



Analysis of (1,2,3,4)-Type Inverses in Neutrosophic Hyper Soft Rough Fuzzy Matrices

G.Punithavalli¹, K. Ramany^{2,3}

¹Department of Mathematics, Annamalai University, Deputed to PG and Research Department of Mathematics, Government Arts College, Chidambaram.

punithavarman78@gmail.com

²Research Scholar, PG & Research Department of Mathematics, Government Arts College, Chidambaram.

³Department of Mathematics, Tagore Government Arts and Science College, Puducherry.

skramany@gmail.com

Abstract: In this paper, we introduce a method for determining the generalized inverse (g-inverse) and the Moore-Penrose inverse of Neutrosophic Hyper Soft Rough Fuzzy Matrices (NHSRFMs), along with the necessary conditions. Furthermore, no algorithm currently exists to find the g-inverse of an NHSRFM. In this study, we present an algorithm to evaluate the g-inverse of an NHSRFM. Several properties and results related to the g-inverse of NHSRFMs are explored. This paper concludes with an application of the g-inverse, supported by numerical examples that illustrate the theorems, algorithm, and application.

Keywords: NHSRFM, generalized inverse (g-inverse), Moore-Penrose inverse, minus ordering.

1. Introduction

The generalized inverse (g-inverse) extends the concept of a matrix inverse to singular or non-square matrices. Similar to the traditional matrix inverse, the g-inverse plays a crucial role in various fields, including control systems, robotics, signal processing, and associative memories. The foundation of fuzzy set theory was laid by Zadeh [1], who introduced the concept of fuzzy sets, paving the way for handling uncertainty in mathematical frameworks. Building on this, Atanassov [2] proposed intuitionistic fuzzy sets, an extension that incorporates both membership and non-membership values. Later, Smarandache [3] generalized these ideas by introducing neutrosophic sets, which consider three components: truth, indeterminacy, and falsity.

The study of fuzzy matrices began with investigations into generalized inverses by Cen [4,5], followed by research on matrix partial orders through generalized inverses by Mitra [8]. Khan and Pal [6] introduced intuitionistic fuzzy tautological matrices, while Shyamal and Pal [14] explored new operators on fuzzy matrices. Advances in interval-valued intuitionistic fuzzy matrices were made by Shyamal and Pal [22,23], while Bhowmik and Pal [26,28] contributed significantly to the generalization of interval-valued intuitionistic fuzzy matrices and their properties.

In recent years, neutrosophic fuzzy matrices have gained prominence. Anandhkumar et al. introduced pseudo similarity measures for neutrosophic fuzzy matrices [9], explored various types of inverses [10], and developed concepts like reverse sharp and left-T right-T partial ordering [11]. Further advancements include k-idempotent neutrosophic fuzzy matrices [19], secondary k-range symmetric matrices [20,31,32], and generalized symmetric Fermatean neutrosophic fuzzy matrices [33]. These studies demonstrate the flexibility and applicability of neutrosophic fuzzy matrices in handling indeterminate and inconsistent information. Other notable contributions include the introduction of the minus partial order in fuzzy matrices by Meenakhi and Inbam [7], distance measures for intuitionistic fuzzy matrices by Shyamal and Pal [22,23], and work on controllable fuzzy matrices by Xin [12].

Al-Quran et al. [42] proposed a novel approach by combining neutrosophic sets with soft rough sets under uncertainty. Their framework provided a powerful mechanism for dealing with both indeterminacy and approximation, offering more flexible tools for information processing. This advancement was further expanded by Das et al. [43], who explored the integration of neutrosophic soft sets with rough set theory. Their contribution established foundational insights into how these theories could be effectively merged to address multi-faceted uncertainties in real-world applications.

Additionally, Kamacı [44] introduced hybrid structures involving hypersoft sets and rough sets, enhancing the structural granularity and extending the applicability of soft set theory. This development aligns with Smarandache's [45] pioneering work on extending soft sets to hypersoft sets and subsequently to plithogenic hypersoft sets, where attributes, values, and parameters themselves possess degrees of truth, indeterminacy, and falsity. These extended models facilitate a richer and more nuanced understanding of complex decision-making scenarios.

The fusion of soft set theory with fuzzy, rough, and neutrosophic concepts has led to the development of powerful decision-making tools. Cagman and Enginoglu [36] and Borah et al. [41] introduced the concept of fuzzy soft matrix theory and demonstrated its application in decision-making. Muthukumar and Krishnan [34] extended this by developing generalized fuzzy soft rough matrices with practical applications. Al-Quran et al. [40] proposed a neutrosophic soft rough set model to manage uncertainty more effectively. Das et al. [37] presented an approach that integrates neutrosophic soft sets with rough set theory. Jafar and Saeed [38] contributed a matrix-based framework for neutrosophic hypersoft sets and applied it to multi-attributive and multi-criteria decision-making problems. Martina and Deepa [35] explored the algebraic structures of rough neutrosophic matrices and highlighted their decision-making relevance. Jayasudha and Raghavi [39] further enriched this domain by introducing operations on neutrosophic hypersoft matrices with validated applications.

The authors of [4, 5] introduced the concept of the generalized inverse for fuzzy matrices and derived numerous theoretical results based on these definitions. However, to the best of our knowledge, there are no existing algorithms specifically designed for determining the generalized inverse of a fuzzy matrix. In this study, we extend this concept to Neutrosophic Fuzzy Matrices (HSRFMs) and propose a straightforward algorithm to compute their generalized inverse. Additionally, we demonstrate the application of this method by solving a rectangular system of Neutrosophic fuzzy relational equations using the g-inverse of HSRFMs.

In recent years, extensive research has been carried out on neutrosophic fuzzy matrices (NFM) and their various extensions, addressing both theoretical developments and applications. Harikrishnan et al. [46] introduced novel compositions of NFM using Min(Max)-Min(Max)-Max(Min)(*) operators, demonstrating their utility in medical diagnosis problems. Shyamala et al. [47] focused on the decomposition of NFM using α -cuts, enriching the structural analysis of these matrices. Anandhkumar et al. [48] developed the determinant theory of quadri-partitioned NFM (QPNFM) and applied it to multi-criteria decision-making (MCDM), while Radhika et al. [49] explored the Schur complement in k-kernel symmetric block QPNFM, extending matrix theory in the neutrosophic framework.

Further, Radhika et al. [50] investigated interval-valued secondary k-range symmetric QPNFM and highlighted their decision-making applications. Punithavalli and Anandhkumar [51, 52] studied partial orderings, kernel, and k-kernel symmetric structures in intuitionistic fuzzy matrices (IFM), bridging intuitionistic and neutrosophic frameworks. Expanding on these, Anandhkumar et al. [53] examined transitive and strongly transitive NFM, providing fundamental properties for theoretical advancements. Murugadas et al. [54] extended the study to interval-valued kernel symmetric, k-kernel symmetric, range symmetric, and column symmetric NFM, showing their versatility. Additionally, Prathab et al. [55] analyzed interval-valued secondary k-range symmetric fuzzy matrices with generalized inverses, emphasizing the role of inverse operations in fuzzy and neutrosophic contexts.

1.1 Abbreviations

FM: Fuzzy Matrices

IFM: Intuitionistic Fuzzy Matrices

IFSs: Intuitionistic Fuzzy Sets

NFSs: Neutrosophic fuzzy Sets

NFM: Neutrosophic fuzzy matrices.

NHSRFM = HSRFM = Neutrosophic Hyper Soft Rough Fuzzy Matrices

HSRFPM = Neutrosophic Hyper Soft Rough Fuzzy Permutation Matrices

2. Contribution of Our Work:

- (i) **Novel Algorithm:** We propose a straightforward and efficient algorithm for computing the generalized inverse (g-inverse) of Neutrosophic Fuzzy Matrices (HSRFMs), filling the gap in existing literature that lacks such computational methods.
- (ii) **Extension to Moore-Penrose Inverse:** The study extends the concept of the g-inverse to include the Moore-Penrose inverse for HSRFMs, providing a comprehensive approach to matrix inverses in neutrosophic fuzzy environments.
- (iii) **Exploration of Properties:** We examine and establish several theoretical results regarding the g-inverse of HSRFMs, contributing to the foundational understanding of this concept.
- (iv) **Application to Relational Equations:** An application of the g-inverse is demonstrated by solving a rectangular system of neutrosophic fuzzy relational equations, showcasing its practical utility.
- (v) **Numerical Validation:** The theorems, algorithm, and application are supported by detailed numerical examples, ensuring clarity and practical relevance for researchers and practitioners.

3. Literature Review

Fuzzy matrix theory has evolved significantly since the inception of fuzzy set theory by Zadeh [1], which introduced a mathematical framework for handling uncertainty. The subsequent development of intuitionistic fuzzy sets by Atanassov [2] added an additional layer of complexity by incorporating both membership and non-membership values. Further extending these concepts, Smarandache [3] introduced neutrosophic sets, which include truth, indeterminacy, and falsity components, offering greater flexibility for modeling uncertain and inconsistent data. The foundational work on fuzzy matrices includes the exploration of their generalized inverses by Cen [4,5] and matrix partial orders through generalized inverses by Mitra [8]. Shyamal and Pal [14] expanded this domain by introducing new operators on fuzzy matrices, while Meenakhi and Inbam [7] proposed the minus partial order for fuzzy matrices. Additionally, the convergence properties of controllable fuzzy matrices were examined by Xin [12].

Intuitionistic fuzzy matrices have also been a focus of research. Khan and Pal [6] studied tautological matrices, and Pal [17] introduced intuitionistic fuzzy determinants. Shyamal and Pal [22,23] developed distance measures for intuitionistic fuzzy matrices, providing tools for comparing these mathematical structures. Bhowmik and Pal [26,28] further extended this field by exploring generalized and interval-valued intuitionistic fuzzy matrices, highlighting their potential applications in uncertainty modeling. Neutrosophic fuzzy matrices, as a generalization of intuitionistic fuzzy matrices, have become a crucial area of study. Recent contributions by Anandhkumar and colleagues include pseudo similarity measures for neutrosophic fuzzy matrices [9], the development of various types of inverses [10], and partial ordering on such matrices [11].

They also introduced k-idempotent neutrosophic fuzzy matrices [19], secondary k-range symmetric neutrosophic fuzzy matrices [20,31,32], and generalized symmetric Fermatean neutrosophic fuzzy matrices [33]. These advancements have expanded the theoretical underpinnings and applications of neutrosophic fuzzy matrices.

Other notable works include Shyamal and Pal’s [13] research on interval-valued fuzzy matrices and their operators, and Khan and Pal’s [29] study on the generalized inverse of intuitionistic fuzzy matrices. The exploration of intuitionistic fuzzy relations by Panigrahi and Nanda [16] and the semiring properties of intuitionistic fuzzy matrices by Sriram and Murugadas [24] have further enriched the field. This literature highlights the progressive development of fuzzy, intuitionistic fuzzy, and neutrosophic fuzzy matrices, showcasing their versatility and applicability across various domains. The present study builds on these contributions, addressing gaps in the literature and introducing new compositions and properties of neutrosophic fuzzy matrices. This work aims to extend the theoretical foundation and practical applications of these mathematical constructs.

4. Comparative of HSRFM model with the existing soft models

Types of soft set	Uncertainty	Falsity	Hesitation	Indeterminacy
FSS [5]	✓	×	×	×
IVFSS [18]	✓	×	×	×
IFSS [2]	✓	✓	✓	×
IVIFSS [3]	✓	✓	✓	×
NSS [34]	✓	✓	×	✓
HSRFMs	✓	✓	✓	✓

Generalized Inverse (g-inverse)

A **generalized inverse** of a matrix A is any matrix G (not necessarily unique) that satisfies: $AGA=A$

- That’s the only condition required.
- Many matrices can satisfy this property, so the generalized inverse is **not unique**.

- It is often used when solving **inconsistent or underdetermined systems of linear equations**.

Moore–Penrose Inverse (MP-inverse or pseudoinverse)

The **Moore–Penrose inverse** A^+ is a **special case** of a generalized inverse.

It is the **unique matrix** that satisfies the following four Penrose conditions:

$$AXA = A, XAX=X, (AX)^T = AX, (XA)^T = XA$$

5. Novelty

The novelty of the references lies in their significant contributions to the development and advancement of fuzzy, intuitionistic fuzzy, and neutrosophic fuzzy matrices. Zadeh [1] laid the foundational work with fuzzy sets, introducing the concept of imprecision in mathematical systems, while Atanassov [2] extended this framework with intuitionistic fuzzy sets, incorporating the idea of non-membership to better handle uncertainty. Smarandache [3] further advanced this by introducing neutrosophic sets, adding an additional dimension of indeterminacy, which later led to the creation of neutrosophic fuzzy matrices (HSRFMs). Cen [4, 5] expanded on the structure of fuzzy matrices, exploring partial ordering and generalized inverses, key for computational matrix applications. Khan and Pal [6] contributed to the theory of intuitionistic fuzzy tautological matrices, offering new insights into their logical properties. Shyamal and Pal [13, 22, 23, 26, 27] introduced interval-valued fuzzy matrices and developed distance metrics and operators, enriching fuzzy system manipulations. Mitra [8] unified partial ordering through generalized inverses, providing a robust framework for linear algebra operations. Dehghan et al. [15] addressed the challenges of inverting fuzzy matrices with fuzzy numbers, offering new computational methods. Finally, the works of Anandhkumar et al. [9-11, 19, 20, 21, 31, 32, 33] significantly expanded the field by introducing pseudo-similarity, new inverse structures, and advanced partial ordering techniques for neutrosophic fuzzy matrices. Their contributions, including secondary k -range symmetry, generalized symmetric matrices, and advanced ordering concepts, have paved the way for the introduction of novel methods for handling uncertainty and improving matrix theory through new compositional operators and symmetry conditions.

6. Preliminaries

Neutrosophic Hyper Soft Rough Matrix (NHSRM)

In this section, the definition, basic operations and various properties of NHSRM are presented.

Definition 6.1 (NHSRM) Let $U = \{u_1, u_2, \dots, u_n\}$ be an non-empty universe set and

$P(U)$ be the power set of U . Let E be the set of parameters $E = \{A_1, A_2, \dots, A_n\}$, where $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let $S_j \subseteq A_j, j \in \{1, 2, \dots, n\}$ then $\prod_{j=1}^n S_j^k \subseteq \prod_{j=1}^n A_j^k$.

The pair $(\varphi, \prod_{j=1}^n S_j^k) = P(U)$, where φ is a mapping defined by $\varphi: \prod_{j=1}^n S_j^k \rightarrow P(U)$ is called neutrosophic hyper soft rough set. Each element $u \in U$ is associated with the values determined by the hyper soft set, where each parameter can take multiple values. For each element $u \in U$ related with a parameter A_j is represented by the triplet (T_{ij}, I_{ij}, F_{ij}) where T_{ij} is the truth membership function, I_{ij} is the indeterminacy membership function and F_{ij} is the falsity membership function, $T_{ij}, I_{ij}, F_{ij} \in [0, 1]$.

If $P_{ij} = \gamma(u_i, A_j^k)$, where $i = 1, 2, 3, \dots, m$, $j = 1, 2, 3, \dots, n$ and $k = q_1, q_2, q_3, \dots, q_n$ then a NHSRM is defined as

$$P = [P_{ij}] = \begin{bmatrix} \langle \underline{P}_{11}; \bar{P}_{11} \rangle & \langle \underline{P}_{12}; \bar{P}_{12} \rangle & \dots & \langle \underline{P}_{1n}; \bar{P}_{1n} \rangle \\ \langle \underline{P}_{21}; \bar{P}_{21} \rangle & \langle \underline{P}_{22}; \bar{P}_{22} \rangle & \dots & \langle \underline{P}_{2n}; \bar{P}_{2n} \rangle \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \langle \underline{P}_{m1}; \bar{P}_{m1} \rangle & \langle \underline{P}_{m2}; \bar{P}_{m2} \rangle & \dots & \langle \underline{P}_{mn}; \bar{P}_{mn} \rangle \end{bmatrix}$$

Lower Approximation matrix is denoted by $\underline{P}_{ij} = (\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P), 0 \leq \underline{T}_{ij}^P + \underline{I}_{ij}^P + \underline{F}_{ij}^P \leq 3$ and

Upper Approximation matrix is denoted by $\bar{P}_{ij} = (\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P), 0 \leq \bar{T}_{ij}^P + \bar{I}_{ij}^P + \bar{F}_{ij}^P \leq 3$

Thus, we can represent any neutrosophic Hyper Soft Rough Set in term of neutrosophic Fuzzy Hyper Soft Rough Matrix.

Definition 6.2 (Addition) Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$ be two NHSRMs of same order, where

$$[P_{ij}] = \left[(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P); (\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P) \right] \text{ and } [Q_{ij}] = \left[(\underline{T}_{ij}^Q, \underline{I}_{ij}^Q, \underline{F}_{ij}^Q); (\bar{T}_{ij}^Q, \bar{I}_{ij}^Q, \bar{F}_{ij}^Q) \right]$$

we define the addition in NHSRM as follows:

$$P + Q = \left[\begin{array}{l} \max(\underline{T}_{ij}^P, \underline{T}_{ij}^Q), \max(\underline{I}_{ij}^P, \underline{I}_{ij}^Q), \min(\underline{F}_{ij}^P, \underline{F}_{ij}^Q); \max(\bar{T}_{ij}^P, \bar{T}_{ij}^Q), \\ \max(\bar{I}_{ij}^P, \bar{I}_{ij}^Q), \min(\bar{F}_{ij}^P, \bar{F}_{ij}^Q) \end{array} \right]$$

Definition 13 (Multiplication) Let $P = [P_{ij}]$ and $Q = [Q_{ij}]$ be two NHSRMs of same order,

where $[P_{ij}] = \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P \right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P \right) \right]$ and

$[Q_{ij}] = \left[\left(\underline{T}_{ij}^Q, \underline{I}_{ij}^Q, \underline{F}_{ij}^Q \right); \left(\bar{T}_{ij}^Q, \bar{I}_{ij}^Q, \bar{F}_{ij}^Q \right) \right]$ we define the multiplication in NHSRM as follows:

$$P.Q = \left[\min(\underline{T}_{ij}^P, \underline{T}_{ij}^Q), \min(\underline{I}_{ij}^P, \underline{I}_{ij}^Q), \max(\underline{F}_{ij}^P, \underline{F}_{ij}^Q); \min(\bar{T}_{ij}^P, \bar{T}_{ij}^Q), \min(\bar{I}_{ij}^P, \bar{I}_{ij}^Q), \max(\bar{F}_{ij}^P, \bar{F}_{ij}^Q) \right]$$

Definition 14 (Transpose) The transpose of a square NHSRM of order $n \times n$ is obtained by switching its rows and columns, similar to the transpose operation in standard matrices. This operation allows us to examine relationships from a different perspective, effectively interchanging the roles of elements and hierarchical parameters.

$$[P_{ij}]^T = \left[\left(\underline{T}_{ji}^P, \underline{I}_{ji}^P, \underline{F}_{ji}^P \right); \left(\bar{T}_{ji}^P, \bar{I}_{ji}^P, \bar{F}_{ji}^P \right) \right]$$

Definition 15 (idempotent)

Let $A = [a_{ij}]$ be a Hexa Symmetric Neutrosophic Fuzzy Matrix (HSNFM) of order n , where each entry $a_{ij} = (T_{ij}, I_{ij}, F_{ij})$ represents the truth-membership, indeterminacy-membership, and falsity-membership degrees, with $T_{ij}, I_{ij}, F_{ij} \in [0, 1], 0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3$. The matrix A is said to be idempotent if $A \circ A = A$

Definition 16 (Diagonal NHSRM) A square NHSRM of order $n \times n$ is said to be a diagonal NHSRM, if all of its non-diagonal elements are $(0, 0, 1)$ and the main diagonal elements are considered as non-zero.

Definition 17 (Scalar Multiplication) Let $P = [P_{ij}]$ be the NHSRFM of order $m \times n$, where

$[P_{ij}] = \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P \right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P \right) \right]$ and $\alpha \in [0, 1]$ is the scalar then scalar multiplication

of NHSRFM P is given by $\alpha P = [\alpha P_{ij}] = \left[\left(\alpha \underline{T}_{ij}^P, \alpha \underline{I}_{ij}^P, \alpha \underline{F}_{ij}^P \right); \left(\alpha \bar{T}_{ij}^P, \alpha \bar{I}_{ij}^P, \alpha \bar{F}_{ij}^P \right) \right]$

Definition 18 (Complement) Let $P = [P_{ij}]$ be the NHSRFM of order $m \times n$, where

$[P_{ij}] = \left[\left(\underline{T}_{ij}^P, \underline{I}_{ij}^P, \underline{F}_{ij}^P \right); \left(\bar{T}_{ij}^P, \bar{I}_{ij}^P, \bar{F}_{ij}^P \right) \right]$

then $P^c = \left[\left(\underline{F}_{ij}^P, 1 - \underline{I}_{ij}^P, \underline{T}_{ij}^P \right); \left(\bar{F}_{ij}^P, 1 - \bar{I}_{ij}^P, \bar{T}_{ij}^P \right) \right]$ is the complement of P .

7. Generalized Inverse:

In this section, the generalized inverse of an NHSRFM is investigated.

Definition 7.1 (Generalized inverse For a NHSRFM)

Let $[P_{ij}] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \in (HSRFM)_{mn}$ is said to be regular or 1-inverse if

there exists another HSRFM, $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \in (HSRFM)_{nm}$

Such-that

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \end{aligned}$$

In this case, $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right]$ is called a generalized inverse (g-inverse) of

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \text{ and it is denoted by } \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]^{-}.$$

The g-inverse of an HSRFM is not unique that is a HSRFM has many g-inverses. The set of all such

g-inverses of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$ are denoted by

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \{1\}.$$

Definition 7.2. For a HSRFM $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \in (NHSRFM)_{mn}$ and another

NHSRFM, $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \in (HSRFM)_{nm}$ is said to be outer inverse or 2-

inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$, if

$$\begin{aligned} & \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \\ & = \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \end{aligned}$$

and is denoted by $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \{2\}.$

The HSRFM $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right]$ is said to be {1,2} inverse or semi inverse of

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right], \text{ if } \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$$

$$\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\overline{g}_{ij}^T, \overline{g}_{ij}^I, \overline{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$$

$$= \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$$

and $\left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right] \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$
 $\left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right]$
 $= \left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right]$ is denoted by $\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \{1,2\}$.

The HSRFM $\left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right]$ is said to be {1,3} inverse or **least square g-inverse** of $\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$ if,

$$\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right] \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$$

$$= \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$$

and $\left[\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right] \right]^T$
 $= \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right]$ and is denoted by
 $\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \{1,3\}$.

Again $\left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right]$ is said to be {1,4} inverse or **minimum norm g-inverse** of $\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$ if,

$$\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right] \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$$

$$= \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$$

and $\left[\left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right] \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \right]^T$
 $= \left[\left(\underline{g}_j^T, \underline{g}_j^I, \underline{g}_j^F \right); \left(\overline{g}_j^T, \overline{g}_j^I, \overline{g}_j^F \right) \right] \left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right]$ is denoted by
 $\left[\left(\underline{u}_j^T, \underline{u}_j^I, \underline{u}_j^F \right); \left(\overline{u}_j^T, \overline{u}_j^I, \overline{u}_j^F \right) \right] \{1,4\}$.

No algorithm is available to find g-inverse of NHSRFM. Here we present a simple algorithm to evaluate g-inverse of an NHSRFM.

7.1 Algorithm (To find the g-inverse of an HSRNFM)

Step 1: Check whether the non-zero rows of HSRFM $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ form a standard basis or not for the row space of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$.

Step 2: If non-zero rows form a standard basis then find some NFPM $\left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right]$ such that

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \\ = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right].$$

Step 3: Choose an HSRFM $\left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right]$ such that

$$\left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right].$$

Step4: Then $\left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right]$ is a g-inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$.

The matrix $\left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right]$ is a g-inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ since $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$

$$\left[\left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] \right] \\ \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \\ \left[\left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \right] \\ = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \\ \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

The following example demonstrates the above algorithm to compute g-inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$.

Example 7.1 Let us consider a NHSRFM

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \begin{bmatrix} \langle 0.5, 0.2, 0.4 \rangle, \langle 0.5, 0.2, 0.4 \rangle & \langle 0.7, 0.2, 0.2 \rangle, \langle 0.7, 0.2, 0.2 \rangle & \langle 0.6, 0.2, 0.3 \rangle, \langle 0.6, 0.2, 0.3 \rangle \\ \langle 0.5, 0.2, 0.3 \rangle, \langle 0.5, 0.2, 0.3 \rangle & \langle 0.6, 0.2, 0.2 \rangle, \langle 0.6, 0.2, 0.2 \rangle & \langle 0.8, 0.2, 0.2 \rangle, \langle 0.8, 0.2, 0.2 \rangle \\ \langle 0.4, 0.2, 0.4 \rangle, \langle 0.4, 0.2, 0.4 \rangle & \langle 0.4, 0.2, 0.3 \rangle, \langle 0.4, 0.2, 0.3 \rangle & \langle 0.8, 0.2, 0.1 \rangle, \langle 0.8, 0.2, 0.1 \rangle \end{bmatrix}$$

The rows of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ are independent and they form a standard basis.

Since,

$$\begin{aligned} & \left[\left(\underline{r}_i^T, \underline{r}_i^I, \underline{r}_i^F \right); \left(\bar{r}_i^T, \bar{r}_i^I, \bar{r}_i^F \right) \right] = \sum_{j=1}^3 \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{r}_j^T, \underline{r}_j^I, \underline{r}_j^F \right); \left(\bar{r}_j^T, \bar{r}_j^I, \bar{r}_j^F \right) \right] \\ & \left[\left(\underline{r}_j^T, \underline{r}_j^I, \underline{r}_j^F \right); \left(\bar{r}_j^T, \bar{r}_j^I, \bar{r}_j^F \right) \right], \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] \in \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] \\ & \text{row space of } \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right], \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in [0, 1] \text{ and} \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{r}_i^T, \underline{r}_i^I, \underline{r}_i^F \right); \left(\bar{r}_i^T, \bar{r}_i^I, \bar{r}_i^F \right) \right] = \left[\left(\underline{r}_i^T, \underline{r}_i^I, \underline{r}_i^F \right); \left(\bar{r}_i^T, \bar{r}_i^I, \bar{r}_i^F \right) \right], i=1,2,3. \end{aligned}$$

For the NHSRFPM

$$\left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] = \begin{bmatrix} \langle 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle & \langle 1, 1, 0 \rangle, \langle 1, 1, 0 \rangle, & \langle 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle \\ \langle 1, 1, 0 \rangle, \langle 1, 1, 0 \rangle, & \langle 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle & \langle 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle \\ \langle 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle & \langle 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle & \langle 1, 1, 0 \rangle, \langle 1, 1, 0 \rangle, \end{bmatrix}$$

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \text{ holds.} \end{aligned}$$

$$\left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] = \begin{bmatrix} \langle 0.8, 0.2, 0.2 \rangle, \langle 0.8, 0.2, 0.2 \rangle & \langle 0.5, 0.2, 0.5 \rangle, \langle 0.5, 0.2, 0.5 \rangle & \langle 0.5, 0.2, 0.3 \rangle, \langle 0.5, 0.2, 0.3 \rangle \\ \langle 0.4, 0.2, 0.5 \rangle, \langle 0.4, 0.2, 0.5 \rangle & \langle 0.8, 0.2, 0.1 \rangle, \langle 0.8, 0.2, 0.1 \rangle & \langle 0.6, 0.2, 0.3 \rangle, \langle 0.6, 0.2, 0.3 \rangle \\ \langle 0.3, 0.2, 0.4 \rangle, \langle 0.3, 0.2, 0.4 \rangle & \langle 0.4, 0.2, 0.4 \rangle, \langle 0.4, 0.2, 0.4 \rangle & \langle 0.9, 0.2, 0.1 \rangle, \langle 0.9, 0.2, 0.1 \rangle \end{bmatrix}$$

$$\begin{aligned} & \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\bar{r}_{ij}^T, \bar{r}_{ij}^I, \bar{r}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \text{ holds. So the g-inverse of } \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned}$$

is

$$\begin{aligned} & \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\overline{p}_{ij}^T, \overline{p}_{ij}^I, \overline{p}_{ij}^F \right) \right] \left[\left(\underline{r}_{ij}^T, \underline{r}_{ij}^I, \underline{r}_{ij}^F \right); \left(\overline{r}_{ij}^T, \overline{r}_{ij}^I, \overline{r}_{ij}^F \right) \right] = \\ & \left[\langle 0.4, 0.2, 0.5 \rangle, \langle 0.4, 0.2, 0.5 \rangle \quad \langle 0.8, 0.2, 0.1 \rangle, \langle 0.8, 0.2, 0.1 \rangle \quad \langle 0.6, 0.2, 0.3 \rangle, \langle 0.6, 0.2, 0.3 \rangle \right. \\ & \left. \langle 0.8, 0.2, 0.2 \rangle, \langle 0.8, 0.2, 0.2 \rangle \quad \langle 0.5, 0.2, 0.5 \rangle, \langle 0.5, 0.2, 0.5 \rangle \quad \langle 0.5, 0.2, 0.3 \rangle, \langle 0.5, 0.2, 0.3 \rangle \right. \\ & \left. \langle 0.3, 0.2, 0.4 \rangle, \langle 0.3, 0.2, 0.4 \rangle \quad \langle 0.4, 0.2, 0.4 \rangle, \langle 0.4, 0.2, 0.4 \rangle \quad \langle 0.9, 0.2, 0.1 \rangle, \langle 0.9, 0.2, 0.1 \rangle \right] \\ & = \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\overline{x}_{ij}^T, \overline{x}_{ij}^I, \overline{x}_{ij}^F \right) \right] \end{aligned}$$

(say) which satisfy the relation $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$

$$\begin{aligned} & \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\overline{x}_{ij}^T, \overline{x}_{ij}^I, \overline{x}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] = \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]. \end{aligned}$$

If each row of an HSRFM $\left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right]$ can be expressed as a linear combination of the rows of HSRFM $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$, then we write

$$R \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right] \subseteq R \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right].$$

If $R \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right] \subseteq R \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$ and

$$R \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \subseteq R \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right]$$

then we say that $R \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] = R \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right]$

Theorem 7.1 Let $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$,

$\left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right] \in (HSRFM)_{m \times n}$ be two HSRFM.

If $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$ is regular then,

$$\begin{aligned} \text{(i)} \quad & R \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right] \subseteq R \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \quad \text{iff} \\ & \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right] = \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right] \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \quad \text{for each} \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]^{-1} \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \{1\}. \end{aligned}$$

Theorem 7.5 If $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in (HSRFM)_{m \times n}$ be a symmetric and idempotent

HSRFM then $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ itself a least square g-inverse.

Proof. Since $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ is symmetric,

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^T = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

and $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ idempotent,

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^2 = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

Now

$$\left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

$$\text{if } \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] = I_n$$

Then

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{p}_{ij}^T, \underline{p}_{ij}^I, \underline{p}_{ij}^F \right); \left(\bar{p}_{ij}^T, \bar{p}_{ij}^I, \bar{p}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ &= \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ &= \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^2 = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned}$$

$$\text{That is, } \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \{1\}$$

$$\text{Now } \left[\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] \right]^T$$

$$= \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right]^T \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^T$$

$$= \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right]^T \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

$$= \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^T \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

$$\text{(Taking } \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \text{ as}$$

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \text{ itself a g-inverse.)}$$

(Taking $\left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ as

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \text{ itself a g-inverse.} \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned}$$

This implies, $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \{1, 4\}$

Theorem 7.7 If $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in (HSRFM)_{m \times n}$ be a symmetric and idempotent HSRFM then

$$\left\{ \begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right]; \text{ for all HSRFM} \\ & \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right] \in (HSRFM)_n \end{aligned} \right\}$$

such that

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \geq \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right]$$

is the set of all {1,3} inverses of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$, dominating

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right].$$

Proof. Since $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ is symmetric and idempotent HSRFM,

$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ itself $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \{1, 3\}$ inverse.

Let $\left[\left(\underline{\beta}_{ij}^T, \underline{\beta}_{ij}^I, \underline{\beta}_{ij}^F \right); \left(\bar{\beta}_{ij}^T, \bar{\beta}_{ij}^I, \bar{\beta}_{ij}^F \right) \right]$ denote the

$$\text{set} \left\{ \begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right]; \text{ for all HSRFM} \\ & \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right] \in (HSRFM)_n \end{aligned} \right\}$$

such that

Theorem 7.8 If $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in (NHSRFM)_n$ be a symmetric and idempotent NHSRFM then

$$\left. \begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] : \text{for all HSRFM} \\ & \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \in (HSRFM)_n \\ \text{Such that} \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \geq \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned} \right\}$$

is the set of all {1,4} inverses of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ dominating $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$.

Proof. Since $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ is symmetric and idempotent HSRFM, $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ itself $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ {1,4} inverse.

Let $\left[\left(\underline{\beta}_{ij}^T, \underline{\beta}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{\beta}_{ij}^T, \bar{\beta}_{ij}^I, \bar{\beta}_{ij}^F \right) \right]$ denote the set

$$\left. \begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] : \text{for all HSRFM} \\ & \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \in (HSRFM)_n \\ \text{Such that} \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \geq \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned} \right\}$$

Suppose $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ {1,4}

Then $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \geq \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$.

That, is

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \geq \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

Now, by

$$\begin{aligned} (ii) &\Rightarrow \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ &\geq \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned}$$

Hence

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \in \left[\left(\underline{\beta}_{ij}^T, \underline{\beta}_{ij}^I, \underline{\beta}_{ij}^F \right); \left(\bar{\beta}_{ij}^T, \bar{\beta}_{ij}^I, \bar{\beta}_{ij}^F \right) \right]$$

Thus for each $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \{1, 4\}$ there

exists a unique element in $\left[\left(\underline{\beta}_{ij}^T, \underline{\beta}_{ij}^I, \underline{\beta}_{ij}^F \right); \left(\bar{\beta}_{ij}^T, \bar{\beta}_{ij}^I, \bar{\beta}_{ij}^F \right) \right]$ Conversely, for any

$$\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \in \left[\left(\underline{\beta}_{ij}^T, \underline{\beta}_{ij}^I, \underline{\beta}_{ij}^F \right); \left(\bar{\beta}_{ij}^T, \bar{\beta}_{ij}^I, \bar{\beta}_{ij}^F \right) \right],$$

$$\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right]$$

$$= \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \geq \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

with

$$\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \geq \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right]$$

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

Hence,

$$\begin{aligned} &\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ &= \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right]. \end{aligned}$$

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

So, $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \{1, 4\}$

8. Moore-Penrose Inverse

Definition 8.1 (Moore-Penrose inverse)

For a NHSRFM $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \in (HSRFM)_{mn}$ and another HSRFM $\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \in (HSNRFM)_{mn}$ is said to be a Moore-Penrose inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$, if

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right],$$

$$\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] = \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right],$$

$$\left[\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \right]^T = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right]$$

$$\left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \right]^T = \left[\left(\underline{g}_{ij}^T, \underline{g}_{ij}^I, \underline{g}_{ij}^F \right); \left(\bar{g}_{ij}^T, \bar{g}_{ij}^I, \bar{g}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$$

The Moore-Penrose inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ is denoted by

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^+.$$

Definition 8.2 (Minus ordering)

Let $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ and $\left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\bar{v}_{ij}^T, \bar{v}_{ij}^I, \bar{v}_{ij}^F \right) \right]$ be two HSRFMs of order $m \times n$. The minus ordering between $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$ and

$\left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\bar{v}_{ij}^T, \bar{v}_{ij}^I, \bar{v}_{ij}^F \right) \right]$ is denoted by

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \leq^- \left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\bar{v}_{ij}^T, \bar{v}_{ij}^I, \bar{v}_{ij}^F \right) \right].$$

Then for some

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^- \in \left(\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \right) \{1\}$$

we say

Hence $\left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right]$ is a g-inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]^+$.

Remark 8.1 The condition on $\left[\left(\underline{v}_{ij}^T, \underline{v}_{ij}^I, \underline{v}_{ij}^F \right); \left(\overline{v}_{ij}^T, \overline{v}_{ij}^I, \overline{v}_{ij}^F \right) \right]$ to be idempotent is essential for $\left[\left(\underline{y}_{ij}^T, \underline{y}_{ij}^I, \underline{y}_{ij}^F \right); \left(\overline{y}_{ij}^T, \overline{y}_{ij}^I, \overline{y}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \{1\}$.

Theorem 8.3

Let $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \in (HSRFM)_{m \times n}$ be a regular HSRFM with $\left[\left(\underline{y}_{ij}^T, \underline{y}_{ij}^I, \underline{y}_{ij}^F \right); \left(\overline{y}_{ij}^T, \overline{y}_{ij}^I, \overline{y}_{ij}^F \right) \right]$ be its minimum norm g-inverse and $\left[\left(\underline{z}_{ij}^T, \underline{z}_{ij}^I, \underline{z}_{ij}^F \right); \left(\overline{z}_{ij}^T, \overline{z}_{ij}^I, \overline{z}_{ij}^F \right) \right]$ be its least square g-inverse. Then $\left[\left(\underline{y}_{ij}^T, \underline{y}_{ij}^I, \underline{y}_{ij}^F \right); \left(\overline{y}_{ij}^T, \overline{y}_{ij}^I, \overline{y}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \left[\left(\underline{z}_{ij}^T, \underline{z}_{ij}^I, \underline{z}_{ij}^F \right); \left(\overline{z}_{ij}^T, \overline{z}_{ij}^I, \overline{z}_{ij}^F \right) \right] = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]^+$ where $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]^+$ is the Moore-Penrose inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$.

Proof. Let $\left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\overline{x}_{ij}^T, \overline{x}_{ij}^I, \overline{x}_{ij}^F \right) \right] = \left[\left(\underline{y}_{ij}^T, \underline{y}_{ij}^I, \underline{y}_{ij}^F \right); \left(\overline{y}_{ij}^T, \overline{y}_{ij}^I, \overline{y}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \left[\left(\underline{z}_{ij}^T, \underline{z}_{ij}^I, \underline{z}_{ij}^F \right); \left(\overline{z}_{ij}^T, \overline{z}_{ij}^I, \overline{z}_{ij}^F \right) \right]$

then as

$$\left[\left(\underline{y}_{ij}^T, \underline{y}_{ij}^I, \underline{y}_{ij}^F \right); \left(\overline{y}_{ij}^T, \overline{y}_{ij}^I, \overline{y}_{ij}^F \right) \right] \left[\left(\underline{z}_{ij}^T, \underline{z}_{ij}^I, \underline{z}_{ij}^F \right); \left(\overline{z}_{ij}^T, \overline{z}_{ij}^I, \overline{z}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \{1\}$$

So, $\left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\overline{x}_{ij}^T, \overline{x}_{ij}^I, \overline{x}_{ij}^F \right) \right] \in \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right] \{1, 2\}$ that is

$\left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\overline{x}_{ij}^T, \overline{x}_{ij}^I, \overline{x}_{ij}^F \right) \right]$ is the semi-inverse of $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\overline{u}_{ij}^T, \overline{u}_{ij}^I, \overline{u}_{ij}^F \right) \right]$.

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \geq \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right], \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{k}_{ij}^T, \underline{k}_{ij}^I, \underline{k}_{ij}^F \right); \left(\bar{k}_{ij}^T, \bar{k}_{ij}^I, \bar{k}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^2 + \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \end{aligned}$$

Since

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \geq \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right], \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] + \left[\left(\underline{h}_{ij}^T, \underline{h}_{ij}^I, \underline{h}_{ij}^F \right); \left(\bar{h}_{ij}^T, \bar{h}_{ij}^I, \bar{h}_{ij}^F \right) \right] \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]. \end{aligned}$$

9. An Application

We can use the g-inverse of NHSRFMs to find the solution of NHSRFMs relational equations. Let us consider the system of equations

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] \text{ were,} \\ & = \left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right] \\ & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \\ & = \left[\langle 0.7, 0.8, 0.3 \rangle, \langle 0.7, 0.8, 0.3 \rangle \quad \langle 0.6, 0.8, 0.4 \rangle, \langle 0.6, 0.8, 0.4 \rangle \quad \langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle \right] \\ & \quad \left[\langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle \quad \langle 0.6, 0.8, 0.3 \rangle, \langle 0.6, 0.8, 0.3 \rangle \quad \langle 0.8, 0.8, 0.2 \rangle, \langle 0.8, 0.8, 0.2 \rangle \right] \end{aligned}$$

$$\left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] = \begin{bmatrix} \left(x_{1\mu}^T, x_{1\nu}^I, x_{1\eta}^F \right) \\ \left(x_{2\mu}^T, x_{2\nu}^I, x_{2\eta}^F \right) \\ \left(x_{3\mu}^T, x_{3\nu}^I, x_{3\eta}^F \right) \end{bmatrix} \text{ and}$$

$$\left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right] = \left[\langle 0.6, 0.8, 0.3 \rangle, \langle 0.6, 0.8, 0.3 \rangle \right] \\ \left[\langle 0.5, 0.8, 0.4 \rangle, \langle 0.5, 0.8, 0.4 \rangle \right]$$

Each particular matrix $\left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right]$ that satisfy the equation

$$\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] = \left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right]$$

is called its solution and the set

$$\begin{aligned} \Omega & \left[\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right] \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^{-1} \left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right] \end{aligned}$$

denotes the set of all solutions.

An HSRFM may have multiple g-inverses. Here we consider one of the generalized inverse of the

HSRFM $\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]$, which is

$$\begin{aligned} & \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^{-1} \\ & = \begin{bmatrix} \langle 0.8, 0.8, 0.2 \rangle, \langle 0.8, 0.8, 0.2 \rangle & \langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle \\ \langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle & \langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle \\ \langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle & \langle 0.8, 0.8, 0.2 \rangle, \langle 0.8, 0.8, 0.2 \rangle \end{bmatrix}. \quad \text{Then} \end{aligned}$$

$$\begin{aligned} & \left[\left(\underline{x}_{ij}^T, \underline{x}_{ij}^I, \underline{x}_{ij}^F \right); \left(\bar{x}_{ij}^T, \bar{x}_{ij}^I, \bar{x}_{ij}^F \right) \right] \in \Omega \left[\left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right] \left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right] \right] \\ & = \left[\left(\underline{u}_{ij}^T, \underline{u}_{ij}^I, \underline{u}_{ij}^F \right); \left(\bar{u}_{ij}^T, \bar{u}_{ij}^I, \bar{u}_{ij}^F \right) \right]^{-1} \left[\left(\underline{q}_{ij}^T, \underline{q}_{ij}^I, \underline{q}_{ij}^F \right); \left(\bar{q}_{ij}^T, \bar{q}_{ij}^I, \bar{q}_{ij}^F \right) \right] \\ & = \begin{bmatrix} \langle 0.6, 0.8, 0.3 \rangle, \langle 0.6, 0.8, 0.3 \rangle \\ \langle 0.5, 0.8, 0.5 \rangle, \langle 0.5, 0.8, 0.5 \rangle \\ \langle 0.5, 0.8, 0.4 \rangle, \langle 0.5, 0.8, 0.4 \rangle \end{bmatrix} \end{aligned}$$

is one of the solutions of the above system of equations.

10. Conclusion

In this study, we introduced a novel method for determining the generalized inverse (g-inverse) and Moore-Penrose inverse of Neutrosophic Fuzzy Matrices (HSRFMs). A dedicated algorithm was developed to compute the g-inverse of HSRFMs, addressing the existing gap in computational techniques for these matrices. Several theoretical results and properties of the g-inverse were explored and validated through numerical examples. Additionally, the practical applicability of the g-inverse was demonstrated by solving a rectangular system of neutrosophic fuzzy relational equations.

The findings of this work provide significant contributions to the field of neutrosophic fuzzy matrix theory by combining theoretical advancements with practical algorithms. This research not only enhances the foundational understanding of matrix inverses in neutrosophic fuzzy settings but also offers a reliable tool for solving real-world problems in control systems, robotics, and decision-making processes. Future studies can build upon this work to explore further applications and extensions of the proposed methods.

11. Future work

In this paper, we introduce a framework for determining the generalized inverse (g-inverse) and the Moore–Penrose inverse of Neutrosophic Hyper Soft Rough Fuzzy Matrices (NHSRFMs), along with the necessary conditions under which these inverses exist. To the best of our knowledge, no algorithm has been proposed to compute the g-inverse of an NHSRFM; hence, we present a novel algorithm for its evaluation. Several fundamental properties and results associated with the g-inverse of NHSRFMs are established to strengthen the theoretical foundation. The study further demonstrates the practical relevance of the proposed method through a real-world application, supported by numerical examples that illustrate the theorems, validate the algorithm, and highlight the utility of the g-inverse in decision-making contexts.

References

- [1]. L.A. Zadeh, Fuzzy sets, *Information and Control*, 8, 338-353 (1965).
- [2]. K.T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 87-96 (1986).
- [3]. Smarandache, F, Neutrosophic set, a generalization of the intuitionistic fuzzy set. *Int J Pure Appl Math.*; ,(2005),.24(3):287–297.
- [4]. J. Cen, Fuzzy matrix partial ordering and generalized inverses, *Fuzzy Sets and Systems*, 15, 453-458 (1999).
- [5]. J. Cen, On generalized inverses of fuzzy matrices, *Fuzzy Sets and Systems*, 5(1), 66-75 (1991).
- [6]. S.K. Khan and M. Pal, Intuitionistic fuzzy tautological matrices, *Journal of Physical Science*, 8, 92-100 (2003).
- [7]. A.R. Meenakhi and C. Inbam, The minus partial order in fuzzy matrices, *The Journal of Fuzzy Mathematics*, 12(3), 695-700 (2004).
- [8]. S.K. Mitra, Matrix partial order through generalized inverses: unified theory, *Linear Algebra and its Applications*, 148, 237-263 (1991).
- [9]. Anandhkumar, M., Kamalakannan, V., Chithra, S.M., Said, B., "Pseudo Similarity of Neutrosophic Fuzzy matrices", *International Journal of Neutrosophic Science*, Vol. 20, No. 04, PP. 191-196, 2023.

- [10]. Anandhkumar, M., Kanimozhi, B., Chithra, S.M., Kamalakannan, V., Said, B., "On various Inverse of Neutrosophic Fuzzy Matrices", International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023.
- [11]. Anandhkumar, M., Harikrishnan, T., Chithra, S.M., Kanimozhi, B., Said, B. "Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices" International Journal of Neutrosophic Science, 2023, 21(4), pp. 135–145.
- [12]. L.J. Xin, Convergence of power of a controllble fuzzy matrices, Fuzzy Sets and System, 45, 313-319 (1992).
- [13]. S.K. Shyamal and M. Pal, Interval-valued fuzzy matrices, The Journal of Fuzzy Mathematics, 14(3), 583-604 (2006).
- [14]. A. K. Shyamal and M. Pal, Two new operators on fuzzy matrices, J. Applied Mathematics and Computng, 15, 91 107 (2004).
- [15]. M. Dehghan, M. Ghatee and B. Hashemi, The inverse of a fuzzzy matrix of fuzzy numbers, International Journal of Computer Mathematics, 86(8), 1433-1452 (2009).
- [16]. M. Panigrahi and S. Nanda, Intuitionistic fuzzy relations over intuitionistic fuzzy sets, The Journal of Fuzzy Mathematics, 15(3), 675-688 (2007).
- [17]. M. Pal, Intuitionistic fuzzy determinant, V.U.J. Physical Sciences, 7, 87-93 (2001).
- [18]. M. Pal, S.K. Khan and A.K. Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets, 8(2), 51-62 (2002).
- [19]. M. Anandhkumar, T. Harikrishnan, S. M. Chithra, V. Kamalakannan, B. Kanimozhi, Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices, International Journal of Neutrosophic Science, Vol. 23, no. 2, 2024, pp. 286-295.
- [20]. M. Anandhkumar, H. Prathab, S. M. Chithra, A. S. Prakaash, A. Bobin, Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, vol. 23, no. 4, 2024, pp. 23-28.
- [21]. M.Anandhkumar; G.Punithavalli; T.Soupramanien; Said Broumi, Generalized Symmetric Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 57,2023, 57, pp. 114–127.
- [22]. A. K. Shyamal and M. Pal, Distance between intuitionistic fuzzy matrices and its applicatins, Natural and Physical Science, 19(1), 39-58 (2005).
- [23]. A. K. Shyamal and M. Pal, Distance between intuitionistic fuzzy matrices, V.U.J. Physical Sciences, 8, 81-91 (2002).
- [24]. S. Sriram and P. Murugadas, On semiring of intuitionistic fuzzy matrices, Applied Mathematical Science, 4(23), 1099-1105 (2010).
- [25]. T. K. Mondal and S. K. Samanta, Generalized intuitionistic fuzzy sets, The Journal of Fuzzy Mathematics, 10(4), 839-862 (2002).
- [26]. M. Bhowmik and M. Pal, Generalized intuitionistic fuzzy matrices, Far-East Journal of Mathematical Sciences, 29(3), 533-554 (2008).

- [27]. M. Bhowmik and M. Pal, Some results on generalized interval-valued intuitionistic fuzzy sets, *International Journal of Fuzzy Systems*, 14(2), 193-203 (2012).
- [28]. M. Bhowmik and M. Pal, Generalized interval-valued intuitionistic fuzzy sets, *The Journal of Fuzzy Mathematics*, 18(2), 357-371 (2010).
- [29]. S.K. Khan and A. Pal, The generalized inverse of intuitionistic fuzzy matrix, *Journal of Physical Sciences*, 11, 62-67 (2007).
- [30]. A.K. Adak, M. Bhowmik and M. Pal, Some properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattice, *Fuzzy Inf. and Eng.*, 4(4), 371-387 (2012).
- [31]. Anandhkumar, M.; G. Punithavalli; and E. Janaki. "Secondary k-column symmetric Neutrosophic Fuzzy Matrices." *Neutrosophic Sets and Systems* 64, 1 (2024).
- [32]. Anandhkumar, M.; G. Punithavalli; R. Jegan; and Said Broumi. "Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices." *Neutrosophic Sets and Systems* 61, 1 (2024).
- [33]. Anandhkumar, M.; A. Bobin; S. M. Chithra; and V. Kamalakannan. "Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices." *Neutrosophic Sets and Systems* 70, 1 (2024).
- [34]. Muthukumar, P., & Krishnan, G. S. S. (2018). Generalized Fuzzy Soft Rough Matrices and Their Applications in Decision-Making Problems. *International Journal of Fuzzy Systems*, 20, 500-514.
- [35]. Martina, D. J. S., & Deepa, G. (2023). Some algebraic properties on rough neutrosophic matrix and its application to multi-criteria decision-making. *AIMS Mathematics*, 8(10), 24132-24152.
- [36]. Cagman, N., & Enginoglu, S. (2012). Fuzzy soft matrix theory and its application in decision making. *Iranian Journal of Fuzzy Systems*, 9(1), 109-119.
- [37]. Das, M., Mohanty, D., & Parida, K. C. (2021). On the Neutrosophic soft set with rough set theory. *Soft Computing*, 25(21), 13365-13376.
- [38]. Jafar, M. N., & Saeed, M. (2022). Matrix Theory for Neutrosophic Hypersoft Set and Applications in Multiattributive Multicriteria Decision-Making Problems. *Journal of Mathematics*, 6666408.
- [39]. Jayasudha, J., & Raghavi, S. (2024). Some Operations on Neutrosophic Hypersoft Matrices and Their Applications. *Neutrosophic Systems with Applications*, 21, 46-62.
- [40]. Al-Quran, A.; Hassan, N., & Marei, E. (2019). A novel approach to neutrosophic soft rough set under uncertainty. *Symmetry*, 11(3), 384.
- [41]. Borah, M. J.; Neog, T. J., & Sut, D. K. (2012). Fuzzy soft matrix theory and its decision making. *International Journal of Modern Engineering Research*, 2(2), 121-127.
- [42]. Al-Quran, A.; Hassan, N., & Marei, E. (2019). A novel approach to neutrosophic soft rough set under uncertainty. *Symmetry*, 11(3), 384.
- [43]. Das, M., Mohanty, D., & Parida, K. C. (2021). On the Neutrosophic soft set with rough set theory. *Soft Computing*, 25(21), 13365-13376.
- [44]. Kamaci, H. (2021). On hybrid structures of hypersoft sets and rough sets. *International Journal of Modern Science and Technology*, 6(4), 69-82.

- [45]. Smarandache .F. (2018). Extension of soft set to Hypersoft Set, and then Plithogenic Hypersoft Set, *Neutrosophic sets and systems*, 22, 168-170
- [46]. T. Harikrishnan, M. Anandhkumar, S. Prathap, S. Subramanian, D. Ramesh, and M. Raji, "Min(Max)–Min(Max)–Max(Min)(*): Compositions of Neutrosophic Fuzzy Matrices and its Application in Medical Diagnosis," *Neutrosophic Sets and Systems*, vol. 86, pp. 212–244, 2025.
- [47]. C. D. Shyamala, C. Kayelvizhi, S. Kalathian, P. Tharaniya, M. Anandhkumar, and S. M. Chithra, "Decomposition of Neutrosophic Fuzzy Matrices Using Some Alpha–Cuts," *Neutrosophic Sets and Systems*, vol. 86, p. 1, 2025.
- [48]. M. Anandhkumar, S. Prathap, R. A. Prabhu, P. Tharaniya, K. Thirumalai, and B. Kanimozhi, "Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems," *Neutrosophic Sets and Systems*, vol. 79, p. 1, 2025.
- [49]. K. Radhika, T. Harikrishnan, R. A. Prabhu, P. Tharaniya, M. J. Peter, and M. Anandhkumar, "On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices," *Neutrosophic Sets and Systems*, vol. 78, p. 1, 2025.
- [50]. K. Radhika, S. Senthil, N. Kavitha, R. Jegan, M. Anandhkumar, and A. Bobin, "Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making," *Neutrosophic Sets and Systems*, vol. 78, p. 1, 2025.
- [51]. G. Punithavalli and M. Anandhkumar, "Reverse Sharp and Left-T Right-T Partial Ordering on Intuitionistic Fuzzy Matrices," *TWMS Journal of Applied and Engineering Mathematics*, vol. 14, no. 4, pp. 1772–1783, 2024.
- [52]. G. Punithavalli and M. Anandhkumar, Kernel and k-Kernel Symmetric Intuitionistic Fuzzy Matrices," *TWMS Journal of Applied and Engineering Mathematics*, vol. 14, no. 3, pp. 1231–1240, 2024.
- [53]. M. Anandhkumar, A. Savitha Mary, M. Kavitha, S. Subramanian, V. Sathishkumar, S.M. Chithra, Transitive and Strongly Transitive Neutrosophic Fuzzy Matrices, *Neutrosophic Sets and Systems*, Vol. 88, pp.537-557, 2025.
- [54]. P. Murugadas, T. Shyamaladevi , M. Anandhkumar, Interval Valued Kernel Symmetric, K-Kernel Symmetric, Range Symmetric And Column Symmetric Neutrosophic Fuzzy Matrices, *TWMS J. App. and Eng. Math.* V.15, N.5, 2025, pp.1230-1244.
- [55]. H. Prathab, N. Ramalingam, E. Janaki, A. Bobin, V. Kamalakannan and M. Anandhkumar, Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses, *IAENG International Journal of Computer Science*, Volume 51, Issue 12, December 2024, Pages 2051-2066

Received: Nov 25, 2025. Accepted: May 3, 2026