



Solution of Fully Fuzzy Neutrosophic Lyapunov Matrix Equation

Myra Suphelea Asmizal¹, Wan Suhana Wan Daud^{2,*}, Khairu Azlan Abd Aziz³ and Siti Rohana Goh Abdullah⁴

^{1,2,4}Department of Mathematical Sciences Faculty of Intelligent Computing Universiti Malaysia Perlis

³Universiti Teknologi Mara, Perlis Branch, Arau Campus, 02600, Perlis, Malaysia

¹myrasuphelea@studentmail.unimap.edu.my

²wsuhana@unimap.edu.my

³khairu493@uitm.edu.my

⁴sitirohana@unimap.edu.my

*Correspondence: wsuhana@unimap.edu.my

Abstract. The classical Lyapunov matrix equation (LME) is important for checking system stability, but it assumes we know exact numbers. It cannot handle unclear, conflicting, or missing information. Most of real-world application of control systems have uncertainty and unpredictability that makes it hard to analyze the stability precisely using conventional mathematical models. The classical Lyapunov matrix equation is important in determining the system stability but it fails to accommodate vague, inconsistent, or incomplete information. Therefore, this study introduces a solution for the Fully Fuzzy Neutrosophic Lyapunov Matrix Equation (FFNLME) to address the limitation. The proposed model extends the classical model of LME into the neutrosophic fuzzy domain by incorporating three independent membership components which are Truth (T), Indeterminacy (I), and Falsity (F). Left right-triangular neutrosophic fuzzy numbers (LR-TriNFN) is utilized to represent uncertain system parameters. Associated linear systems (ALS) and score function method are employed for deneutrosophication, allowing for computational analysis while preserving the embedded uncertainty. The study demonstrates that the FFNLME framework provides a more comprehensive and flexible representation of uncertain systems, enhancing the robustness of stability analysis compared to classical or fuzzy Lyapunov equations. This formulation can be applied to both linear and nonlinear systems in control engineering, offering an effective means to evaluate stability under ambiguous or incomplete information. Future research may focus on developing optimized numerical algorithms and extending this approach to large-scale and time-varying systems.

Keywords: left right-triangular fuzzy numbers; fully fuzzy neutrosophic numbers; Lyapunov; matrix equations; neutrosophic.

1. Introduction

A matrix equation represents the mathematical relationships between variables and constants in matrix form. These equations are widely used in various fields, such as physics, engineering, computer science and control systems to model and solve problems. Matrix addition, multiplication or inversion are the common operations involved in solving matrix equations [1]. In addition, the matrix equations provide a compact and efficient way to represent and compute systems of linear equations, transformation and other operations involving numerical arrays [2].

There are a few types of matrix equations that are widely applied in various mathematical formulations such as the Lyapunov matrix equation and they are ranging from linear to nonlinear forms [3]. Lyapunov equation is crucial in analyzing the stability of control systems, especially for verifying the stability of linear time-invariant (LTI) systems. The solutions are obtained through different numerical method such as direct and iterative methods. Bartels–Stewart is the example direct methods that provide an exact numerical solution by transforming the system into a simpler equivalent form [4]. Besides, the iterative methods such as the Krylov subspace approach are often employed for large-scale systems and offering efficient approximations with reduced computational complexity [5].

Many real-world problems involve uncertainty, vagueness and incomplete information, which makes it difficult to obtain precise solutions using classical numerical methods. To address these limitations, Zadeh et al. [6] introduced the concept of fuzzy sets, providing a mathematical framework to handle problems that cannot be effectively solved by conventional techniques.

Fuzzy sets allow the measurement of the membership of an element within a set that rely on interval from 0 to 1 instead of saying true or false. This approach had been applied in many areas where the information is imprecise or incomplete. Various forms of fuzzy numbers have been developed to represent uncertainty in a structured way. Following that, interval-valued fuzzy sets (IVFS) were proposed as an extension, offering a broader representation of uncertainty [7]. Subsequently, Jun et al. [8] combined fuzzy sets and IVFS to create a more flexible and robust framework for modeling vagueness and indeterminacy. This development further evolved into concepts such as cubic sets [8], soft sets [9] and cubic soft sets [10]. Overall, fuzzy numbers provide a mechanism to express partial truth and probabilistic reasoning which are effectively in capturing the inherent ambiguity and complexity of real-world systems.

Nevertheless, fuzzy numbers do not fully deal with the indeterminate or contradictory information that often arises in complex real-world systems. In order to overcome this limitation, Smarandache et al. [11] introduced the concept of neutrosophic sets as an extension of both fuzzy sets to handle incomplete, inconsistent and indeterminate information in practical applications. In neutrosophic theory, each element is characterized by three independent

membership functions which are Truth (T), Indeterminacy (I) and Falsity (F). This structure enables a more flexible representation of uncertainty, allowing simultaneous consideration of what is true, indeterminate and false. Neutrosophic sets have been successfully applied in various fields where uncertain, inconsistent or contradictory data must be modeled and analyzed such as decision making, image processing and pattern recognition. Similar to fuzzy numbers, neutrosophic numbers can take different forms, including triangular and trapezoidal neutrosophic numbers, among others developed by subsequent researchers. From a paper reviewed by Asmizal et al. [12], it stated each role of fuzzy, intuitionistic and neutrosophic numbers in solving matrix equations and the uncertain matrix equations can be handled better in the neutrosophic environment. Overall, the neutrosophic framework provides a more comprehensive approach to representing real-life uncertainty and imprecision compared to classical fuzzy models.

Hence, this study aims to solve the solution of fully fuzzy neutrosophic Lyapunov matrix equation (FFNLME),

$$\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} = \tilde{C} \tag{1}$$

and can be defined in matrix form of

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mm} \end{pmatrix}^T \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \dots & \tilde{x}_{mn} \end{pmatrix} + \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \dots & \tilde{x}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \dots & \tilde{x}_{mn} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mm} \end{pmatrix} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \dots & \tilde{c}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{m1} & \tilde{c}_{m2} & \dots & \tilde{c}_{mn} \end{pmatrix}$$

where $\tilde{A} = (a_{ij})$, $1 \leq i, j \leq m$ and $\tilde{C} = (c_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq n$ are the neutrosophic fuzzy matrices. While the solution $\tilde{X} = (x_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq n$ is the unknown matrix. The coefficients are in the form of left right-triangular neutrosophic fuzzy numbers (LR-TriNFN) of $(m, \alpha, \beta; T, I, F)$.

The remaining part of this paper is structured as follows. Section 2 presents some preliminaries that discuss the foundational concepts of fuzzy numbers and neutrosophic numbers with their mathematical properties. Section 3 presents the theoretical formulation of the FFNLME meanwhile section 4 discusses the solution algorithm for FFNLME. Next, section 5 provides illustrative examples followed by its verification of the solution in Section 6. Lastly, section 7 concludes the study with recommendations for future research.

2. Preliminaries

This section recalls some fundamental concepts including definitions and theorems related to this study.

2.1. Theory of Fuzzy Numbers

The definitions of fuzzy numbers are given as follows.

Definition 2.1. [6] Let X be a nonempty set, the fuzzy set \tilde{A} in X is characterized by its membership function,

$$\mu_{\tilde{A}} : X \rightarrow [0, 1] \quad (2)$$

and $\mu_{\tilde{A}}(x)$ represents the degree of membership of the element x in fuzzy numbers \tilde{A} for each $x \in X$.

Definition 2.2. [6] The fuzzy set \tilde{A} is represented by a set of ordered pairs of element x and grade $\mu_{\tilde{A}}$ which can be written as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}. \quad (3)$$

Definition 2.3. [6] A fuzzy set \tilde{A} in X has the following properties as described belows,

- (1) $\mu_{\tilde{A}}$ is upper semicontinuous.
- (2) $\mu_{\tilde{A}} = 0$ is outside of some interval $[c, d]$.
- (3) There are real numbers of a and b , such that $c \leq a \leq b \leq d$ and
 - (a) $\mu_{\tilde{A}}$ is monotonic increasing on $[c, a]$,
 - (b) $\mu_{\tilde{A}}$ is monotonic decreasing on $[b, d]$,
 - (c) $\mu_{\tilde{A}} = 1$, for $a \leq x \leq b$.

Various types of fuzzy numbers appear frequently in literature, especially triangular fuzzy numbers (TriFN) being the most commonly used and there is a form of TriFN called LR-TriFN. For clarity and easy reference, the definition of LR-TriFN is provided below.

Definition 2.4. [13] The LR-TriFN $\tilde{M} = (m, \alpha, \beta)$ has the membership function as follows,

$$\tilde{\mu}_{\tilde{M}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha} & m - \alpha \leq x \leq m \\ 1 - \frac{x-m}{\beta} & m \leq x \leq m + \beta \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where m is the mean value while α and β are the spreads on the left and right sides, respectively.

Meanwhile, in this study, the multiplication operator proposed by Babbar et. al [14] is adopted, as it is more comprehensive and suitable for use with all types of fuzzy numbers. The definition for the Babbar's multiplication operator is given below.

Definition 2.5. [14] The product of two fuzzy numbers $A = (m, \alpha, \beta)$ and $B = (n, \gamma, \delta)$ is defined as

$$\tilde{A} \otimes \tilde{B} = \{(mn, f_1, f_2) \mid (m, \alpha, \beta) \geq 0\} \quad (5)$$

where

$$\begin{aligned} f_1 &= mn - \text{Min}((m - \alpha)(n - \gamma), (m + \beta)(n - \gamma)), \\ f_2 &= \text{Max}((m - \alpha)(n + \delta), (m + \beta)(n - \delta)) - mn. \end{aligned} \quad (6)$$

Nevertheless, this method overlooks non-membership degrees and the hesitation that often occurs in real-life scenarios plus remain limited in handling incomplete or inconsistent data effectively. To address these challenges, neutrosophic fuzzy numbers (NFN) were developed as an extension of both fuzzy numbers. Next subsection will discussed on the theory of neutrosophic numbers.

2.2. Theory of Neutrosophic numbers

Neutrosophic fuzzy numbers (NFN) offered a more flexible framework than traditional fuzzy and intuitionistic fuzzy numbers. They can manage not only incomplete and uncertain information but also inconsistent and indeterminate data, making them particularly effective for addressing complex real-world problems [15]. The definition of neutrosophic numbers is provided below:

Definition 2.6. [15] Let X be a universal set and let $x \in X$. The neutrosophic set A in X is characterized by a Truth (T), Indeterminacy (I) and Falsity (F) membership functions, then neutrosophic numbers is given as,

$$A = \{(x, T(x), I(x), F(x)) : x \in X\}. \quad (7)$$

There is a restriction on the sum of T , I and F which must be satisfied, which is

$$0^+ \leq T + I + F \leq 3^+. \quad (8)$$

In terms of philosophy, the neutrosophic set is derived from actual standard or non-standard subsets of $]0^-, 1^+[$. However, for technical applications the interval $[0, 1]$ is more appropriate rather than $]0^-, 1^+[$ [16]. Same goes to fuzzy numbers, neutrosophic fuzzy numbers also classify into Triangular neutrosophic fuzzy numbers (TriNFN). For instance, a TriNFN is defined by three parameters representing the lower, middle, and upper values, each associated with truth membership, indeterminacy membership and falsity membership degrees. This classification helps in modeling uncertainty more effectively by capturing the nuances of indeterminate and inconsistent information within a triangular framework. The detailed definition and operations of TriNFN are described as follows.

Definition 2.7. [17] Let $N = (a, b, c; T, I, F)$ is single-valued TriNFN, consist of truth, indeterminacy and falsity membership function which defined as,

$$T(x) = \begin{cases} \frac{x-a}{b-a}T, & a \leq x < b \\ T, & x = b \\ \frac{c-x}{c-b}T, & b < x \leq c \\ 0, & \text{otherwise,} \end{cases}$$

$$I(x) = \begin{cases} \frac{b-x+I(x-a)}{b-a}, & a \leq x < b \\ I, & x = b \\ \frac{x-b+I(c-x)}{c-b}, & b < x \leq c \\ 1, & \text{otherwise,} \end{cases}$$

$$F(x) = \begin{cases} \frac{b-x+F(x-a)}{b-a}, & a \leq x < b \\ F, & x = b \\ \frac{x-b+F(c-x)}{c-b}, & b < x \leq c \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.8. [18] Let $X_1 = (a_1, b_1, c_1; T_1, I_1, F_1)$ and $X_2 = (a_2, b_2, c_2; T_2, I_2, F_2)$ be TriNFN. The addition and multiplication operations are as below:

$$X_1 + X_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2; T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2) \tag{9}$$

$$X_1 \times X_2 = (a_1a_2, b_1b_2, c_1c_2; T_1 + T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2) \tag{10}$$

2.3. Kronecker products and Vec-operator

Kronecker products and Vec-operator are widely used in solving matrix equations and in this study, these operators are used to convert the neutrosophic fuzzy matrix equations to a simpler form of neutrosophic fuzzy linear equations. Basically, the symbol of the Kronecker product is \otimes_k , and the definition is given as follows.

Definition 2.9. [19] Let matrix $A = [a_{ij}]$ is $m \times n$ and $B = [b_{ij}]$ is $p \times q$ then the Kronecker product of $A \otimes_k B$ is given by,

$$A_{m \times n} \otimes_k B_{p \times q} = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \tag{11}$$

which is a $mp \times nq$ matrix.

On the other hand, the Vec -operator is an operator that the matrix will be transformed into a column vector by stacking its column on top of each other, which can be defined as

Definition 2.10. [19] Let A is a $m \times n$ matrix then the vectorization of A is denoted by $Vec(A)$, which given as follows,

$$Vec(A)_{m \times n} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ a_{12} \\ \vdots \\ a_{mn} \end{pmatrix} = A_{mn \times 1} \quad (12)$$

Theorem 2.11. [20] If $\tilde{A} = (\tilde{a}_{ij})_{m \times m}$ is a fuzzy matrix and $\tilde{U} = (\tilde{u}_{ij})_{p \times p}$ is a unitary fuzzy matrix written as

$$\tilde{U} = \begin{pmatrix} (1, 0, 0) & (0, 0, 0) & \dots & (0, 0, 0) \\ (0, 0, 0) & (1, 0, 0) & \dots & (0, 0, 0) \\ \vdots & \vdots & \ddots & \vdots \\ (0, 0, 0) & (0, 0, 0) & \dots & (1, 0, 0) \end{pmatrix} \quad (13)$$

then

- (i) $\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}$
- (ii) $\tilde{U}^T = \tilde{U}$.

The following definitions demonstrate how the fuzzy neutrosophic Kronecker product and Vec -operator are utilized in the fuzzy environment.

Definition 2.12. [20] Let \tilde{A}, \tilde{B} and \tilde{X} be the fuzzy matrices, while \tilde{U} is a unitary fuzzy matrix, then

- (i) $Vec[\tilde{A}\tilde{X}] = [\tilde{U} \otimes_k \tilde{A}]Vec(\tilde{X})$
- (ii) $Vec[\tilde{X}\tilde{B}] = [\tilde{B}^T \otimes_k \tilde{U}]Vec(\tilde{X})$
- (iii) $Vec[\tilde{A} \pm \tilde{B}] = Vec[\tilde{A}] \pm Vec[\tilde{B}]$

The size of \tilde{U} in the Definition 2.12 depends on the size of \tilde{A} and \tilde{B} .

Definition 2.13. [21] Let $\tilde{A} = (\tilde{a}_{ij})_{m \times m}$, $\tilde{C} = (\tilde{c}_{ij})_{m \times n}$ and $\tilde{X} = (\tilde{x}_{ij})_{m \times n}$, then using fuzzy Kronecker product and fuzzy Vec -operator, the equation of Lyapunov matrix equation, $\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} = \tilde{C}$ can be written as

$$[(\tilde{U}_{n \times n} \otimes_k \tilde{A}_{m \times m}^T) + (\tilde{A}_{n \times n}^T \otimes_k \tilde{U}_{m \times m})]Vec(\tilde{X}_{m \times n}) = Vec(\tilde{C})_{m \times n} \quad (14)$$

where $\tilde{U}_{m \times m}$ and $\tilde{U}_{n \times n}$ denote the fuzzy identity matrices with the order of $m \times m$ and $n \times n$, respectively.

For the purpose of this study, the Kronecker product and *Vec*-operator are adapted and referred to as the fuzzy neutrosophic Kronecker product and fuzzy neutrosophic *Vec*-operator respectively.

2.4. Associated Linear System

Associated linear systems (ALS) are developed to form a crisp form of the matrix equation. The definitions of ALS are given as follows,

Definition 2.14. Let $\tilde{S} = (m^{\tilde{S}}, \alpha^{\tilde{S}}, \beta^{\tilde{S}})$, $\tilde{X} = (m^{\tilde{X}}, \alpha^{\tilde{X}}, \beta^{\tilde{X}})$ and $\tilde{C} = (m^{\tilde{C}}, \alpha^{\tilde{C}}, \beta^{\tilde{C}})$. Consider a fully fuzzy linear systems (FFLS) in the form of

$$\tilde{S}\tilde{X} = \tilde{C} \tag{15}$$

where $\tilde{S} = (m^{\tilde{S}}_{ij}, \alpha^{\tilde{S}}_{ij}, \beta^{\tilde{S}}_{ij})$, for $1 \leq i \leq p$, $1 \leq j \leq q$, $\tilde{X} = (m^{\tilde{X}}_j, \alpha^{\tilde{X}}_j, \beta^{\tilde{X}}_j)$ with $1 \leq j \leq q$ and $\tilde{C} = (m^{\tilde{C}}_i, \alpha^{\tilde{C}}_i, \beta^{\tilde{C}}_i)$ with $1 \leq i \leq p$, which is equivalent to

$$\sum_{j=1, \dots, q}^{\oplus} (m^{\tilde{S}}_{ij}, \alpha^{\tilde{S}}_{ij}, \beta^{\tilde{S}}_{ij}) \otimes (m^{\tilde{X}}_j, \alpha^{\tilde{X}}_j, \beta^{\tilde{X}}_j) = (m^{\tilde{C}}_i, \alpha^{\tilde{C}}_i, \beta^{\tilde{C}}_i). \tag{16}$$

Definition 2.15. [22] Let $\tilde{S} = (m^{\tilde{S}}, \alpha^{\tilde{S}}, \beta^{\tilde{S}})$ and $\tilde{C} = (m^{\tilde{C}}, \alpha^{\tilde{C}}, \beta^{\tilde{C}})$ be positive, negative or near-zero fuzzy matrices, and $\tilde{X} = (m^{\tilde{X}}, \alpha^{\tilde{X}}, \beta^{\tilde{X}})$ be the solution which is in a positive fuzzy matrix. If \tilde{S} is positive, then the forms of ALS obtained, such that,

$$\begin{cases} m^{\tilde{S}}m^{\tilde{X}} & = m^{\tilde{C}} \\ \alpha^{\tilde{S}}m^{\tilde{X}} + (m^{\tilde{S}} - \alpha^{\tilde{S}})\alpha^{\tilde{X}} & = \alpha^{\tilde{C}} \\ \beta^{\tilde{S}}m^{\tilde{X}} + (m^{\tilde{S}} + \beta^{\tilde{S}})\beta^{\tilde{X}} & = \beta^{\tilde{C}} \end{cases} \tag{17}$$

which also can be represented as

$$\left(\begin{array}{c|c|c} m^{\tilde{S}} & 0 & 0 \\ \alpha^{\tilde{S}} & (m^{\tilde{S}} - \alpha^{\tilde{S}}) & 0 \\ \beta^{\tilde{S}} & 0 & (m^{\tilde{S}} + \beta^{\tilde{S}}) \end{array} \right) \begin{pmatrix} m^{\tilde{X}} \\ \alpha^{\tilde{X}} \\ \beta^{\tilde{X}} \end{pmatrix} = \begin{pmatrix} m^{\tilde{C}} \\ \alpha^{\tilde{C}} \\ \beta^{\tilde{C}} \end{pmatrix}. \tag{18}$$

Subsequently, a brief definition of the score function method is provided.

2.5. Score Function method

The score function method is a common approach used to transform neutrosophic fuzzy numbers into crisp numerical values. This method assigns a single representative value to each neutrosophic number, enabling straightforward comparison and ranking among multiple alternatives. Because neutrosophic fuzzy numbers consist of truth, indeterminacy, and falsity components, direct comparisons between them can be complex. The score function simplifies this process by integrating these components into one distinct and interpretable value [23]. The formula of score function is given in the following definition,

Definition 2.16. [24] Let $M = (T, I, F)$ be the single-valued neutrosophic numbers, where $T, I, F \in [0, 1], 0^+ \leq T + I + F \leq 3^+$. Then, the single-valued neutrosophic score function is defined as,

$$s(M) = \frac{T + (1 - I) + (1 - F)}{3} = \frac{2 + T - I - F}{3} \quad (19)$$

where $s : M \rightarrow [0, 1]$.

2.6. Relative Residual Error

The accuracy of the solution for solving linear systems, $Ax = b$ is commonly determined by relative residual error [25]. The relative residual error is defined as the magnitude of residual over the magnitude of original vector b and given by

$$\frac{\|Ax - b\|}{\|b\|}. \quad (20)$$

However, in this study of FFNLME the relative residual error measures how well the calculated solution \tilde{X} satisfies the original equation [26].

Definition 2.17. [27] Given that \tilde{X} is an approximate solution of FFNLME, then the absolute residual, $\|R\|$ is given as follows,

$$\|R\| = \|\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} - \tilde{C}\|, \quad (21)$$

then, the relative residual, ϵ is equal to

$$\begin{aligned} \epsilon &= \frac{\|R\|_F}{\|C\|_F} \\ &= \frac{\|\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} - \tilde{C}\|_F}{\|C\|_F} \end{aligned} \quad (22)$$

$\|\cdot\|_F$ denotes the Frobenius norm, which used to calculate the distance of each element from neutrosophic zero [27].

3. Theoretical Development

In this section, there are several definitions and theorems are presented to support the solution of FFNLME. They are presented as follows.

First, Theorem 3.1 is established and proved by contradiction with aimed to show that the fuzzy neutrosophic coefficients of FFNLME must always be square matrices.

Theorem 3.1. *The fuzzy neutrosophic coefficients \tilde{A} and \tilde{X} of FFNLME $\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} = \tilde{C}$ must be square matrices of the same order.*

Proof. Let fuzzy neutrosophic coefficients \tilde{A} be a square matrix while \tilde{X} be a non-square matrix, such that $\tilde{A}_{p \times p}$ and $\tilde{X}_{p \times q}$, then

$$\tilde{A}_{p \times p}^T \tilde{X}_{p \times q} \oplus \tilde{X}_{p \times q} \tilde{A}_{p \times p}. \tag{23}$$

However, the product of $\tilde{X}_{p \times q}$ and $\tilde{A}_{p \times p}$ cannot be performed since the order of both matrices do not fulfill the condition of matrix multiplication. While, if fuzzy neutrosophic \tilde{A} and \tilde{X} both be the square matrices but in different order, such that $\tilde{A}_{p \times p}$ while $\tilde{X}_{q \times q}$, then

$$\tilde{A}_{p \times p}^T \tilde{X}_{q \times q} \oplus \tilde{X}_{q \times q} \tilde{A}_{p \times p}. \tag{24}$$

The product of both $\tilde{A}_{p \times p}^T \tilde{X}_{q \times q}$ and $\tilde{X}_{p \times q} \tilde{A}_{p \times p}$ also cannot be computed since their dimensions do not satisfy the requirement for matrix multiplication. Meanwhile, if fuzzy neutrosophic \tilde{A} and \tilde{X} be the square matrix with the same order, such that $\tilde{A}_{p \times p}$ and $\tilde{X}_{p \times p}$, then

$$\tilde{A}_{p \times p}^T \tilde{X}_{p \times p} \oplus \tilde{X}_{p \times p} \tilde{A}_{p \times p} = (\tilde{A}^T \tilde{X})_{p \times p} \oplus (\tilde{X} \tilde{A})_{p \times p}. \tag{25}$$

$\tilde{A}_{p \times p}$ can be multiplied by $\tilde{X}_{p \times p}$ because their sizes match the rules for matrix multiplication and they can also be added since both have the same size. Thus, in all cases, \tilde{A} and \tilde{X} in FFNLME must be square matrices with the same order. \square

Next, the unitary fuzzy matrix of Theorem 3.2 are extended to include truth, indeterminacy, and falsity components as in the following theorem.

Theorem 3.2. *Let \tilde{U} be the fuzzy neutrosophic identity matrix which can be described as below,*

$$\tilde{U} = \begin{pmatrix} (1, 0, 0; 1, 0, 0) & (0, 0, 0; 0, 1, 1) & \dots & (0, 0, 0; 0, 1, 1) \\ (0, 0, 0; 0, 1, 1) & (1, 0, 0; 1, 0, 0) & \dots & (0, 0, 0; 0, 1, 1) \\ \vdots & \vdots & \ddots & \vdots \\ (0, 0, 0; 0, 1, 1) & (0, 0, 0; 0, 1, 1) & \dots & (1, 0, 0; 1, 0, 0) \end{pmatrix}$$

\tilde{U} satisfies the condition of

- (i) $\tilde{A} \tilde{U} = \tilde{U} \tilde{A} = \tilde{A}$
- (ii) $\tilde{U}^T = \tilde{U}$

Proof. For condition (i), consider

$$\tilde{A} = \begin{pmatrix} (m_{11}, \alpha_{11}, \beta_{11}; T_{11}, I_{11}, F_{11}) & (m_{12}, \alpha_{12}, \beta_{12}; T_{12}, I_{12}, F_{12}) \\ (m_{21}, \alpha_{21}, \beta_{21}; T_{21}, I_{21}, F_{21}) & (m_{22}, \alpha_{22}, \beta_{22}; T_{22}, I_{22}, F_{22}) \end{pmatrix},$$

and

$$\tilde{U} = \begin{pmatrix} (1, 0, 0; 1, 0, 0) & (0, 0, 0; 0, 1, 1) \\ (0, 0, 0; 0, 1, 1) & (1, 0, 0; 1, 0, 0) \end{pmatrix}. \tag{26}$$

By applying the multiplication arithmetic operator of positive LR-TriNFN as defined in Definition 3.4,

$$\tilde{A}\tilde{U} = \begin{pmatrix} (m_{11}, \alpha_{11}, \beta_{11}; T_{11}, I_{11}, F_{11}) & (m_{12}, \alpha_{12}, \beta_{12}; T_{12}, I_{12}, F_{12}) \\ (m_{21}, \alpha_{21}, \beta_{21}; T_{21}, I_{21}, F_{21}) & (m_{22}, \alpha_{22}, \beta_{22}; T_{22}, I_{22}, F_{22}) \end{pmatrix},$$

meanwhile,

$$\tilde{U}\tilde{A} = \begin{pmatrix} (m_{11}, \alpha_{11}, \beta_{11}; T_{11}, I_{11}, F_{11}) & (m_{12}, \alpha_{12}, \beta_{12}; T_{12}, I_{12}, F_{12}) \\ (m_{21}, \alpha_{21}, \beta_{21}; T_{21}, I_{21}, F_{21}) & (m_{22}, \alpha_{22}, \beta_{22}; T_{22}, I_{22}, F_{22}) \end{pmatrix},$$

therefore,

$$\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}.$$

On the other hand, for the condition (ii), by implementing method of matrix transpose, such that

$$\tilde{U}^T = \begin{pmatrix} (1, 0, 0; 1, 0, 0) & (0, 0, 0; 0, 1, 1) \\ (0, 0, 0; 0, 1, 1) & (1, 0, 0; 1, 0, 0) \end{pmatrix}$$

which is equivalent to Equation (26). Therefore,

$$\tilde{U}^T = \tilde{U}.$$

□

Theorem 3.3. Let \tilde{A} and \tilde{X} be the square matrices with the same order, such that $\tilde{A} = (\tilde{a}_{ij})_{m \times m}$, $\tilde{X} = (\tilde{x}_{ij})_{m \times m}$ and $\tilde{C} = (\tilde{c}_{ij})_{m \times m}$, the fully fuzzy neutrosophic linear systems (FFNLS) of $\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} = \tilde{C}$ is equal to

$$\tilde{D} \tilde{X} = \tilde{C} \tag{27}$$

where $\tilde{D} = (\tilde{U}_{n \times n} \otimes_k \tilde{A}_{m \times m}^T + \tilde{A}_{m \times m}^T \otimes_k \tilde{U}_{n \times n})$, $\tilde{X} = \text{Vec}(\tilde{X})_{mn \times 1}$ and $\tilde{C} = \text{Vec}(\tilde{C})_{mn \times 1}$.

Proof. The FFNLME of Equation (1) is reformulated into linear equation by vectorizing the equation as follows,

$$\text{Vec}(\tilde{A}_{m \times m}^T \tilde{X}_{m \times m} + \tilde{X}_{m \times m} \tilde{A}_{m \times m}) = \text{Vec}(\tilde{C})_{m \times m}. \tag{28}$$

Applying the properties of vectorization operator of Definition 2.12 (iii),

$$\text{Vec}(\tilde{A}_{m \times m}^T \tilde{X}_{m \times m}) + \text{Vec}(\tilde{X}_{m \times m} \tilde{A}_{m \times m}) = \text{Vec}(\tilde{C})_{m \times m}. \tag{29}$$

Then, the following equation is obtained by adapting Definition 2.12 (ii) and (i), where

$$(\tilde{U}_{n \times n} \otimes_k \tilde{A}_{m \times m}^T + \tilde{A}_{m \times m}^T \otimes_k \tilde{U}_{n \times n}) \text{Vec}(\tilde{X})_{mn \times 1} = \text{Vec}(\tilde{C})_{mn \times 1} \tag{30}$$

and equivalent to

$$\tilde{D} \tilde{X} = \tilde{C} \tag{31}$$

where $\tilde{D} = (\tilde{U}_{n \times n} \otimes_k \tilde{A}_{m \times m}^T + \tilde{A}_{m \times m}^T \otimes_k \tilde{U}_{n \times n})$, $\tilde{X} = \text{Vec}(\tilde{X})_{mn \times 1}$ and $\tilde{C} = \text{Vec}(\tilde{C})_{mn \times 1}$. □

Previously, the multiplication operation of TriNFN is calculated based on the formula as in Equation (10). In this study, since the LR-TriNFN is used as the coefficient of FFNLME which in the form of $(m, \alpha, \beta; T, I, F)$, then the multiplication operation of Equation (5) is adapted into the existing operator of Equation (10). Therefore, the new multiplication operator is defined as follows.

Definition 3.4. Let $X_1 = (m_1, \alpha_1, \beta_1; T_1, I_1, F_1)$ and $X_2 = (m_2, \alpha_2, \beta_2; T_2, I_2, F_2)$ be the positive LR-TriNFN. The multiplication arithmetic operator is defined as belows.

$$\begin{aligned}
 X_1 \times X_2 = & (m_1 m_2, m_1 m_2 - \text{Min}[(m_1 - \alpha_1)(m_2 - \alpha_2), (m_1 + \beta_1)(m_2 - \alpha_2)], \\
 & \text{Max}[(m_1, \alpha_1)(m_2 + \beta_2), (m_1 + \beta_1)(m_2 - \beta_2)]); \\
 & T_1 T_2, I_1 + I_2 - I_1 I_2, F_1 + F_2 - F_1 F_2).
 \end{aligned} \tag{32}$$

Besides that, the additional operation of LR-TriNFN is executed according to the Equation (9), such that

$$X_1 + X_2 = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2; T_1 + T_2 - T_1 T_2, I_1 I_2, F_1 F_2). \tag{33}$$

4. Algorithm for solving FFNLME

Step 1: FFNLME is reduced to FFNLS

The FFNLME is reduced to FFNLS, $\tilde{D}\tilde{X} = \tilde{C}$ by using the fuzzy neutrosophic Kronecker product and neutrosophic fuzzy Vec-operator as defined in Definition 3.3.

Step 2: FFNLS is deneutrosophicated to crisp form

The conversion of FFNLS to a crisp form is carried out by treating separately the component of LR-TFN, (m, α, β) and the neutrosophic components, (T, I, F) .

Consider FFNLS $\tilde{D}\tilde{X} = \tilde{C}$ in the following form of

$$\sum_{j=1, \dots, q}^{\oplus} (m_{ij}^{\tilde{D}}, \alpha_{ij}^{\tilde{D}}, \beta_{ij}^{\tilde{D}}; T_{ij}^{\tilde{D}}, I_{ij}^{\tilde{D}}, F_{ij}^{\tilde{D}}) \otimes (m_j^{\tilde{X}}, \alpha_j^{\tilde{X}}, \beta_j^{\tilde{X}}; T_j^{\tilde{X}}, I_j^{\tilde{X}}, F_j^{\tilde{X}}) = (m_i^{\tilde{C}}, \alpha_i^{\tilde{C}}, \beta_i^{\tilde{C}}; T_i^{\tilde{C}}, I_i^{\tilde{C}}, F_i^{\tilde{C}}); \tag{34}$$

for $i = 1, \dots, p$, where $\tilde{D} = (\tilde{D}_{ij})_{mn \times mn}$, $\tilde{X} = (\tilde{X}_j)_{mn \times 1}$ and $\tilde{C} = (\tilde{C}_i)_{mn \times 1}$. Subsequently, the implementation is described as follows:

i. Component of LR-TriFN of (m, α, β)

The conversion of the LR-TriFN is aimed to form an associated linear system as defined in Definition 2.15. Next, the following sub-matrices of $(m^{\tilde{D}} - \alpha^D)$, $(m^{\tilde{D}} - \alpha^D)^+$, $(m^{\tilde{D}} - \alpha^D)^-$, as well as $(m^{\tilde{D}} + \alpha^D)$, $(m^{\tilde{D}} + \alpha^D)^+$ and $(m^{\tilde{D}} + \alpha^D)^-$ are defined. All the sub-matrices are in the form of crisp matrices. Subsequently, all sub-matrices are substituted into a general form of an associated linear system as given in the Equation (18). Thereby a crisp form of linear system

$$D_1 X_1 = C_1 \tag{35}$$

is established, which can be equivalent to

$$\left(\begin{array}{c|cc} m^{\tilde{D}_1} & 0 & 0 \\ \hline \alpha^{\tilde{D}_1} & (m^{\tilde{D}_1} - \alpha^{\tilde{D}_1}) & 0 \\ \hline \beta^{\tilde{D}_1} & 0 & (m^{\tilde{D}_1} + \beta^{\tilde{D}_1}) \end{array} \right) \left(\begin{array}{c} m^{\tilde{X}_1} \\ \alpha^{\tilde{X}_1} \\ \beta^{\tilde{X}_1} \end{array} \right) = \left(\begin{array}{c} m^{\tilde{C}_1} \\ \alpha^{\tilde{C}_1} \\ \beta^{\tilde{C}_1} \end{array} \right). \tag{36}$$

ii. **Component of neutrosophic numbers of (T, I, F)**

On the other hand, the conversion of the neutrosophic component (T, I, F) is executed by applying the score function method as stated in Equation (19). Generally, the conversion of $(T_{11}^{\tilde{D}}, I_{11}^{\tilde{D}}, F_{11}^{\tilde{D}})$ to a crisp form is

$$s(T_{11}^{\tilde{D}}, I_{11}^{\tilde{D}}, F_{11}^{\tilde{D}}) = \frac{T_{11}^{\tilde{D}} + (1 - I_{11}^{\tilde{D}}) + (1 - F_{11}^{\tilde{D}})}{3} = D_{11} \tag{37}$$

Hence the following crisp form of linear system is obtained,

$$\left(\begin{array}{cccc} D_{11} & D_{12} & \dots & D_{1n} \\ D_{21} & D_{22} & \dots & D_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ D_{nn} & D_{nn} & \dots & D_{nn} \end{array} \right) \left(\begin{array}{c} x_{11} \\ x_{12} \\ \vdots \\ x_{nn} \end{array} \right) = \left(\begin{array}{cccc} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{nn} & c_{nn} & \dots & c_{nn} \end{array} \right), \tag{38}$$

which can be represented as

$$D_2 X_2 = C_2. \tag{39}$$

Step 3: Obtaining the solution \tilde{X}

Hence, based on Equation (36), the solution X_1 can obtained by

$$\left(\begin{array}{c} m^{\tilde{X}_1} \\ \alpha^{\tilde{X}_1} \\ \beta^{\tilde{X}_1} \end{array} \right) = \left(\begin{array}{c|cc} m^{\tilde{D}_1} & 0 & 0 \\ \hline \alpha^{\tilde{D}_1} & (m^{\tilde{D}_1} - \alpha^{\tilde{D}_1}) & 0 \\ \hline \beta^{\tilde{D}_1} & 0 & (m^{\tilde{D}_1} + \beta^{\tilde{D}_1}) \end{array} \right)^{-1} \left(\begin{array}{c} m^{\tilde{C}_1} \\ \alpha^{\tilde{C}_1} \\ \beta^{\tilde{C}_1} \end{array} \right), \tag{40}$$

or in fuzzy form,

$$\tilde{X}_1 = \left(\begin{array}{cccc} (m_{11}^{\tilde{X}_1}, \alpha_{11}^{\tilde{X}_1}, \beta_{11}^{\tilde{X}_1}) & (m_{12}^{\tilde{X}_1}, \alpha_{12}^{\tilde{X}_1}, \beta_{12}^{\tilde{X}_1}) & \dots & (m_{1n}^{\tilde{X}_1}, \alpha_{1n}^{\tilde{X}_1}, \beta_{1n}^{\tilde{X}_1}) \\ \vdots & \vdots & \ddots & \vdots \\ (m_{m1}^{\tilde{X}_1}, \alpha_{m1}^{\tilde{X}_1}, \beta_{m1}^{\tilde{X}_1}) & (m_{m2}^{\tilde{X}_1}, \alpha_{m2}^{\tilde{X}_1}, \beta_{m2}^{\tilde{X}_1}) & \dots & (m_{mn}^{\tilde{X}_1}, \alpha_{mn}^{\tilde{X}_1}, \beta_{mn}^{\tilde{X}_1}) \end{array} \right). \tag{41}$$

Meanwhile, for X_2 which is the solution of the Equation (39), obtained by

$$\left(\begin{array}{c} x_{11} \\ x_{12} \\ \vdots \\ x_{nn} \end{array} \right) = \left(\begin{array}{cccc} D_{11} & D_{12} & \dots & D_{1n} \\ D_{21} & D_{22} & \dots & D_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ D_{nn} & D_{nn} & \dots & D_{nn} \end{array} \right)^{-1} \left(\begin{array}{c} c_{11} \\ c_{12} \\ \vdots \\ c_{nn} \end{array} \right). \tag{42}$$

As the obtained solution X_2 is a crisp matrix, conversion to a neutrosophic number is required. However, to the best of our knowledge, there is currently no literature that provides a formal approach for converting a crisp number into a neutrosophic representation. Therefore,

for the sake of simplicity and illustrative purposes, this study considers the obtained value of X_2 as the truth-membership value (T), while the indeterminacy (I) and falsity (F) components are assigned based on reasonable assumptions or approximations. It is noted that, the assumption and approximations value must ensure that the basic fundamental of neutrosophic number is imposed, which allows numbers within $[0, 1]$, with $0^+ < T + I + F \leq 3^+$ as stated earlier in the Equation (8).

Therefore, the solution \tilde{X}_2 after represented in a neutrosophic form, can be generally written as,

$$\tilde{X}_2 = \begin{pmatrix} (T_{11}^{\tilde{X}_2}, I_{11}^{\tilde{X}_2}, F_{11}^{\tilde{X}_2}) & \dots & (T_{1n}^{\tilde{X}_2}, I_{1n}^{\tilde{X}_2}, F_{1n}^{\tilde{X}_2}) \\ \vdots & \ddots & \vdots \\ (T_{m1}^{\tilde{X}_2}, I_{m1}^{\tilde{X}_2}, F_{m1}^{\tilde{X}_2}) & \dots & (T_{mn}^{\tilde{X}_2}, I_{mn}^{\tilde{X}_2}, F_{mn}^{\tilde{X}_2}) \end{pmatrix}. \tag{43}$$

Subsequently, the final solution, \tilde{X} therefore is obtained by assemble both answers obtained in Equation (41) and Equation (43) to be in the following form,

$$\tilde{X} = \begin{pmatrix} (m_{11}^{\tilde{X}_1}, \alpha_{11}^{\tilde{X}_1}, \beta_{11}^{\tilde{X}_1}; T_{11}^{\tilde{X}_2}, I_{11}^{\tilde{X}_2}, F_{11}^{\tilde{X}_2}) & \dots & (m_{1n}^{\tilde{X}_1}, \alpha_{1n}^{\tilde{X}_1}, \beta_{1n}^{\tilde{X}_1}; T_{1n}^{\tilde{X}_2}, I_{1n}^{\tilde{X}_2}, F_{1n}^{\tilde{X}_2}) \\ \vdots & \ddots & \vdots \\ (m_{m1}^{\tilde{X}_1}, \alpha_{m1}^{\tilde{X}_1}, \beta_{m1}^{\tilde{X}_1}; T_{m1}^{\tilde{X}_2}, I_{m1}^{\tilde{X}_2}, F_{m1}^{\tilde{X}_2}) & \dots & (m_{mn}^{\tilde{X}_1}, \alpha_{mn}^{\tilde{X}_1}, \beta_{mn}^{\tilde{X}_1}; T_{mn}^{\tilde{X}_2}, I_{mn}^{\tilde{X}_2}, F_{mn}^{\tilde{X}_2}) \end{pmatrix} \tag{44}$$

5. Numerical Example

Example 5.1. Consider a FFNLME of $\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} = \tilde{C}$,

$$\begin{pmatrix} (11, 4, 3; 0.3, 0.5, 0.1) & (5, 1, 3; 0.9, 0.7, 0.2) \\ (7, 2, 1; 0.8, 0.2, 0.4) & (8, 6, 9; 0.6, 0.4, 0.3) \end{pmatrix}^T \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} + \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} \begin{pmatrix} (11, 4, 3; 0.3, 0.5, 0.1) & (5, 1, 3; 0.9, 0.7, 0.2) \\ (7, 2, 1; 0.8, 0.2, 0.4) & (8, 6, 9; 0.6, 0.4, 0.3) \end{pmatrix} \\ = \begin{pmatrix} (157, 89, 223; 0.7, 0.13, 0.2) & (140, 84, 415; 0.83, 0.13, 0.22) \\ (235, 143, 444; 0.87, 0.14, 0.08) & (177, 121, 539; 0.89, 0.18, 0.09) \end{pmatrix}$$

Step 1

The FFNLME of $\tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} = \tilde{C}$ is reduced into FFNLS, $\tilde{D} \tilde{X} = \tilde{C}$ where $\tilde{D} = (\tilde{U}_{n \times n} \otimes_k \tilde{A}_{m \times m}^T + \tilde{A}_{m \times m}^T \otimes_k \tilde{U}_{n \times n})$, $\tilde{X} = Vec(\tilde{X})_{mn \times 1}$ and $\tilde{C} = Vec(\tilde{C})_{mn \times 1}$ as shown in Equation (28).

First, find $\tilde{U}_{2 \times 2} \otimes_k \tilde{A}_{2 \times 2}^T$. Here, the size of the fuzzy neutrosophic identity matrix of 2×2 is chosen to match the order of \tilde{A}^T which also in the same order. The corresponding calculations

are presented as follows.

$$\begin{aligned}
 & (\tilde{U}_{2 \times 2} \otimes_k \tilde{A}_{2 \times 2}^T) \\
 &= \begin{pmatrix} (11, 4, 3; 0.3, 0.5, 0.1) & (7, 2, 1; 0.8, 0.2, 0.4) & (0, 0, 0; 0, 1, 1) & (0, 0, 0; 0, 1, 1) \\ (5, 1, 3; 0.9, 0.7, 0.2) & (8, 6, 9; 0.6, 0.4, 0.3) & (0, 0, 0; 0, 1, 1) & (0, 0, 0; 0, 1, 1) \\ (0, 0, 0; 0, 1, 1) & (0, 0, 0; 0, 1, 1) & (11, 4, 3; 0.3, 0.5, 0.1) & (7, 2, 1; 0.8, 0.2, 0.4) \\ (0, 0, 0; 0, 1, 1) & (0, 0, 0; 0, 1, 1) & (5, 1, 3; 0.9, 0.7, 0.2) & (8, 6, 9; 0.6, 0.4, 0.3) \end{pmatrix}
 \end{aligned}$$

Similarly for $(\tilde{A}_{2 \times 2}^T \otimes_k \tilde{U}_{2 \times 2})$,

$$\begin{aligned}
 & \tilde{A}_{2 \times 2}^T \otimes_k \tilde{U}_{2 \times 2} \\
 &= \begin{pmatrix} (11, 4, 3; 0.3, 0.5, 0.1) & (0, 0, 0; 0, 1, 1) & (7, 2, 1; 0.8, 0.2, 0.4) & (0, 0, 0; 0, 1, 1) \\ (0, 0, 0; 0, 1, 1) & (11, 4, 3; 0.3, 0.5, 0.1) & (0, 0, 0; 0, 1, 1) & (7, 2, 1; 0.8, 0.2, 0.4) \\ (5, 1, 3; 0.9, 0.7, 0.2) & (0, 0, 0; 0, 1, 1) & (8, 6, 9; 0.6, 0.4, 0.3) & (0, 0, 0; 0, 1, 1) \\ (0, 0, 0; 0, 1, 1) & (5, 1, 3; 0.9, 0.7, 0.2) & (0, 0, 0; 0, 1, 1) & (8, 6, 9; 0.6, 0.4, 0.3) \end{pmatrix}
 \end{aligned}$$

Next, execute $(\tilde{U} \otimes_k \tilde{A}^T) \oplus (\tilde{A}^T \otimes_k \tilde{U})$ to obtain \tilde{D} . Hence, the FFNLS, $\tilde{D}\tilde{X} = \tilde{C}$ is formed as follows,

$$\begin{aligned}
 & \begin{pmatrix} (22, 8, 6; 0.51, 0.25, 0.01) & (7, 2, 1; 0.8, 0.2, 0.4) & (7, 2, 1; 0.8, 0.2, 0.4) & (0, 0, 0; 0, 1, 1) \\ (5, 1, 3; 0.9, 0.7, 0.2) & (19, 10, 12; 0.72, 0.2, 0.03) & (0, 0, 0; 0, 1, 1) & (7, 2, 1; 0.8, 0.2, 0.4) \\ (5, 1, 3; 0.9, 0.7, 0.2) & (0, 0, 0; 0, 1, 1) & (19, 10, 12; 0.72, 0.2, 0.03) & (7, 2, 1; 0.8, 0.2, 0.4) \\ (0, 0, 0; 0, 1, 1) & (5, 1, 3; 0.9, 0.7, 0.2) & (5, 1, 3; 0.9, 0.7, 0.2) & (16, 12, 18; 0.84, 0.16, 0.09) \end{pmatrix} \\
 & \begin{pmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \tilde{x}_{12} \\ \tilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (157, 89, 223; 0.7, 0.13, 0.2) \\ (235, 143, 444; 0.87, 0.14, 0.08) \\ (140, 84, 415; 0.83, 0.13, 0.22) \\ (177, 121, 539; 0.89, 0.18, 0.09) \end{pmatrix}
 \end{aligned}$$

Step 2:

Deneutrosophication of the FFNLS $\tilde{D}\tilde{X} = \tilde{C}$ is carried out, starting with the LR-TriFN (m, α, β) . Based on Equation (18), all the coefficients are collected in the crisp matrices as follows,

$$\begin{aligned}
 m^{\tilde{D}} &= \begin{pmatrix} 22 & 7 & 7 & 0 \\ 5 & 19 & 0 & 7 \\ 5 & 0 & 19 & 7 \\ 0 & 5 & 5 & 16 \end{pmatrix}, \alpha^{\tilde{D}} = \begin{pmatrix} 8 & 2 & 2 & 0 \\ 1 & 10 & 0 & 2 \\ 1 & 0 & 10 & 2 \\ 0 & 1 & 1 & 12 \end{pmatrix}, \beta^{\tilde{D}} = \begin{pmatrix} 6 & 1 & 1 & 0 \\ 3 & 12 & 0 & 1 \\ 3 & 0 & 12 & 1 \\ 0 & 3 & 3 & 18 \end{pmatrix}, \\
 m^{\tilde{C}} &= \begin{pmatrix} 157 \\ 235 \\ 140 \\ 177 \end{pmatrix}, \alpha^{\tilde{C}} = \begin{pmatrix} 89 \\ 143 \\ 84 \\ 121 \end{pmatrix}, \beta^{\tilde{C}} = \begin{pmatrix} 223 \\ 444 \\ 415 \\ 539 \end{pmatrix},
 \end{aligned}$$

and the sub-matrices are obtained as follows,

$$\begin{aligned}
 (m^{\tilde{D}} - \alpha^{\tilde{D}}) &= \begin{pmatrix} 14 & 5 & 5 & 0 \\ 4 & 9 & 0 & 5 \\ 4 & 0 & 9 & 5 \\ 0 & 4 & 4 & 4 \end{pmatrix}, (m^{\tilde{D}} - \alpha^{\tilde{D}})^+ = \begin{pmatrix} 14 & 5 & 5 & 0 \\ 4 & 9 & 0 & 5 \\ 0 & 0 & 9 & 5 \\ 0 & 4 & 4 & 4 \end{pmatrix}, (m^{\tilde{D}} - \alpha^{\tilde{D}})^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 (m^{\tilde{D}} + \beta^{\tilde{D}}) &= \begin{pmatrix} 28 & 8 & 8 & 0 \\ 8 & 31 & 0 & 8 \\ 8 & 0 & 31 & 8 \\ 0 & 8 & 8 & 34 \end{pmatrix}, (m^{\tilde{D}} + \beta^{\tilde{D}})^+ = \begin{pmatrix} 28 & 8 & 8 & 0 \\ 8 & 31 & 0 & 8 \\ 8 & 0 & 31 & 8 \\ 0 & 8 & 8 & 34 \end{pmatrix}, (m^{\tilde{D}} + \beta^{\tilde{D}})^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Hence the crisp form of $D_1X_1 = C_1$ is presented as,

$$\left(\begin{array}{cccc|cccc} 22 & 7 & 7 & 0 & 0 & 0 & 0 & 0 \\ 5 & 19 & 0 & 7 & 0 & 0 & 0 & 0 \\ 5 & 0 & 19 & 7 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 16 & 0 & 0 & 0 & 0 \\ \hline 8 & 2 & 2 & 0 & 14 & 5 & 5 & 0 \\ 1 & 10 & 0 & 2 & 4 & 9 & 0 & 5 \\ 1 & 0 & 10 & 9 & 4 & 0 & 9 & 5 \\ 0 & 1 & 1 & 12 & 0 & 4 & 4 & 4 \\ \hline 6 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 12 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 12 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 18 & 0 & 0 & 0 & 0 \\ \hline & & & & 28 & 8 & 8 & 0 \\ & & & & 8 & 31 & 0 & 8 \\ & & & & 8 & 0 & 31 & 8 \\ & & & & 0 & 8 & 8 & 34 \end{array} \right) \begin{pmatrix} m_{11}^{\tilde{X}_1} \\ m_{21}^{\tilde{X}_1} \\ m_{21}^{\tilde{X}_1} \\ m_{22}^{\tilde{X}_1} \\ \hline \alpha_{11}^{\tilde{X}_1} \\ \alpha_{12}^{\tilde{X}_1} \\ \alpha_{21}^{\tilde{X}_1} \\ \alpha_{22}^{\tilde{X}_1} \\ \hline \beta_{11}^{\tilde{X}_1} \\ \beta_{12}^{\tilde{X}_1} \\ \beta_{21}^{\tilde{X}_1} \\ \beta_{22}^{\tilde{X}_1} \end{pmatrix} = \begin{pmatrix} 157 \\ 235 \\ 140 \\ 177 \\ \hline 89 \\ 143 \\ 84 \\ 121 \\ \hline 223 \\ 444 \\ 415 \\ 539 \end{pmatrix}. \tag{45}$$

On the other hand, the crisp form of the neutrosophic elements, $(T, I, F) = (0.51, 0.25, 0.01)$ is obtained using the score function method described in Definition 2.16. For illustration, the conversion of the element (T, I, F) of the coefficient \tilde{D} is presented as follows.

$$\begin{aligned}
 s(T_{11}, I_{11}, F_{11}) &= \frac{T_{11} + (1 - I_{11}) + (1 - F_{11})}{3} \\
 &= \frac{0.51 + (1 - 0.25) + (1 - 0.01)}{3} \\
 &= 0.75.
 \end{aligned}$$

Hence, the crisp linear system in the form of $D_2X_2 = C_2$ is obtained as follows,

$$\begin{pmatrix} 0.75 & 0.73 & 0.73 & 0 \\ 0.67 & 0.83 & 0 & 0.73 \\ 0.67 & 0 & 0.75 & 0.73 \\ 0 & 0.67 & 0.67 & 0.73 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0.79 \\ 0.88 \\ 0.83 \\ 0.87 \end{pmatrix}. \tag{46}$$

Step 3

According to the crisp form of $D_1X_1 = C_1$ as shown in Equation 45 , the solution of X_1 is

determined by taking the inverse coefficient of D_1 . Hence,

$$\begin{pmatrix} m_{11}^{\tilde{X}_1} \\ m_{21}^{\tilde{X}_1} \\ m_{12}^{\tilde{X}_1} \\ m_{22}^{\tilde{X}_1} \\ \alpha_{11}^{\tilde{X}_1} \\ \alpha_{21}^{\tilde{X}_1} \\ \alpha_{12}^{\tilde{X}_1} \\ \alpha_{22}^{\tilde{X}_1} \\ \beta_{11}^{\tilde{X}_1} \\ \beta_{21}^{\tilde{X}_1} \\ \beta_{12}^{\tilde{X}_1} \\ \beta_{22}^{\tilde{X}_1} \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 4 \\ 7 \\ 1 \\ 3 \\ 2 \\ 1 \\ 2 \\ 8 \\ 9 \\ 7 \end{pmatrix}$$

which can be expressed and written in the following form,

$$\tilde{X}_1 = \begin{pmatrix} (m_{11}^{\tilde{X}_1}, \alpha_{11}^{\tilde{X}_1}, \beta_{11}^{\tilde{X}_1}) & (m_{12}^{\tilde{X}_1}, \alpha_{12}^{\tilde{X}_1}, \beta_{12}^{\tilde{X}_1}) \\ (m_{21}^{\tilde{X}_1}, \alpha_{21}^{\tilde{X}_1}, \beta_{21}^{\tilde{X}_1}) & (m_{22}^{\tilde{X}_1}, \alpha_{22}^{\tilde{X}_1}, \beta_{22}^{\tilde{X}_1}) \end{pmatrix} = \begin{pmatrix} (3, 1, 2) & (4, 2, 9) \\ (9, 3, 8) & (7, 1, 7) \end{pmatrix}. \tag{47}$$

In addition to that, the solution of the crisp linear system $D_2X_2 = C_2$ is as follows,

$$\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0.296 \\ 0.4 \\ 0.38 \\ 0.48 \end{pmatrix}$$

obtained via the inverse method. The solution can thus be expressed as follows,

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 0.296 & 0.38 \\ 0.4 & 0.48 \end{pmatrix}. \tag{48}$$

The solutions are converted to neutrosophic form (T, I, F) by directly assigning the value obtained in Equation (48) as the component T . While for the other component, I and F , the values are determined by assuming or approximating, which ensures that the resulting satisfy the fundamental condition of neutrosophic as stated in Equation (8),

$$\tilde{X}_2 = \begin{pmatrix} (0.3, 0.4, 0.3) & (0.4, 0.15, 0.45) \\ (0.38, 0.42, 0.2) & (0.48, 0.27, 0.25) \end{pmatrix}. \tag{49}$$

Finally, both fuzzy and neutrosophic elements as stated in Equation (47) and Equation (49) are combined into a single matrix to form a triangular neutrosophic fuzzy matrix, \tilde{X} as the following,

$$\tilde{X} = \begin{pmatrix} (3, 1, 2; 0.3, 0.4, 0.3) & (4, 2, 9; 0.4, 0.15, 0.45) \\ (9, 3, 8; 0.38, 0.42, 0.2) & (7, 1, 7; 0.48, 0.27, 0.25) \end{pmatrix}.$$

Each component of the solution \tilde{X} is illustrated in the following graphs.

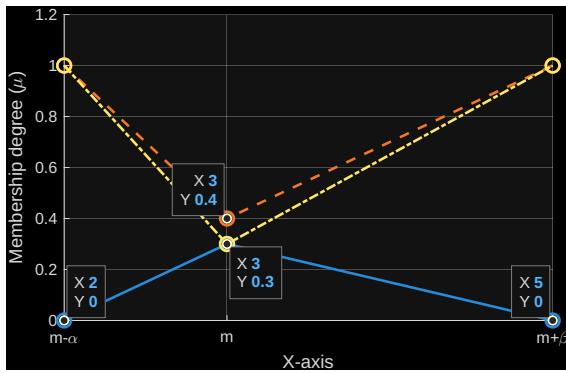


FIGURE 1. Component $\tilde{X}_{11} = (3, 1, 2; 0.3, 0.4, 0.3)$

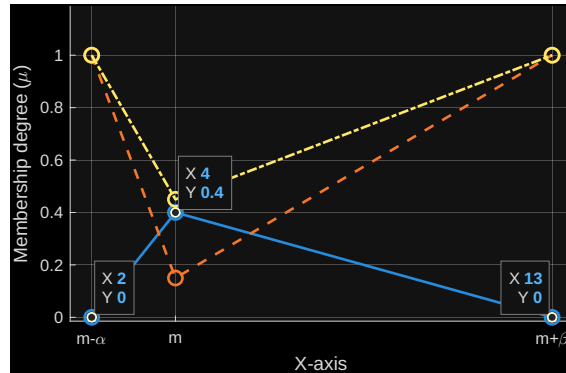


FIGURE 2. Component $\tilde{X}_{12} = (4, 2, 9; 0.4, 0.15, 0.45)$

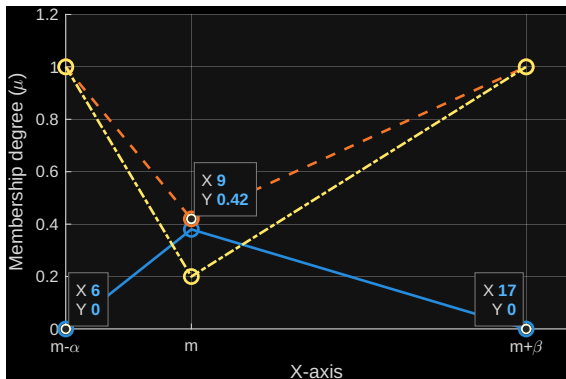


FIGURE 3. Component $\tilde{X}_{21} = (9, 3, 8; 0.38, 0.42, 0.2)$

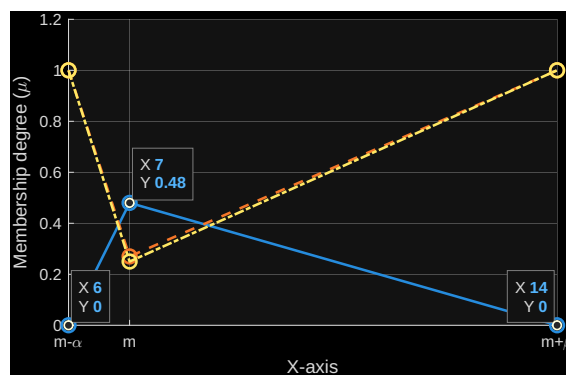


FIGURE 4. Component $\tilde{X}_{22} = (7, 1, 7; 0.48, 0.27, 0.25)$

Figures 1 to 4 illustrate the graphical representations of solutions for fuzzy neutrosophic numbers. In these figures, the horizontal axis corresponds to the domain of the triangular fuzzy numbers, while the vertical axis represents the degree of membership, which ranges from 0 to 1. The three distinct membership functions which are Truth (T), Indeterminacy (I) and Falsity (F) are depicted simultaneously to provide a comprehensive understanding of their behavior. For instance, as shown in Figure 1, the neutrosophic fuzzy element $\tilde{X}_{11} = (3, 1, 2; 0.3, 0.4, 0.3)$ is defined over the interval $[2, 5]$. The truth membership function increases linearly from 0 at $m - \alpha = 2$ to its peak value of 0.3 at $m = 3$ and then decreases linearly back to 0 at $m + \beta = 5$. The indeterminacy and falsity membership functions follow the same interval but reach different peak values of 0.4 and 0.3, respectively. By plotting these three membership functions within the same interval, the figure effectively visualizes how truth, indeterminacy and falsity coexist and interact in representing a neutrosophic fuzzy element.

6. Verification of the Solution

In this section, the verification of the obtained solution from the previous section is outlined. It is carried out by substituting the solution of \tilde{X} into the FFNLME of Example 5.1, as follows,

$$\begin{aligned}
 & \tilde{A}^T \tilde{X} + \tilde{X} \tilde{A} \\
 &= \begin{pmatrix} (11, 4, 3; 0.3, 0.5, 0.1) & (7, 2, 1; 0.8, 0.2, 0.4) \\ (5, 1, 3; 0.9, 0.7, 0.2) & (8, 6, 9; 0.6, 0.4, 0.3) \end{pmatrix} \begin{pmatrix} (3, 1, 2; 0.3, 0.4, 0.3) & (4, 2, 9; 0.4, 0.15, 0.45) \\ (9, 3, 8; 0.38, 0.42, 0.2) & (7, 1, 7; 0.48, 0.27, 0.25) \end{pmatrix} \\
 &+ \begin{pmatrix} (3, 1, 2; 0.3, 0.4, 0.3) & (4, 2, 9; 0.4, 0.15, 0.45) \\ (9, 3, 8; 0.38, 0.42, 0.2) & (7, 1, 7; 0.48, 0.27, 0.25) \end{pmatrix} \begin{pmatrix} (11, 4, 3; 0.3, 0.5, 0.1) & (5, 1, 3; 0.9, 0.7, 0.2) \\ (7, 2, 1; 0.8, 0.2, 0.4) & (8, 6, 9; 0.6, 0.4, 0.3) \end{pmatrix} \\
 &= \begin{pmatrix} (157, 89, 223; 0.6, 0.1, 0.04) & (140, 84, 415; 0.7, 0.1, 0.1) \\ (235, 143, 444; 0.7, 0.15, 0.03) & (177, 121, 539; 0.8, 0.2, 0.05) \end{pmatrix} \\
 &= \tilde{C}^*.
 \end{aligned} \tag{50}$$

The solution matrix \tilde{C}^* obtained in Equation(50) after substituting into Example 5.1 is found to be approximately equal to the original matrix \tilde{C} stated in the Example 5.1. The minor discrepancies are likely attributable to round-off errors introduced during multiplication operations of neutrosophic numbers.

Hence, the relative residual error, ϵ stated in Equation (20) is employed to quantify the deviation between C and C^* , as follows

$$\epsilon = \frac{\|\tilde{C}^* - \tilde{C}\|}{\|\tilde{C}\|}.$$

Firstly, by using score function approach, the crisp form of \tilde{C}^* and \tilde{C} are obtained as below,

$$\tilde{C}^* = \begin{pmatrix} 0.82 & 0.83 \\ 0.84 & 0.85 \end{pmatrix}, \tilde{C} = \begin{pmatrix} 0.79 & 0.83 \\ 0.83 & 0.87 \end{pmatrix} \tag{51}$$

Next, norm which is the single value of the matrix in Equation (51) is calculated as

$$\begin{aligned}
 \|\tilde{C}^* - \tilde{C}\|_F &= \sqrt{(0.82 - 0.79)^2 + (0.83 - 0.83)^2 + (0.84 - 0.83)^2 + (0.85 - 0.87)^2} \\
 &= \sqrt{(0.03)^2 + (0)^2 + (0.01)^2 + (-0.02)^2} \\
 &= 0.0374 \\
 \|\tilde{C}\|_F &= \sqrt{(0.79)^2 + (0.83)^2 + (0.83)^2 + (0.87)^2} \\
 &= 1.661
 \end{aligned}$$

then,

$$\begin{aligned}
 \epsilon &= \frac{0.0374}{1.661} \\
 &= 0.02.
 \end{aligned}$$

The tolerance value of $\epsilon = 0.02$ should be considered due to the preliminary approximations made for the component I and F . Currently, no formal literature provides a standard

for converting crisp results back into neutrosophic form, requiring assumptions that naturally influence the final residual. Furthermore, these deviations are attributed to the algebraic complexities of neutrosophic multiplication and inherent floating-point round-off errors. Despite this, the proposed algorithm can still be acceptable and applicable, offering a flexible representation of uncertainty that extends the capabilities of classical or fuzzy Lyapunov approaches.

7. Conclusion

In conclusion, this study has developed a framework to solve FFNLME, which generalizes the classical Lyapunov formulation to incorporate truth, indeterminacy and falsity components. By applying neutrosophic representations with correspondent solution methods, the new approach is capable to accommodate uncertainties as well as capturing inconsistencies that cannot be presented using conventional or fuzzy models. It is shown that the neutrosophic Lyapunov formulation provides a comprehensive and flexible tools for analyzing the dynamic systems that deal with uncertainties. In the future, the focus on other neutrosophic numbers such as trapezoidal and pentagonal neutrosophic numbers can be implemented and extend the framework to large-scale systems. This is important in developing efficient numerical algorithms and integrating optimization-based or iterative methods to enhance computational efficiency.

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