9331, 103.2609)
$T_{bt_{3N}}$	(100.4551, 98.3674)	(101.4616, 99.9273)	(102.4728, 101.5710)	(103.2931, 103.2419)
$T_{p_{3N}}$	(100.4869, 100.3521)	(101.5948, 100.9843)	(103.3861, 101.9475)	(105.9331, 103.2609)
$T_{k_{3N}}$	(103.7446, 101.1953)	(104.7063, 101.8354)	(106.5596, 102.8260)	(109.5933, 104.2104)
$T_{k_{6N}}$	(103.7374, 101.1339)	(104.7245, 101.7940)	(106.5858, 102.8034)	(109.5978, 104.2066)

ratio estimators $T_{p_{jN}}$. Furthermore, the performance of the suggested and adapted neutrosophic estimators is higher in category II compared to I and III, highlighting the importance of using respondent auxiliary information over the full sample.

Table 6 also presents a comparison of the neutrosophic and classical approaches based on the MSEs and PREs of the proposed estimators for the population mean estimation. The findings clearly indicate that the neutrosophic results are more efficient and reliable than their classical counterparts. Classical estimators, which rely on crisp or single-valued outcomes, fail to capture the uncertainty and indeterminacy present in real-world data. In contrast, the neutrosophic estimators effectively accommodate uncertainty information, thereby providing more realistic and accurate results. Consequently, the neutrosophic-based results in Table 6 provide superior estimation for uncertain data, representing the true values more faithfully than the classical methods.

Table 5. Descriptive statistics of real data set

Neutrosophic parameters	Neutrosophic values	Classical parameters	Classical values
N_N	(22, 22)	N	22
n_N	(8, 8)	n	8
r_N	(6, 6)	r	6
$\bar{Y}_N$	(31.77, 93.86)	$\bar{Y}$	62.95
$\bar{X}_N$	(22.55, 62.23)	$\bar{X}$	42.38
C_{y_N}	(0.126, 0.689)	C_y	0.150
C_{x_N}	(0.139, 0.579)	C_x	0.162
ρ_{xy_N}	(0.885, 0.948)	ρ_{xy}	0.859

Table 6. Real data findings for neutrosophic vs classical estimators

Neutrosophic estimators	MSE_N	PRE_N	Classical estimators	MSE	PRE
T_{mN}	(1.9423, 506.9272)	(100.0000, 100.0000)	T_m	10.8073	100.0000
Category I					
$T_{r_{1N}}$	(1.0046, 211.8097)	(193.3315, 239.3314)	T_{r_1}	5.9204	182.5425
$T_{lr_{1N}}$	(1.0856, 300.6366)	(178.9037, 168.6179)	T_{lr_1}	6.2957	171.6602
$T_{bt_{1N}}$	(1.0856, 293.3595)	(179.0820, 172.8006)	T_{bt_1}	6.2863	171.9169
$T_{p_{1N}}$	(0.9439, 207.9545)	(205.7577, 243.7684)	T_{p_1}	5.5740	193.8865
$T_{k_{1N}}$	(0.9428, 203.8945)	(206.0036, 248.6222)	T_{k_1}	5.5642	194.2291
$T_{k_{4N}}$	(0.9424, 202.6574)	(206.0850, 250.1400)	T_{k_4}	5.5613	194.3305
Category II					
$T_{r_{2N}}$	(0.5135, 57.2244)	(378.2504, 885.8582)	T_{r_2}	3.3606	321.5848
$T_{lr_{2N}}$	(0.6369, 192.5797)	(304.9347, 263.2299)	T_{lr_2}	3.9325	274.8159
$T_{bt_{2N}}$	(0.6366, 190.8203)	(305.0917, 265.6569)	T_{bt_2}	3.9292	275.0504
$T_{p_{2N}}$	(0.4210, 51.3497)	(461.3078, 987.2058)	T_{p_2}	2.8328	381.5061
$T_{k_{2N}}$	(0.4206, 51.2759)	(461.7099, 988.6250)	T_{k_2}	2.8290	382.0122
$T_{k_{5N}}$	(0.4203, 50.8376)	(462.1078, 997.1495)	T_{k_5}	2.8260	382.4165
Category III					
$T_{r_{3N}}$	(1.4511, 352.3419)	(133.8457, 143.8737)	T_{r_3}	8.2475	131.0371
$T_{lr_{3N}}$	(1.4936, 398.8702)	(130.0426, 127.0908)	T_{lr_3}	8.4441	127.9863
$T_{bt_{3N}}$	(1.4914, 383.4719)	(130.2296, 132.1941)	T_{bt_3}	8.4266	128.2527
$T_{p_{3N}}$	(1.4193, 350.3224)	(136.8426, 144.7030)	T_{p_3}	8.0661	133.9848
$T_{k_{3N}}$	(1.4172, 337.5712)	(137.0529, 150.1690)	T_{k_3}	8.0482	134.2819
$T_{k_{6N}}$	(1.4169, 336.4928)	(137.0780, 150.6502)	T_{k_6}	8.0463	134.3147

7. Conclusions

Traditional imputation approaches in survey sampling frequently fail to successfully deal with the inherent complexity and ambiguity of missing data, resulting in biased and sometimes misleading results. These approaches often make stringent assumptions about data behaviour, ignoring the uncertainty and variability that are common with real-world datasets. Neutrosophic imputation frameworks, founded on neutrosophic logic principles, offer a more complete approach by taking into account three critical dimensions, namely, truth, indeterminacy, and falsity. In this work, we presented several neutrosophic exponential imputation approaches and the resulting neutrosophic exponential estimators of population mean as well as their related properties. The suggested neutrosophic exponential estimators exhibit first-order approximation properties such as bias and MSE. The suggested neutrosophic exponential imputation approaches consistently yield more accurate and reliable estimates than the adapted neutrosophic imputation approaches under the conditions stated in the mathematical comparison. A rigorous simulation investigation of symmetric and asymmetric neutrosophic datasets supported the theoretical results. The simulation results demonstrated the superiority of the proposed neutrosophic exponential imputations over the adapted ones. Insights from neutrosophic real-world ambiguous climatic data show the applicability and practical value of these approaches, emphasising their importance in improving the quality and reliability of statistical conclusions. Thus, as long as the data is neutrosophic, the proposed neutrosophic exponential imputation approaches may be used to estimate the population mean in the situation of missing data.

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Appendix A

Proof of Theorem 2.1:

Proof. Consider the resultant neutrosophic mean estimator as

$$T_{mN} = \bar{y}_N.$$

Utilizing the notations given in (1), we express T_{mN} as follows:

$$\begin{aligned} T_{mN} &= \bar{Y}_N(1 + e_{0N}), \\ \implies T_{mN} - \bar{Y}_N &= \bar{Y}_N e_{0N}. \end{aligned} \tag{A.9}$$

Taking expectation both sides to (A.9), we get the bias of T_{mN} as

$$Bias(T_{mN}) = 0.$$

This shows that the estimator $\bar{y}_N$ is unbiased. Again, squaring and taking expectation both sides to (A.9), we get the variance of T_{mN} as

$$V(T_{mN}) = f_{r_N} \bar{Y}_N^2 C_{y_N}^2.$$

□

Appendix B

Proof of Theorem 2.2:

Proof. Consider the resulting neutrosophic ratio estimator under category I as

$$T_{r_{1N}} = \bar{y}_{r_N} \left(\frac{\bar{X}_N}{\bar{x}_{n_N}} \right).$$

Taking the notations given in (1), using Taylor series expansion, and ignoring error terms having power more than two, we express $T_{r_{1N}}$ as follows:

$$\begin{aligned} T_{r_{1N}} &\approx \bar{Y}_N(1 + e_{0N} - e_{2N} + e_{2N}^2 - e_{0N}e_{2N}), \\ \implies T_{r_{1N}} - \bar{Y}_N &\approx \bar{Y}_N(e_{0N} - e_{2N} + e_{2N}^2 - e_{0N}e_{2N}). \end{aligned} \tag{B.10}$$

Taking expectation both sides to (B.10), we get the bias of $T_{r_{1N}}$ as

$$Bias(T_{r_{1N}}) \approx \bar{Y}_N f_{n_N}(C_{x_N}^2 - \rho_{xy_N} C_{x_N} C_{y_N}).$$

Using Taylor series expansion, squaring and taking expectation both sides to (B.10), we can get the MSE of $T_{r_{1N}}$ up to the first order approximation as

$$MSE(T_{r_{1N}}) \approx f_{r_N}\bar{Y}_N^2 C_{y_N}^2 + f_{n_N}\bar{Y}_N^2(C_{x_N}^2 - 2\rho_{xy_N}C_{x_N}C_{y_N}). \tag{B.11}$$

Similarly, we can find out the bias and MSE of the resulting neutrosophic ratio estimators $T_{r_{2N}}$ and $T_{r_{3N}}$ under categories II and III. □

Appendix C

Proof of Theorem 2.3:

Proof. Consider the first resulting neutrosophic regression estimator under category I as

$$T_{lr_{1N}} = \bar{y}_{r_N} + \beta_{1N}(\bar{X}_N - \bar{x}_{n_N}).$$

Utilizing the notations given in (1), we can express $T_{lr_{1N}}$ as follows:

$$\begin{aligned} T_{lr_{1N}} &= \bar{Y}_N(1+e_{0N}) - \beta_{1N}\bar{X}_N e_{2N}, \\ \implies T_{lr_{1N}} - \bar{Y}_N &= \bar{Y}_N\left(e_{0N} - \frac{\beta_{1N}}{R_N}e_{2N}\right), \end{aligned} \tag{C.12}$$

where $R_N = \bar{Y}_N/\bar{X}_N$. Taking expectation both sides to (C.12), we get the bias of $T_{lr_{1N}}$ as

$$Bias(T_{lr_{1N}}) = 0.$$

Squaring and taking expectation both sides to (C.12), we can get the MSE of $T_{lr_{1N}}$ as

$$MSE(T_{lr_{1N}}) = \bar{Y}_N^2\left(f_{r_N}C_{y_N}^2 + \frac{\beta_{1N}^2}{R_N^2}f_{n_N}C_{x_N}^2 - 2\frac{\beta_{1N}}{R_N}f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\right). \tag{C.13}$$

Minimizing (C.13) with respect to β_{1N}, we have

$$\beta_{1N_{(opt)}} = R_N\rho_{xy_N}\frac{C_{y_N}}{C_{x_N}}.$$

The minimum MSE can be obtained by putting optimum value of β_{1N} in (C.13) as

$$min.MSE(T_{lr_{1N}}) = \bar{Y}_N^2 C_{y_N}^2(f_{r_N} - f_{n_N}\rho_{xy_N}^2).$$

Similarly, we can find out the bias and minimum MSE of the resulting neutrosophic regression estimators $T_{lr_{2N}}$ and $T_{lr_{3N}}$ under categories II and III. □

Appendix D

Proof of Theorem 2.4:

Proof. Consider the resulting neutrosophic exponential ratio estimator $T_{bt_{1N}}$ under category I as

$$T_{bt_{1N}} = \bar{y}_{r_N} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right),$$

Utilizing the notations given in (1), using Taylor series expansion, and ignoring error terms having power more than two, we express the estimator $T_{bt_{1N}}$ as

$$T_{bt_{1N}} \approx \bar{Y}_N(1+e_{0N}) \exp\left(\frac{\bar{X}_N - \bar{X}_N(1+e_{2N})}{\bar{X}_N + \bar{X}_N(1+e_{2N})}\right)$$

$$T_{bt_{1N}} \approx \bar{Y}_N(1+e_{0N}) \exp\left(\frac{-\bar{X}_N e_{2N}}{2\bar{X}_N + \bar{X}_N e_{2N}}\right)$$

$$T_{bt_{1N}} - \bar{Y}_N \approx \bar{Y}_N\left(e_{0N} - \frac{e_{2N}}{2} + \frac{3}{8}e_{2N}^2 - \frac{1}{2}e_{0N}e_{2N}\right) \tag{D.14}$$

Taking expectation both sides to (D.14), we get the bias of the estimator $T_{bt_{1N}}$ as

$$Bias(T_{bt_{1N}}) \approx f_{n_N}\bar{Y}_N\left(\frac{3}{8}C_{x_N}^2 - \frac{1}{2}\rho_{xy_N}C_{x_N}C_{y_N}\right). \tag{D.15}$$

Using Taylor series expansion, squaring and taking expectation on both sides of (D.14), we can get the MSE of the estimator $T_{bt_{1N}}$ up to first order approximation as

$$MSE(T_{bt_{1N}}) \approx \bar{Y}_N^2\left(f_{r_N}C_{y_N}^2 + \frac{1}{4}f_{n_N}C_{x_N}^2 - f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\right). \tag{D.16}$$

Similarly, we can find out the bias and minimum MSE of the resulting neutrosophic exponential ratio estimators $T_{bt_{2N}}$ and $T_{bt_{3N}}$ under category II and III. □

Appendix E

Proof of Theorem 2.5:

Proof. Consider the resulting neutrosophic power ratio-cum exponential ratio estimator $T_{p_{1N}}$ under category I as

$$T_{p_{1N}} = \bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^{\kappa_{1N}} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right),$$

Utilizing the notations given in (1), using Taylor series expansion, and ignoring error terms having power more than two, we can express the estimator $T_{p_{1N}}$ as

$$T_{p_{1N}} \approx \bar{Y}_N \left(\frac{\bar{X}_N}{\bar{X}_N(1+e_{2N})} \right)^{\kappa_{1N}} \exp\left(\frac{\bar{X}_N - \bar{X}_N(1+e_{2N})}{\bar{X}_N + \bar{X}_N(1+e_{2N})} \right)$$

$$T_{p_{1N}} \approx \bar{Y}_N (1+e_{0N})(1+e_{2N})^{-\kappa_{1N}} \exp\left(\frac{-\bar{X}_N e_{2N}}{2\bar{X}_N + \bar{X}_N e_{2N}} \right)$$

$$T_{p_{1N}} - \bar{Y}_N \approx \bar{Y}_N \left[\begin{array}{l} e_{0N} - \kappa_{1N} e_{2N} - \frac{e_{2N}}{2} + \left\{ \frac{\kappa_{1N}(\kappa_{1N}+1)}{2} + \frac{\kappa_{1N}}{2} + \frac{3}{8} \right\} e_{2N}^2 \\ - \left(\kappa_{1N} + \frac{1}{2} \right) e_{0N} e_{2N} \end{array} \right] \tag{E.17}$$

Taking expectation both sides to (E.17), we get the bias of the estimator $T_{p_{1N}}$ as

$$Bias(T_{p_{1N}}) \approx \bar{Y}_N \left[\left(\frac{\kappa_{1N}(\kappa_{1N}+1)}{2} + \frac{\kappa_{1N}}{2} + \frac{3}{8} \right) f_{n_N} C_{x_N}^2 - \left(\kappa_{1N} + \frac{1}{2} \right) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N} \right].$$

Using Taylor series expansion, squaring and taking expectation both sides to (E.17), we can get the MSE of the estimator $T_{p_{1N}}$ up to first order approximation as

$$MSE(T_{p_{1N}}) \approx \bar{Y}_N^2 \left[1 + f_{r_N} C_{y_N}^2 + \left(\kappa_{1N}^2 + \kappa_{1N} + \frac{1}{4} \right) f_{n_N} C_{x_N}^2 - (2\kappa_{1N}+1) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N} \right] \tag{E.18}$$

Minimizing (E.18) with respect to κ_{1N}, we have

$$\kappa_{1N(opt)} = \rho_{xy_N} \frac{C_{y_N}}{C_{x_N}} - \frac{1}{2}.$$

The minimum MSE can be obtained by putting optimum value of κ_{1N} in (E.18) as

$$min.MSE(T_{p_{1N}}) \approx \bar{Y}_N^2 C_{y_N}^2 (f_{r_N} - f_{n_N} \rho_{xy_N}^2). \tag{E.19}$$

Similarly, we can find out the bias and minimum MSE of the resulting neutrosophic power ratio-cum exponential ratio estimators $T_{p_{2N}}$ and $T_{p_{3N}}$ under categories II and III. □

Appendix F

Proof of Theorem 3.1:

Proof. Consider the proposed resulting neutrosophic exponential estimator $T_{k_{1N}}$ under category I as

$$T_{k_{1N}} = \Delta_{1N} \bar{y}_{r_N} \left(\frac{\bar{X}_N}{\bar{x}_{n_N}} \right)^{\zeta_{1N}} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}} \right).$$

Taking the notations given in (1), using Taylor series expansion, and ignoring the error terms having power more than two, we can express the estimator $T_{k_{1N}}$ as

$$T_{k_{1N}} \approx \Delta_{1N} \bar{Y}_N \left(\frac{\bar{X}_N}{\bar{X}_N(1+e_{2N})} \right)^{\zeta_{1N}} \exp\left(\frac{\bar{X}_N - \bar{X}_N(1+e_{2N})}{\bar{X}_N + \bar{X}_N(1+e_{2N})} \right)$$

$$T_{k_{1N}} \approx \Delta_{1N}\bar{Y}_N(1+e_{0N})(1+e_{2N})^{-\zeta_{1N}} \exp\left(\frac{-\bar{X}_N e_{2N}}{2\bar{X}_N + \bar{X}_N e_{2N}}\right)$$

$$T_{k_{1N}} - \bar{Y}_N \approx \bar{Y}_N \left[\Delta_{1N} \left\{ \begin{array}{l} 1 + e_{0N} - \zeta_{1N}e_{2N} - \frac{e_{2N}}{2} + \left(\frac{\zeta_{1N}(\zeta_{1N}+1)}{2} + \frac{\zeta_{1N}}{2} + \frac{3}{8}\right) e_{2N}^2 \\ -\left(\zeta_{1N} + \frac{1}{2}\right) e_{0N}e_{2N} \end{array} \right\} - 1\right] \tag{F.20}$$

Taking expectation both sides to (F.20), we get the bias of the estimator $T_{k_{1N}}$ as

$$Bias(T_{k_{1N}}) = \bar{Y}_N \left(\Delta_{1N}G_{1N} - 1\right).$$

where

$$G_{1N} = \bar{Y}_N \left[1 + \left\{\frac{\zeta_{1N}(\zeta_{1N}+1)}{2} + \frac{\zeta_{1N}}{2} + \frac{3}{8}\right\} f_{n_N} C_{x_N}^2 - \left(\zeta_{1N} + \frac{1}{2}\right) f_{n_N}\rho_{xy_N} C_{x_N} C_{y_N}\right].$$

Squaring and taking expectation both sides to (F.20), we can get the MSE of the estimator $T_{k_{1N}}$ as

$$MSE(T_{k_{1N}}) \approx \bar{Y}_N^2 \left[\begin{array}{l} 1 + \Delta_{1N}^2 \left\{ \begin{array}{l} 1 + f_{r_N}C_{y_N}^2 + (2\zeta_{1N}^2 + 3\zeta_{1N} + 1) f_{n_N} C_{x_N}^2 \\ -2(2\zeta_{1N}+1) f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N} \end{array} \right\} \\ -2\Delta_{1N} \left\{ \begin{array}{l} 1 + \left(\frac{\zeta_{1N}(\zeta_{1N}+1)}{2} + \frac{\zeta_{1N}}{2} + \frac{3}{8}\right) f_{n_N} C_{x_N}^2 \\ -\left(\zeta_{1N} + \frac{1}{2}\right) f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N} \end{array} \right\} \end{array} \right]$$

The above expression can further be written as

$$MSE(T_{k_{1N}}) = \bar{Y}_N^2 \left(1 + \Delta_{1N}^2 F_{1N} - 2\Delta_{1N}G_{1N}\right). \tag{F.21}$$

where

$$F_{1N} = 1 + f_{r_N}C_{y_N}^2 + (2\zeta_{1N}^2 + 3\zeta_{1N} + 1) f_{n_N} C_{x_N}^2 - 2(2\zeta_{1N}+1) f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N},$$
$$G_{1N} = 1 + \left(\frac{\zeta_{1N}(\zeta_{1N}+1)}{2} + \frac{\zeta_{1N}}{2} + \frac{3}{8}\right) f_{n_N} C_{x_N}^2 - \left(\zeta_{1N} + \frac{1}{2}\right) f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}.$$

Minimizing (F.21) with respect to Δ_{1N} and ζ_{1N}, we have

$$\Delta_{1N(opt)} = \frac{G_{1N}}{F_{1N}},\ \zeta_{1N(opt)} = \rho_{xy_N}\frac{C_{y_N}}{C_{x_N}} - \frac{1}{2}.$$

The minimum MSE can be obtained by putting optimum values of Δ_{1N} and ζ_{1N} in (F.21) as

$$min.MSE(T_{k_{1N}}) = \bar{Y}_{1N}^2 \left(1 - \frac{G_{1N}^2}{F_{1N}}\right).$$

Similarly, we can find out the bias, MSE, and minimum MSE of the resulting proposed neutrosophic estimators T_{k_2} and T_{k_3} under categories II and III. □

Appendix G

Proof of Theorem 3.2:

Proof. Consider the proposed resulting neutrosophic exponential estimator $T_{k_{4N}}$ under category I as

$$T_{pk4N} = \left\{ \Delta_{4N}\bar{y}_{r_N} + \zeta_{4N}\bar{y}_{r_N}\left(\frac{\bar{X}_N}{\bar{x}_{n_N}}\right)^g \right\} \exp\left(\frac{\bar{X}_N - \bar{x}_{n_N}}{\bar{X}_N + \bar{x}_{n_N}}\right).$$

Taking the notations given in (1), using Taylor series expansion, and ignoring the error terms having power more than two, we can express the estimator $T_{k_{4N}}$ as

$$T_{k_{4N}} \approx \bar{Y}_N(1+e_{0N})\left(\Delta_{4N} + \zeta_{4N}(1+e_{2N})^{-g}\right)\left(1 - \frac{e_{2N}}{2} + \frac{3}{8}e_{2N}^2\right),$$

$$T_{k_{4N}} \approx \bar{Y}_N(1+e_{0N})\left(\Delta_{4N} + \zeta_{4N} - \zeta_{4N} g e_{2N} + \frac{\zeta_{4N} g(g+1)}{2}e_{2N}^2\right)\left(1 - \frac{e_{2N}}{2} + \frac{3}{8}e_{2N}^2\right),$$

$$T_{k_{4N}} - \bar{Y}_N \approx \bar{Y}_N \left[\begin{array}{l} \Delta_{4N}\left\{\ 1 + e_{0N} - \frac{e_{2N}}{2} - \frac{e_{0_N}e_{2N}}{2} + \frac{3}{8}e_{2N}^2\ \right\} \\ +\zeta_{4N}\left\{\begin{array}{l} 1 + e_{0N} - g e_{2N} - \frac{e_{2N}}{2} + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8}\right)e_{2N}^2 \\ -\left(g+\frac{1}{2}\right)e_{0_N}e_{2N} \end{array}\right\} - 1 \end{array}\right] \quad \text{(G.22)}$$

Taking expectation both sides to (G.22), we get the bias of the estimator $T_{k_{4N}}$ as

$$Bias(T_{p_{4N}}) \approx \bar{Y}_N(\Delta_{4N}L_{4N} + \zeta_{4N}M_{4N} - 1).$$

where

$$L_{4N} = 1 + \frac{3}{8}f_{n_N}C_{x_N}^2 - \frac{1}{2}f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N},$$

$$M_{4N} = 1 + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8}\right)f_{n_N}C_{x_N}^2 - \left(g + \frac{1}{2}\right)f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}.$$

Squaring and taking expectation both sides to (G.22), we can get the MSE of the estimator $T_{k_{4N}}$ up to first order approximation as

$$MSE(T_{k_{4N}}) \approx \bar{Y}_N^2 \left[\begin{array}{l} 1 + \Delta_{4N}^2\left\{1 + f_{r_N}C_{y_N}^2 + f_{n_N}C_{x_N}^2 - 2f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\right\} \\ +\zeta_{4N}^2\left\{\ 1 + f_{r_N}C_{y_N}^2 + (2g^2+3g+1)f_{n_N}C_{x_N}^2 - (4g+2)f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\ \right\} \\ +2\Delta_{4N}\zeta_{4N}\left\{\begin{array}{l} 1 + f_{r_N}C_{y_N}^2 + \left(\frac{g(g+1)}{2} + g + 1\right)f_{n_N}C_{x_N}^2 \\ -2(g+1)f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N} \end{array}\right\} \\ -2\Delta_{4N}\left\{\ 1 + \frac{3}{8}f_{n_N}C_{x_N}^2 - \frac{1}{2}f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\ \right\} \\ -2\zeta_{4N}\left\{\ 1 + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8}\right)f_{n_N}C_{x_N}^2 - \left(g+\frac{1}{2}\right)f_{n_N}\rho_{xy_N}C_{x_N}C_{y_N}\ \right\} \end{array}\right].$$

The above expression can further be written as

$$MSE(T_{k_{4N}}) \approx \bar{Y}_N^2(1 + \Delta_{4N}^2 I_{4N} + \theta_{4N}^2 J_{4N} + 2\Delta_{4N}\zeta_{4N}K_{4N} - 2\Delta_{4N}L_{4N} - 2\zeta_{4N}M_{4N}). \quad \text{(G.23)}$$

where

$$I_{4N} = 1 + f_{r_N} C_{y_N}^2 + f_{n_N} C_{x_N}^2 - 2 f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$J_{4N} = 1 + f_{r_N} C_{y_N}^2 + (2g^2 + 3g + 1) f_{n_N} C_{x_N}^2 - (4g + 2) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$K_{4N} = 1 + f_{r_N} C_{y_N}^2 + \left(\frac{g(g+1)}{2} + g + 1 \right) f_{n_N} C_{x_N}^2 - 2(g+1) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$L_{4N} = 1 + \frac{3}{8} f_{n_N} C_{x_N}^2 - \frac{1}{2} f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N},$$
$$M_{4N} = 1 + \left(\frac{g(g+1)}{2} + \frac{g}{2} + \frac{3}{8} \right) f_{n_N} C_{x_N}^2 - \left(g + \frac{1}{2} \right) f_{n_N} \rho_{xy_N} C_{x_N} C_{y_N}.$$

Minimizing (G.23) with respect to Δ_{4N} and ζ_{4N}, we have

$$\Delta_{4N(opt)} = \frac{J_{4N} L_{4N} - K_{4N} M_{4N}}{I_{4N} J_{4N} - K_{4N}^2} \text{ and } \theta_{4N(opt)} = \frac{I_{4N} M_{4N} - K_{4N} L_{4N}}{I_{4N} J_{4N} - K_{4N}^2}.$$

The minimum MSE can be obtained by putting optimum values of Δ_{4N} and ζ_{4N} in (G.23) as

$$min.MSE(T_{k_{4N}}) \approx \bar{Y}_N^2 \left(1 - \frac{I_{4N} M_{4N}^2 + J_{4N} L_{4N}^2 - 2 K_{4N} L_{4N} M_{4N}}{I_{4N} J_{4N} - K_{4N}^2} \right).$$

Similarly, we can find out the bias, MSE, and minimum MSE of the estimators $T_{k_{5N}}$ and $T_{k_{6N}}$ under categories II and III. □

Appendix H

The data set utilized in Section 6 is displayed in Table 7 for reference.

TABLE 7. The climate data is taken for November month from Georgia state of USA

Station Name	Min Dew Point Temp (F)	Max Dew Point Temp (F)	Min RH (%)	Max RH (%)
ALBANY SW GA RGNL AP	19	68	22	100
AUGUSTA BUSH FLD AP	13	66	16	100
ATHENS BEN EPPS AP	11	64	16	100
ALMA BACON CO AP	21	67	21	100
ATLANTA HARTSFIELD INTL AP	14	64	22	94
COLUMBUS METRO AP	16	66	18	100
AUGUSTA DANIEL FLD AP	13	65	16	100
PEACHTREE CITY FALCON FLD	10	65	17	100
ATLANTA FULTON CO AP	14	64	22	100
GAINESVILLE GILMER AP	12	61	19	100
FT BENNING LAWSON FLD	23	66	28	100
MACON MIDDLE GA RGNL AP	16	67	19	100
MARIETTA DOBBINS AFB	36	47	59	87
ATLANTA PEACHTREE AP	10	64	20	100
ROME R B RUSSELL AP	13	63	19	94
SAVANNAH INT'L AP	22	67	15	100
BRUNSWICK MALCOLM MCKINNON AP	47	56	59	73
VALDOSTA MOODY AFB	50	66	88	94
VALDOSTA RGNL AP	24	71	26	100
WARNER ROBINS' AFB	43	66	72	94
BRUNSWICK 23 S	46	55	67	74
NEWTON 8 W	23	31	38	55

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