NSS Neutrosophic Sets and Systems, Vol. 100, 2026

University of New Mexico

An Approach to Bilevel Multi-Objective Linear Fractional Probabilistic Optimization in Neutrosophic Environments with Applications

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Abstract: In real-life organization sectors, decision-makers (DMs) always encounter several complexities like hierarchical decision-process with multiple conflict ratio goals, constraints associated with probability and uncertainty parameter with hesitance scope. Despite the important role of optimization method for real-life problem, a very few success has been achieved to deal with this applicable complex problem. Thus, the current study has focused on investigating an efficient method consisting of a goal-programming (GP), chance-constrained optimization (CCO) and intuitionistic fuzzy mathematical (IFM) approach for solving hierarchical multi-objective linear fractional probabilistic optimization (HMOLFPO) problems under neutrosophic environment. Here each hierarchical decision-making optimize numerous conflict affain-ratio objective functions under chance constraints with uncertain coefficients and quantity defined as neutrosophic linear numbers of the form $\gamma + \omega I$ where $\gamma, \omega \in \mathbb{R}$ and I represents indeterminacy. In the proposed method, first the crisp linear model of neutrosophic HMOLFPO problem is formulated using the concept of interval ratio, CCO and linearized strategy. Then, the membership and non-memberships are investigated to measure the degree of satisfaction and dissatisfaction for each imprecise goal of DMs. Finally, GP with an IFM method is developed to generate a non-dominant solution for all DMs at both levels. Unlike existing methods, these strategy helps to avoid a decision-locks between upper and lower DMs. The applicability and efficiency of the suggested method are illustrated by solving numerical examples and industrial production real-life problem.

Keywords: Multi-Objective Fractional Problem · Chance-Constrained Optimization · Bi-level Decision-Making Problem · Neutrosophic Linear Number · Intuitionistic Fuzzy Goal Programming.

1 Introduction

Decentralized bi-level decision-making (DBLDM) problems, stated as mathematical programming with a sole DM at the leader-level and numerous DMs at the follower-level, are frequently found in the hierarchical structures of real-life applications [36, 37]. In DBLDM problem, the ULDM (standing for Upper-Level Decision-Maker) decides his goals and/or decision-control choices first, then each LLDM (standing for Lower-Level Decision-Maker) react by selecting their optimal decision variables that optimize the LLDM problem. That means, even though the ULDM independently maximizes its own benefits, the LLDM's response could influence the decision. As a result, most real-world decision scenarios involve the issue of distributing decision authority appropriately, which commonly leads to decision deadlock [8, 25, 39, 41].

In most real-world decision-making problems, the values to be allocated to the parameters cannot be exactly determined due to ambiguous and insufficient information. In order to solve this issue, fuzzy sets (FSs) have been introduced by Bellman and Zadhe [10] to simplify the task of demonstrating the imprecise value of parameters with the aid of fuzzy numbers. Since fuzzy numbers only take into account the truth-membership function, one of their main drawbacks is their inability to effectively capture ambiguous and inconsistent data with hesitation aspects [11, 15, 35, 38]. An extension of FSs that takes into account the membership for truth and falsity function is called an intuitionistic fuzzy set (IFS) introduced by Atanassov [7] to overcome these drawbacks. In the last decade, a large number of studies and applications on the domain of fuzzy and intuitionistic fuzzy bi-level multi-objective decision-making problems have been presented. Among these studies, we can mention the works of [5, 13, 20–22, 25, 36, 39, 46, 52, 56].

Even with IFS and FS-based optimization methods, the ability to cover imprecise and inconsistent information in various areas is still trailing behind the more realistic scenarios of decision-making, where neutral ideas which neither affirmation nor negation cannot be addressed [3, 37]. As a result, handling the unpredictable conditions of ambiguous information and experience becomes difficult. To address this difficulty, Smarandache [55] proposed the neutrosophic set (NS). This novel set type was designed to handle decision-making scenarios including incomplete, inconsistent, and ambiguous data. In this case, indeterminacy is regarded as an explicitly element that significantly influences decision-making. Many researchers have used the basic concept of NS, such as [23, 28, 29, 37] in decision-making processes. NS uses membership functions for truth, falsehood, and indeterminacy to aid in human-like decision-making.

Significant progress in the research of neutrosophic optimization environments and related practical concerns has been made in recent years. For instance, a neutosophic-GP (NGP) technique was devised by Ahmad et.al. [4] for the multiple objectives in a hierarchical mathematical problem with a rough parameter in objective functions. GP was used by Pramenilk and Dey [43] to solve the multilevel mathematical problem utilizing a neutrosophic number. To address transportation application concerns, Hosseinzadeh and Tayyebi [23] searched for a compromise optimal solution to the multi-objective neutrosophic linear optimization (MONLO) problem. Buvaneshwari and Anuradha [12] created a novel method based on score and accuracy ranking approaches to solve a bi-objective interval-valued neutrosophic assignment problem.

A NGP approach was presented by Maiti et al. [33] to solve the hierarchical with MONLO problem with coefficients provided in the $c + Id$ form, where I denotes indeterminacy and $c, d \in \Re$. Emam et al. [18,19] addressed various neutrosophic optimization problem by considering trapezoidal neutrosophic parameter and quadratic objective type.

In a real-life decision-making environment, despite having vast experience, the DMs cannot always articulate the goals precisely due to the uncertain behavior of the input parameter and the probabilistic nature of the constraints in the problem's model formulation process [14]. For example, the variability of resources—such as time, power, financial resources, and machinery—typically affected from irregular fluctuations or stochastic factors. These unpredictable oscillations are not always foreseeable or preventable; however, they generally adhere to a random process, making the representation of known random variables a logical approach. The discipline of mathematics that examines the theory, and methodologies related to optimization problems that incorporate random variables is known as chance constraint optimization. A limited number of methods have been explored in the literature to address real-world application problems characterized by chance constraints and imprecise input data. A concise overview of existing studies aimed at resolving HMOLFPO problems within FS, IFS, or NS-based optimization methods, along with the contributions of the current study, is presented in Table 1. To the best of our knowledge, there has been no algorithm developed in the literature specifically for addressing HMOLFPO problems under neutrosophic environment.

This study aims to present an efficient methodology for the HMOLFPO problems under neutrosophic environment, characterized by neutrosophic linear numbers for all input parameters and random variables for some right-hand resources. The DMs at hierarchical levels need optimize multiple linear-fractional objective functions. The proposed solution method transforms the introduced problem into a crisp bi-level multi-objective linear optimization problem using interval ratio ranking, stochastic optimization, and modified general linear-transformation techniques, thereby streamlining the process. The degree of achievable for the intuitionistic fuzzy aspiration level of the objective function for each DM, along with the decision-control variable under the upper-level's supervision, is quantified using membership and non-membership functions. Finally, we employ an interactive GP method on the intuitionistic fuzzy decision set to derive a preferred optimal solution.

The paper is structured as follows: Section 2 introduces the theory of the neutrosophic linear number concept. In Section 3, we present the mathematical formulation of the HMOLFPO problem within a neutrosophic optimization environment, along with its crisp and linear model. Section 4 details our proposed method for addressing the newly introduced HMOLFPO problems under neutrosophic environment, using the interactive goal programming method and intuitionistic fuzzy decision sets. The algorithm for the proposed solution method is outlined in Section 5. Finally, Section 6 demonstrates the applicability of the proposed methodology through two numerical examples and production planning problems. Sect. 7 discusses conclusions and further scope based on the work conducted thus far.

2 Basic concepts and preliminaries

This section briefly discusses essential concepts related to neutrosophic sets, neutrosophic linear numbers, and their arithmetic operations needed for this task.

Definition 2.1. [12, 37] Let X be a non-empty general set and $x \in X$. Then a neutrosophic set, $\tilde{\mathbf{A}}^N$ is defined as $\tilde{\mathbf{A}}^N = \{(x, T_{\tilde{\mathbf{A}}^N}(x), I_{\tilde{\mathbf{A}}^N}(x), F_{\tilde{\mathbf{A}}^N}(x))|x \in X\}$ where the membership function $T_{\tilde{\mathbf{A}}^N}|X \to [0,1], I_{\tilde{\mathbf{A}}^N}|X \to [0,1]$ and $F_{\tilde{\mathbf{A}}^N}|X \to [0,1]$ are illustrates the degree of truth, indeterminacy and falsity of member/ element $x \in X$ in $\tilde{\mathbf{A}}^N$ so that $0 \leq T_{\tilde{\mathbf{A}}^N}(x) + I_{\tilde{\mathbf{A}}^N}(x) + F_{\tilde{\mathbf{A}}^N}(x) \leq 3$, unlike IFS where restricted as $T_{\tilde{\mathbf{A}}^N}(x) + I_{\tilde{\mathbf{A}}^N}(x) + F_{\tilde{\mathbf{A}}^N}(x) = 1$

Definition 2.2. [33] A neutrosophic linear number (NLN) is denoted as $N = \gamma + \omega I$ where γ and ω are real/ complex numbers and $I \in [I^L, I^R]$ represents indeterminacy. Here, γ and ω are called determinate and indeterminate part of N, respectively. Thus, the NLN, N can be written as:

$$N = \gamma + \omega I = \gamma + \omega[I^L, I^R] = [\gamma + \omega I^L, \gamma + \omega I^R] = [N^L, N^R](\text{say}).$$

Here, N is transferred into an interval-number $[N^L, N^R]$. Suppose a NLN considered as $N = 3 + 5I$ where 3 is determinate and 5I is indeterminate part. Here, let it be $I \in [0.3, 0.7]$, so N transforms into an interval-number of the form [4.5, 6.5].

Now, the arithmetic operation on NLN discussed as follows: [33, 43] Suppose that two NLNs given as $N_1 = \gamma_1 + \omega_1 I_1 = [\gamma_1 + \omega_1 I_1^L, \gamma_1 + \omega_1 I_1^R] = [N_1^L, N_1^R]$ and $N_2 = \gamma_2 + \omega_2 I_2 = [\gamma_2 + \omega_2 I_2^L, \gamma_2 + \omega_2 I_2^R] = [N_2^L, N_2^R]$ where, $I_1 \in [I_1^L, I_1^R]$ and $I_2 = [I_2^L, I_2^R]$, then

3 Mathematical Development of the Problem

In this section, the mathematical formulation of the introduced neutrosophic bi-level multi-objective linear fractional chance-constraint optimization (NBL-MOLFCCo) problems and its corresponding crisp parameteric, deterministic constraints and linear-models are developed.

3.1 Problem Definition

Assume that a NBL-MOLFCCo problem is constructed having two stage/ hierarchical level decision-makers (DMs) called as ULDM and LLDM, where the ULDM having a single decision-maker and denoted as DM_1^1 that controlling a decision vector $\mathbf{x}_1 = (x_{11}, x_{12}, \ldots x_{1n_1})$ and LLDM having multiple decision-makers and denoted as DM_i^2 where, $l = 2, 3, \ldots, p$ which control over a different decision vector $\mathbf{x}_l = (x_{l1}, x_{l2}, \ldots x_{ln_l})' \in R^{n_l}, \forall l = 2, 3, \ldots, p$. Also assume that each decision-maker $DM_l, l = 1, 2, \ldots, p$ has several affain ratio objective functions with neutrosophic linear coefficients of the form $\gamma + \omega I$ that are independently optimized under the given feasible sets. Thus, the neutrosophic affain fractional objective functions under $DM_l, l = 1, 2, \ldots, p$ are:

$$\tilde{f}_{lk}(\mathbf{x}) = \tilde{f}_{lk}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) : \Re^{n_1} \times \Re^{n_2} \times \cdots \times \Re^{n_p} \to \Re^{l_l} l = 1, \ldots, p, k = 1, 2, \ldots, t_l.$$

Therefore, the mathematical formulation of the neutrosophic bi-level multi-objective linear fractional chance-constraint optimization (NBL-MOLFCCO) problem is defined as follows:

ULDM

$$\max_{\mathbf{x}_1} \tilde{f}_{1k}(\mathbf{x}) = \max_{\mathbf{x}_1} \frac{\tilde{N}_{1k}(\mathbf{x})}{\tilde{D}_{1k}(\mathbf{x})} = \frac{(c_{1k}^1+\bar{c}_{1k}^1 I_{1k}^1)x_1+(c_{1k}^2+\bar{c}_{1k}^2 I_{1k}^2)x_2+\cdots+(c_{1k}^p+\bar{c}_{1k}^p I_{1k}^p)x_p+(\alpha_{1k}+\bar{\alpha}_{1k} I_{1k})}{(d_{1k}^1+\bar{d}_{1k}^1 J_{1k}^1)x_1+(d_{1k}^2+\bar{d}_{1k}^2 J_{1k}^2)x_2+\cdots+(d_{1k}^p+\bar{d}_{1k}^p J_{1k}^p)x_p+(\beta_{1k}+\bar{\beta}_{1k} J_{1k})},$$

where, $x_2, x_3, \ldots, x_p$ solves

LLDM

$$\begin{cases} \max_{\mathbf{x}_2} \tilde{f}_{2k}(\mathbf{x}) = \max_{\mathbf{x}_2} \frac{\tilde{N}_{2k}(\mathbf{x})}{\tilde{D}_{2k}(\mathbf{x})} = \frac{(c_{2k}^1+\bar{c}_{2k}^1 I_{2k}^1)x_1+(c_{2k}^2+\bar{c}_{2k}^2 I_{2k}^2)x_2+\cdots+(c_{2k}^p+\bar{c}_{2k}^p I_{2k}^p)x_p+(\alpha_{2k}+\bar{\alpha}_{2k} I_{2k})}{(d_{2k}^1+\bar{d}_{2k}^1 J_{2k}^1)x_1+(d_{2k}^2+\bar{d}_{2k}^2 J_{2k}^2)x_2+\cdots+(d_{2k}^p+\bar{d}_{2k}^p J_{2k}^p)x_p+(\beta_{2k}+\bar{\beta}_{2k} J_{2k})}, \\ \ldots \\ \ldots \\ \max_{\mathbf{x}_p} \tilde{f}_{pk}(\mathbf{x}) = \max_{\mathbf{x}_p} \frac{\tilde{N}_{pk}(\mathbf{x})}{\tilde{D}_{pk}(\mathbf{x})} = \frac{(c_{pk}^1+\bar{c}_{pk}^1 I_{pk}^1)x_1+(c_{pk}^2+\bar{c}_{pk}^2 I_{pk}^2)x_2+\cdots+(c_{pk}^p+\bar{c}_{pk}^p I_{pk}^p)x_p+(\alpha_{pk}+\bar{\alpha}_{pk} I_{pk})}{(d_{pk}^1+\bar{d}_{pk}^1 J_{pk}^1)x_1+(d_{pk}^2+\bar{d}_{pk}^2 J_{pk}^2)x_2+\cdots+(d_{pk}^p+\bar{d}_{pk}^p J_{pk}^p)x_p+(\beta_{pk}+\bar{\beta}_{pk} J_{pk})}, \end{cases} \quad (1)$$

Subject to:

$$\mathbf{x} \in \tilde{X}_N = \begin{cases} \sum_{i=1}^n (a_{ti}+\bar{a}_{ti} I_{ti})x_i \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b_t + \bar{b}_t I_t, \quad t = 1, 2, \ldots, m \\ Pr\left\{ \sum_{i=1}^n (a_{ti}+\bar{a}_{ti} I_{ti})x_i \begin{pmatrix} \leq \\ \geq \end{pmatrix} b_t \right\} \geq \alpha_t, \quad t = m+1, \ldots, s \\ x_i \geq 0, \quad i = 1, 2, \ldots, n \end{cases}$$

Where,

- $\tilde{f}_{lk}(\mathbf{x}) = \frac{\tilde{N}_{lk}(\mathbf{x})}{\tilde{D}_{lk}(\mathbf{x})}, k = 1, 2, \ldots, t_l$ indicates the neutrosophic affain ratio objective function under DM_l.
- $(c_{lk}^j + \bar{c}_{lk}^j I_{lk}^j)$ and $(d_{lk}^j + \bar{d}_{lk}^j J_{lk}^j)$, $j = 1, 2, \ldots, p$ are n-dimension row vectors of neutrosophic linear coefficients of $\tilde{N}_{lk}$ (numerator) and $\tilde{D}_{lk}$ (denominator) of $\tilde{f}_{lk}(\mathbf{x}) = \frac{\tilde{N}_{lk}}{\tilde{D}_{lk}}$ respectively.
- Assume that $\tilde{D}_{lk}(\mathbf{x}) = (d_{lk}^1 + \bar{d}_{lk}^1 J_{lk}^1)x_1 + \cdots + (d_{lk}^p + \bar{d}_{lk}^p J_{lk}^p)x_p + (\beta_{lk} + \bar{\beta}_{lk} J_{lk}) > 0, \forall \mathbf{x} \in \tilde{X}_N$.
- $b_t, t = m+1, \ldots, s$ are autonomous random variables with follows a Cauchy distribution having mean θ and standard deviation σ.
- $\alpha_i \in (0, 1)$ is act as the admissible risk of constraint violation. "Pr$\{\cdot\}$" represent probability.
- The scalars $\alpha_{lk} + \bar{\alpha}_{lk} I_{lk}$ and $\beta_{lk} + \bar{\beta}_{lk} J_{lk}$ are neutrosophic linear numbers.
- Each decision vectors monitored by DM_l are $\mathbf{x}_l \in \Re^{n_l}, \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \in \Re^n$ with $n = \sum_{l=1}^p n_l$.
- $\tilde{X}_N \neq \emptyset$ is a compact and convex neutrosophic feasible region for the NBL-MOLFCCO problem (1).
- $a_{ti} + \bar{a}_{ti} I_{ti}$ acts as the neutrosophic linear coefficient of the ith decision variable and its right-side quantity is denoted as $b_t + \bar{b}_t I_t, t = 1, 2, \ldots, m$.
- $I_{lk}^j \in [I_{lk}^{Lj}, I_{lk}^{Rj}], J_{lk}^j \in [J_{lk}^{Lj}, J_{lk}^{Rj}], I_{lk} \in [I_{lk}^L, I_{lk}^R], J_{lk} \in [J_{lk}^L, J_{lk}^R], I_t \in [I_t^L, I_t^R], I \in [I^L, I^R]$ and $I_{lk}^{Lj}, I_{lk}^{Rj}, J_{lk}^{Lj}, J_{lk}^{Rj}, I$ are real numbers ($\Re$).

3.2 Formulation of Crisp Model of NBL-MOLFCCO Problem

Following the discussion given in section 2, we reduce each neutrosophic linear number (NLN) into an interval number. Thus, all uncertain coefficients and quantity in NBL-MOLFCCO problem (1) reduced to:

$$\begin{aligned}
&c^j_{lk} + \bar{c}^j_{lk} I^j_{lk} = c^j_{lk} + \bar{c}^j_{lk}[I^{Lj}_{lk}, I^{Rj}_{lk}] = [c^j_{lk} + \bar{c}^j_{lk} I^{Lj}_{lk}, c^j_{lk} + \bar{c}^j_{lk} I^{Rj}_{lk}] = [m_{lk}, n_{lk}] \text{ (say)}\\
&d^j_{lk} + \bar{d}^j_{lk} J^j_{lk} = d^j_{lk} + \bar{d}^j_{lk}[J^{Lj}_{lk}, J^{Rj}_{lk}] = [d^j_{lk} + \bar{d}^j_{lk} J^{Lj}_{lk}, d^j_{lk} + \bar{d}^j_{lk} J^{Rj}_{lk}] = [e_{lk}, f_{lk}] \text{ (say)}\\
&\alpha_{lk} + \bar{\alpha}_{lk} I_{lk} = \alpha_{lk} + \bar{\alpha}_{lk}[I^L_{lk}, I^R_{lk}] = [\alpha_{lk} + \bar{\alpha}_{lk} I^L_{lk}, \alpha_{lk} + \bar{\alpha}_{lk} I^R_{lk}] = [\alpha^L_{lk}, \alpha^R_{lk}] \text{ (say)}\\
&\beta_{lk} + \bar{\beta}_{lk} J_{lk} = \beta_{lk} + \bar{\beta}_{lk}[J^L_{lk}, J^R_{lk}] = [\beta_{lk} + \bar{\beta}_{lk} J^L_{lk}, \beta_{lk} + \bar{\beta}_{lk} J^R_{lk}] = [\beta^L_{lk}, \beta^R_{lk}] \text{ (say)}\\
&a_{ti} + \bar{a}_{ti} I_{ti} = a_{ti} + \bar{a}_{ti}[I^L_{ti}, I^R_{ti}] = [a_{ti} + \bar{a}_{ti} I^L_{ti}, a_{ti} + \bar{a}_{ti} I^R_{ti}] = [a^L_{ti}, a^R_{ti}] \text{ (say)}\\
&b_t + \bar{b}_t I_t = b_t + \bar{b}_t[I^L_t, I^R_t] = [b_t + \bar{b}_t I^L_t, b_t + \bar{b}_t I^R_t] = [b^L_t, b^R_t] \text{ (say)}
\end{aligned} \tag{2}$$

Based on ratio optimization concept [36, 41] and interval number given in Eqs. (2), the neutrosophic objective function $\tilde{f}_{lk}(\mathbf{x}) = \frac{\tilde{N}_{lk}}{\tilde{D}_{lk}}$ of minimization and maximization-form problems are transformed into crisp form $f_{lk}(\mathbf{x}) = \frac{N^L_{lk}(\mathbf{x})}{D^R_{lk}(\mathbf{x})}$ and $f_{lk}(\mathbf{x}) = \frac{N^R_{lk}(\mathbf{x})}{D^L_{lk}(\mathbf{x})}$ respectively, where defined as:

$$f_{lk}(\mathbf{x}) = \frac{N^L_{lk}(\mathbf{x})}{D^R_{lk}(\mathbf{x})} = \frac{m^1_{lk}x_1 + m^2_{lk}x_2 + \ldots + m^p_{lk}x_p + \alpha^L_{lk}}{n^1_{lk}x_1 + n^2_{lk}x_2 + \ldots + n^p_{lk}x_p + \beta^R_{lk}}, \quad l = 1, 2, \ldots, p, k = 1, 2, \ldots, t_l \tag{3}$$

$$f_{lk}(\mathbf{x}) = \frac{N^R_{lk}(\mathbf{x})}{D^L_{lk}(\mathbf{x})} = \frac{n^1_{lk}x_1 + n^2_{lk}x_2 + \ldots + n^p_{lk}x_p + \alpha^R_{lk}}{m^1_{lk}x_1 + m^2_{lk}x_2 + \ldots + m^p_{lk}x_p + \beta^L_{lk}}, \quad l = 1, 2, \ldots, p, k = 1, 2, \ldots, t_l \tag{4}$$

Suppose an optimization problem have a constraint in the form of $\sum_{i=1}^n [a^L_{ti}, a^R_{ti}]x_i \geq [b^L_t, b^R_t]$, then the minimum and maximum value range are obtained from the inequality of $\sum_{i=1}^n a^L_{ti}x_i \geq b^R_t$ and $\sum_{i=1}^n a^R_{ti}x_i \geq b^L_t$ respectively.

Considering Proposition 3.1 on the constraint sets $\tilde{X}_N$ of NBL-MOLFCCO problem (1) and to develop the maximum relaxed feasible region, the corresponding crisp constraints of neutrosophic linear constraints in the problem (1) are formulated as follows:

$$\begin{aligned}
&\textstyle\sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti})x_i \leq [b_t + \bar{b}_t I_t] \Rightarrow \sum_{i=1}^n a^L_{ti}x_i \leq b^R_t, \quad t = 1, 2, \ldots, m_1\\
&\textstyle\sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti})x_i \geq [b_t + \bar{b}_t I_t] \Rightarrow \sum_{i=1}^n a^R_{ti}x_i \geq b^L_t, \quad t = m_1 + 1, \ldots, m_2\\
&\textstyle\sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti})x_i = [b_t + \bar{b}_t I_t] \Rightarrow \sum_{i=1}^n a^L_{ti}x_i \leq b^R_t \;\&\; \sum_{i=1}^n a^R_{ti}x_i \geq b^L_t, \quad t = m_2 + 1, \ldots, m
\end{aligned} \tag{5}$$

Therefore, following the above discussion the equivalent crisp model of the NBL-MOLFCCO problem (1) is developed as:

$$\text{ULDM}\left\{ \max f_{1k}(\mathbf{x}) = \max \frac{N^R_{1k}(\mathbf{x})}{D^L_{1k}(\mathbf{x})} = \frac{\mathbf{n}^1_{1k}\mathbf{x}_1 + \mathbf{n}^2_{1k}\mathbf{x}_2 + \cdots + \mathbf{n}^p_{1k}\mathbf{x}_p + \alpha^R_{1k}}{\mathbf{m}^1_{1k}\mathbf{x}_1 + \mathbf{m}^2_{1k}\mathbf{x}_2 + \cdots + \mathbf{m}^p_{1k}\mathbf{x}_p + \beta^L_{1k}}, \quad k = 1, 2, \ldots, t_1 \right. \tag{6}$$

where, $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p$ solves

$$\text{LLDM}\left\{ \max f_{2k}(\mathbf{x}) = \max \frac{N^R_{2k}(\mathbf{x})}{D^L_{2k}(\mathbf{x})} = \frac{\mathbf{n}^1_{2k}\mathbf{x}_1 + \mathbf{n}^2_{2k}\mathbf{x}_2 + \cdots + \mathbf{n}^p_{2k}\mathbf{x}_p + \alpha^R_{2k}}{\mathbf{m}^1_{2k}\mathbf{x}_1 + \mathbf{m}^2_{2k}\mathbf{x}_2 + \cdots + \mathbf{m}^p_{2k}\mathbf{x}_p + \beta^L_{2k}}, \quad k = 1, 2, \ldots, t_2 \right. \tag{7}$$

Subject to:

$$\begin{array}{l} \sum_{i=1}^{n} a_{ti}^{L} x_i \le b_t^R, \quad t = 1, 2, \ldots, m_1 \\ \sum_{i=1}^{n} a_{ti}^{R} x_i \ge b_t^L, \quad t = m_1 + 1, \ldots, m_2 \\ \sum_{i=1}^{n} a_{ti}^{L} x_i \le b_t^R \text{ and } \sum_{i=1}^{n} a_{ti}^{R} x_i \ge b_t^L, \quad t = m_2 + 1, \ldots, m \\ Pr\left[\sum_{i=1}^{n} a_{ti}^{L} x_i \le b_t\right] \ge \alpha_t, \quad t = m+1, \ldots, s_0 \\ Pr\left[\sum_{i=1}^{n} a_{ti}^{R} x_i \ge b_t\right] \ge \alpha_t, \quad t = s_0+1, \ldots, s \\ x_i \ge 0, \quad i = 1, 2, \ldots, n \end{array} \tag{8}$$

3.3 Deterministic Constraint Formulation of NBL-MOLFCCO Problem

Given that the right-hand-side (RHS) variable $b_t, t = m+1, \ldots, s$ in problem (3) be an autonomous Cauchy-random variable having arithmetic mean θ_t and standard deviation σ_t. Since $Pr\{g_t(\mathbf{x}) \le b_t\} \ge \alpha$ is equivalent to $Pr\{g_t(\mathbf{x}) \ge b_t\} \le 1 - \alpha$, here we consider the first type of chance constraints. The probability density function, $\Phi(g_t(\mathbf{x}))$ of the Cauchy-random variable b_t is defined as:

$$\Phi(g_t(\mathbf{x})) = Pr[g_t(\mathbf{x}) \le b_t] = \int_{g_t(\mathbf{x})}^{\infty} \frac{\sigma_t}{\pi(\sigma_t^2 + (b_t - \theta_t)^2)} db_t \quad t = m+1, \ldots, s_0. \tag{9}$$

Theorem 3.1. If the RHS b_t is an autonomous Cauchy random variable having arithmetic mean θ_t and standard-deviation σ_t, then the chance constraint, $Pr[\sum_{i=1}^{n} a_{ti}^{L} x_i \le b_t] \ge \alpha_t, t = m+1, \ldots, s_0$ and $Pr[\sum_{i=1}^{n} a_{ti}^{R} x_i \ge b_t] \ge \alpha_t, t = s_0+1, \ldots, s$ of problem (3) is equivalent to

$$\sum_{i=1}^{n} a_{ti}^{L} x_i \le \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right)\pi \quad \text{and} \quad \sum_{i=1}^{n} a_{ti}^{R} x_i \ge \theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right)\pi, \quad \text{respectively.}$$

Proof. Let $g_t(\mathbf{x}) = \sum_{i=1}^{n} a_{ti}^{L} x_i$ for each $t = m+1, \ldots, r_0$, then using Eqs.(4) the chance constraint in problem (3) can be written as:

$$\Phi(g_t(\mathbf{x})) = Pr[g_t(\mathbf{x}) \le b_t] = \int_{g_t(\mathbf{x})}^{\infty} \frac{\sigma_t}{\pi(\sigma_t^2 + (b_t - \theta_t)^2)} db_t \ge \alpha_t \quad \text{for } t = m+1, \ldots, s_0.$$

By determining the integration, we obtain:

$$\frac{1}{\pi}\left[\tan^{-1}\left(\frac{b_t - \theta_t}{\sigma_t}\right)\right]_{g_t(\mathbf{x})}^{\infty} \ge \alpha_t \quad \text{for } t = m+1, \ldots, s_0.$$

Substituting the tolerance limit of integration, we obtain the following results:

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{g_t(\mathbf{x}) - \theta_t}{\sigma_t}\right) \ge \pi\alpha_t \quad \text{for } t = m+1, \ldots, s_0.$$

It can be further simplified as:

$$\tan^{-1}\left(\frac{g_t(\mathbf{x}) - \theta_t}{\sigma_t}\right) \le \frac{\pi}{2} - \pi\alpha_t \quad \text{for } t = m+1, \ldots, s_0. \tag{5}$$

By applying tan (tangent) function on both sides of Eqs. (5) and then solving for $g_t(\mathbf{x})$, we obtain:

$$g_t(\mathbf{x}) \le \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right)\pi \quad \text{for } t = m+1, \ldots, s_0.$$

Now, replacing $g_t(\mathbf{x})$ by $\sum_{i=1}^{n} a_{ti}^{L} x_i$, we obtain the following equivalent deterministic-constraint:

$$\sum_{i=1}^{n} a_{ti}^{L} x_i \leq \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right)\pi, \quad \text{for each } t = m+1, \ldots, s_0.$$

Since $Pr[g_t(\mathbf{x}) \geq b_t] = 1 - Pr[g_t(\mathbf{x}) \leq b_t]$, the analogous deterministic constraint of the chance constraint $Pr[\sum_{i=1}^{n} a_{ti}^{R} x_i \geq b_t] \geq \alpha_t, t = s_0+1, \ldots, s$ of problem (3) in similar fashion formulated as:

$$\sum_{i=1}^{n} a_{ti}^{R} x_i \geq \theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right)\pi \quad \text{for } t = s_0+1, \ldots, s.$$

□

Therefore, following Theorem 3.1 the equivalent deterministic constraint model of problem (3) is formulated as follows:

$$\text{ULDM}\left\{ \max f_{1k}(\mathbf{x}) = \max \frac{N_{1k}^{R}(\mathbf{x})}{D_{1k}^{L}(\mathbf{x})} = \frac{\mathbf{n}_{1k}^{1}\mathbf{x}_1 + \mathbf{n}_{1k}^{2}\mathbf{x}_2 + \cdots + \mathbf{n}_{1k}^{p}\mathbf{x}_p + \alpha_{1k}^{R}}{\mathbf{m}_{1k}^{1}\mathbf{x}_1 + \mathbf{m}_{1k}^{2}\mathbf{x}_2 + \cdots + \mathbf{m}_{1k}^{p}\mathbf{x}_p + \beta_{1k}^{L}}, \quad k = 1, 2, \ldots, t_1 \right. \tag{10}$$

where, $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_p$ solves

$$\text{LLDM}\left\{ \begin{array}{ll} \max f_{2k}(\mathbf{x}) = \max \frac{N_{2k}^{R}(\mathbf{x})}{D_{2k}^{L}(\mathbf{x})} = \frac{\mathbf{n}_{2k}^{1}\mathbf{x}_1 + \mathbf{n}_{2k}^{2}\mathbf{x}_2 + \cdots + \mathbf{n}_{2k}^{p}\mathbf{x}_p + \alpha_{2k}^{R}}{\mathbf{m}_{2k}^{1}\mathbf{x}_1 + \mathbf{m}_{2k}^{2}\mathbf{x}_2 + \cdots + \mathbf{m}_{2k}^{p}\mathbf{x}_p + \beta_{2k}^{L}}, & k = 1, 2, \ldots, t_2 \\ \cdots & \\ \max f_{pk}(\mathbf{x}) = \max \frac{N_{pk}^{R}(\mathbf{x})}{D_{pk}^{L}(\mathbf{x})} = \frac{\mathbf{n}_{pk}^{1}\mathbf{x}_1 + \mathbf{n}_{pk}^{2}\mathbf{x}_2 + \cdots + \mathbf{n}_{pk}^{p}\mathbf{x}_p + \alpha_{pk}^{R}}{\mathbf{m}_{pk}^{1}\mathbf{x}_1 + \mathbf{m}_{pk}^{2}\mathbf{x}_2 + \cdots + \mathbf{m}_{pk}^{p}\mathbf{x}_p + \beta_{pk}^{L}}, & k = 1, 2, \ldots, t_p \end{array} \right. \tag{11}$$

Subject to:

$$\begin{array}{l} \sum_{i=1}^{n} a_{ti}^{L} x_i \leq b_t^{R}, \quad t = 1, 2, \ldots, m_1 \\ \sum_{i=1}^{n} a_{ti}^{R} x_i \geq b_t^{L}, \quad t = m_1+1, \ldots, m_2 \\ \sum_{i=1}^{n} a_{ti}^{L} x_i \leq b_t^{R} \text{ and } \sum_{i=1}^{n} a_{ti}^{R} x_i \geq b_t^{L}, \quad t = m_2+1, \ldots, m \\ \sum_{i=1}^{n} a_{ti}^{L} x_i \leq \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right)\pi, \quad t = m+1, \ldots, s_0 \\ \sum_{i=1}^{n} a_{ti}^{R} x_i \geq \theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right)\pi, \quad t = s_0+1, \ldots, s \\ x_i \geq 0, \quad i = 1, 2, \ldots, n \end{array} \tag{12}$$

Definition 3.1. The DM_l-MOLFO problem (7) is said to be a concave-convex fractional decision-making problem if the numerator, $N_{lk}(\mathbf{x})$ is concave function, the denominator, $D_{lk}(\mathbf{x})$ is convex function with $D_{lk}(\mathbf{x}) > 0, \forall \mathbf{x} \in X$ for each l, k and region X is convex feasible set.

Definition 3.2. For any decision-control vector $\mathbf{x}_1^* \in \{\mathbf{x}_1^* | \mathbf{x} = (\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_p^*) \in X\}$ given by $DM_1, \mathbf{x}_1^*$, $l = 2, 3, \ldots, p$ be a Pareto-optimal (efficient) solution to the LLDM (i.e., $DM_l -$ MOLFO, $l = 2, 3, \ldots, p$) problem, if there does not exist a new-vector $\mathbf{z}_l \in X$, such that $f_{lk}(\mathbf{x}_l^*) \leq f_{lk}(\mathbf{z}_l), \forall k = 1, 2, \ldots, t_l$ and $f_{lk}(\mathbf{x}_l^*) < f_{lk}(\mathbf{z}_l)$ for some k. Hence, $\mathbf{x}^* = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_p^*\}$ is called a feasible-solution to problem (3).

Definition 3.3. Let $\mathbf{x}^* = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_p^*\}$ be a feasible-solution to problem (3). Then, $\mathbf{x}^*$ be an efficient solution to problem (3) if there doesn't exists a new feasible-vector $\mathbf{x} \in X$ such that $f_{lk}(\mathbf{x}^*) \leq f_{lk}(\mathbf{x}), \forall k = 1, 2, \ldots, t_1$ and $f_{lk}(\mathbf{x}^*) < f_{lk}(\mathbf{x})$, for some k.

Assume that in the problem (7), if $D^L_{lk}(\mathbf{x})$ is concave with $D^L_{lk}(\mathbf{x}) > 0, \forall \mathbf{x} \in X$ and $N^R_{lk}(\mathbf{x})$ is concave with $N^R_{lk}(\mathbf{x}) < 0, \forall \mathbf{x} \in X$, then the negative of $N_{lk}(\mathbf{x})$ is positive and convex linear-functional for each feasible decision-vector $\mathbf{x}$ and $\max_{x\in X} \frac{N^R_{lk}(\mathbf{x})}{D^L_{lk}(\mathbf{x})} \iff \min_{x\in X} \frac{-N^R_{lk}(\mathbf{x})}{D^L_{lk}(\mathbf{x})} \iff \max_{x\in X} \frac{D^L_{lk}(\mathbf{x})}{-N^R_{lk}(\mathbf{x})}$. Therefore, this way it converted into concave-convex fractional decision-making problem [35, 41].

Suppose that T be represent an index set such that $T = \{k = 1, 2, \ldots, t_l, |N^R_{lk}(\mathbf{x}) \geq 0 \text{ for some } \mathbf{x} \in X\}$, $T^c = \{k = 1, 2, \ldots, t_l, |N^R_{lk}(\mathbf{x}) < 0, \forall \mathbf{x} \in X\}$, and $T \cup T^c = \{1, 2, \ldots, t_l\}$ for each $l = 1, 2, \ldots, p$. Let us choose a new vector of n-variables $\mathbf{w}$ that related with each component of $\mathbf{x} \in \Re^n_+$ as:

$$\mathbf{w} = r\mathbf{x}, \text{ where } r = \min\left\{\frac{1}{\mathbf{m}^1_{lk}\mathbf{x}_1 + \mathbf{m}^2_{lk}\mathbf{x}_2 + \cdots + \mathbf{m}^p_{lk}\mathbf{x}_p + \beta^L_{lk}}, k \in T\right\} > 0 \text{ and}$$

$$\mathbf{w} = r\mathbf{x}, \text{ where } r = \min\left\{\frac{-1}{\mathbf{n}^1_{lk}\mathbf{x}_1 + \mathbf{n}^2_{lk}\mathbf{x}_2 + \cdots + \mathbf{n}^p_{lk}\mathbf{x}_p + \alpha^R_{lk}}, k \in T^c\right\} > 0 \text{ for each } l.$$

This is equivalent to:

$$\mathbf{m}^1_{lk}\mathbf{x}_1 + \mathbf{m}^2_{lk}\mathbf{x}_2 + \cdots + \mathbf{m}^p_{lk}\mathbf{x}_p + \beta^L_{lk} \geq r \Rightarrow r(\mathbf{m}^1_{lk}\mathbf{x}_1 + \mathbf{m}^2_{lk}\mathbf{x}_2 + \cdots + \mathbf{m}^p_{lk}\mathbf{x}_p + \beta^L_{lk}) \leq 1 \text{ for } k \in T \text{ and}$$

$$\frac{-1}{\mathbf{n}^1_{lk}\mathbf{x}_1 + \mathbf{n}^2_{lk}\mathbf{x}_2 + \cdots + \mathbf{n}^p_{lk}\mathbf{x}_p + \alpha^R_{lk}} \geq r \Rightarrow -r(\mathbf{n}^1_{lk}\mathbf{x}_1 + \mathbf{n}^2_{lk}\mathbf{x}_2 + \cdots + \mathbf{n}^p_{lk}\mathbf{x}_p + \alpha^R_{lk}) \leq 1 \text{ for } k \in T^c.$$

Using this new vector $\mathbf{w} \in \Re^n_+$ with relation operator $\mathbf{w} = r\mathbf{x}$, where, $r > 0, \mathbf{x} \in \Re^n_+$, we obtain:

$$\mathbf{x}_l = \frac{\mathbf{w}_l}{r} \iff x_{lj} = \frac{w_{lj}}{r}, j = 1, 2, \ldots, n_l \iff x_i = \frac{w_i}{r}, i = 1, 2, \ldots, n.$$

Thus, each affian fractional objective function $f_{lk}(\mathbf{x}) = \frac{N^R_{lk}(\mathbf{x})}{D^L_{lk}(\mathbf{x})} = \frac{\mathbf{n}^1_{lk}\mathbf{x}_1+\mathbf{n}^2_{lk}\mathbf{x}_2+\cdots+\mathbf{n}^p_{lk}\mathbf{x}_p+\alpha^R_{lk}}{\mathbf{m}^1_{lk}\mathbf{x}_1+\mathbf{m}^2_{lk}\mathbf{x}_2+\cdots+\mathbf{m}^p_{lk}\mathbf{x}_p+\beta^L_{lk}}$ in the DM_l- MOLFO problem (7) is converted into an equivalent linear-objective function:

$$f_{lk}(\mathbf{w}, r) = (\mathbf{n}^1_{lk}\frac{\mathbf{w}_1}{r} + \mathbf{n}^2_{lk}\frac{\mathbf{w}_2}{r} + \cdots + \mathbf{n}^p_{lk}\frac{\mathbf{w}_p}{r} + \alpha^R_{lk})r = \mathbf{n}^1_{lk}\mathbf{w}_1 + \mathbf{n}^2_{lk}\mathbf{w}_2 + \cdots + \mathbf{n}^p_{lk}\mathbf{w}_p + \alpha^R_{lk}r = rN^R_{lk}(\frac{\mathbf{w}}{r}), \quad k \in T$$

$$f_{lk}(\mathbf{w}, r) = (\mathbf{m}^1_{lk}\frac{\mathbf{w}_1}{r} + \mathbf{m}^2_{lk}\frac{\mathbf{w}_2}{r} + \cdots + \mathbf{m}^p_{lk}\frac{\mathbf{w}_p}{r} + \beta^L_{lk})r = \mathbf{m}^1_{lk}\mathbf{w}_1 + \mathbf{m}^2_{lk}\mathbf{w}_2 + \cdots + \mathbf{m}^p_{lk}\mathbf{w}_p + \beta^L_{lk}r = rD^L_{lk}(\frac{\mathbf{w}}{r}), \quad k \in T^c.$$

Therefore, using these linear-transformation strategies, the bi-level multi-objective linear optimization (BL-MOLO) model (8) of the crisp problem (6) is developed as:

$$\text{ULDM}\begin{cases} \max_{w_1}(f_{1k}(\mathbf{w}, r) = rN^R_{1k}(\mathbf{w}) \text{ for } k \in T; f_{1k}(\mathbf{w}, r) = rD^L_{1k}(\mathbf{w}) \text{ for } k \in T^c) \quad [DM_1], \\ \text{where, } \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_p \text{ solves;} \end{cases} \tag{13}$$

$$\text{LLDM}\begin{cases} \max_{w_2}(f_{2k}(\mathbf{w}, r) = rN^R_{2k}(\mathbf{w}) \text{ for } k \in T; f_{2k}(\mathbf{w}, r) = rD^L_{2k}(\mathbf{w}) \text{ for } k \in T^c) \quad [DM_2], \\ \vdots \\ \max_{w_p}(f_{pk}(\mathbf{w}, r) = rN^R_{pk}(\mathbf{w}), \text{ for } k \in T; f_{pk}(\mathbf{w}, r) = rD^L_{pk}(\mathbf{w}) \text{ for } k \in T^c) \quad [DM_p], \end{cases} \tag{14}$$

Subject to:

$$
\begin{aligned}
&\textstyle\sum_{i=1}^{n} a_{ti}^{L} w_i - r b_t^{L} \leq 0, \quad t = 1,2,\ldots,m_1, m_2+1,\ldots,m \\
&\textstyle\sum_{i=1}^{n} a_{ti}^{R} w_i - r b_t^{L} \geq 0, \quad t = m_1+1,\ldots,m_2, m_2+1,\ldots,m \\
&r D_{lk}(\mathbf{w}) \leq 1, \quad \text{for } k \in T; k = 1,2,\ldots,t_l; l = 1,2,\ldots,p \\
&-r N_{lk}(\mathbf{w}) \leq 1, \quad \text{for } k \in T^c, k = 1,2,\ldots,t_l; l = 1,2,\ldots,p \\
&\textstyle\sum_{i=1}^{n} a_{ti}^{L} w_i - r\left(\theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right)\pi\right) \leq 0, \quad t = m+1,\ldots,s_0 \\
&\textstyle\sum_{i=1}^{n} a_{ti}^{R} w_i - r\left(\theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right)\pi\right) \geq 0, \quad t = s_0+1,\ldots,s \\
&\mathbf{w} \in \Re^n, \mathbf{w} \geq 0, r > 0.
\end{aligned}
\tag{15}
$$

Where, it assumed that $S^{wr} \neq \emptyset$ and a bounded constraint region of BL-MOLO (8) problem.

4 Proposed Solution Method

In this section, we introduced a new hybrid method that combined intuitionistic fuzzy optimization and goal programming is called IFGP (intuitionistic fuzzy goal programming) method.

4.1 Intuitionistic Fuzzy Optimization (IFO)

In order to address real-world decision-making scenarios and solve the resulting BL-MOLO (8) problem, IFO play a crucial role and offer numerous advantageous features. Hence, to solve the problem (8) by considering the real-life decision scenarios, we need to develop the equivalent IFO model of problem (8) by converting each linear-objective function $f_{lk}(\mathbf{w}, r), k = 1,2,\ldots,t_l, l = 1,2,\ldots,p$ into an intuitionistic-fuzzy-goal (IFG) by injecting an aspiration level. To find aspiration levels or goals for objective functions, we solve a sole-objective function under the relaxed feasible sets.

Let $f_{lk}^{I} = f_{lk}(\mathbf{w}_{lk}^{*}, r^{*}) = \max_{(\mathbf{w},r)\in S^{wr}} f_{lk}(\mathbf{w}, r)$ is an Ideal (global optimal) value at the Ideal (best optimal) solution $(\mathbf{w}_{lk}^{*}, r^{*}), l = 1,2,\ldots,p$ obtained from solving a single-objective function under feasible convex region S^{wr}. Thus, the IFO model of problem (8) is formulated according to:

$$\text{Determine } (\mathbf{w}, r) \in S^{wr} \tag{16}$$

$$\text{Subject to: } f_{lk}(\mathbf{w}, r) \gtrsim f_{lk}^{I}, \quad l = 1,2,\ldots,p, k = 1,2,\ldots,t_l \tag{17}$$

Where, $\gtrsim$ is an intuitionistic fuzzy inequality which means that "significantly greater than or equal to".

4.1.1 Membership and Non-membership functions

We will use the membership function $\mu_{lk}(f_{lk}(\mathbf{w}, r))$ to gauge acceptance and satisfaction and the non-membership function $\nu_{lk}(f_{lk}(\mathbf{w}, r))$ to assess rejection and dissatisfaction for each objective function $f_{lk}(\mathbf{w}, r) \gtrsim f_{lk}^{I}$. To establish these functions, we must first define relaxation bounds and the aspired level. This requires calculating a payoff matrix following Eqs. (10), based on individual ideal solutions $(\mathbf{w}_{lk}^{*}, r^{*})$, obtained by solving each lkth single-objective maximization problem, where $f_{lk}(\mathbf{w}_{lk}^{*}, r^{*}) = \max_{(\mathbf{w},r)\in S^{wr}} f_{lk}(\mathbf{w}, r)$.

$$\begin{array}{c|cccc} \max_{(\mathbf{w},r)\in S^{wr}} f_{lk}(\mathbf{w},r) & f_{l1}(\mathbf{w},r) & f_{l2}(\mathbf{w},r) & \dots & f_{lt_l}(\mathbf{w},r) \\ \hline (\mathbf{w}^*_{l1},r^*) & f_{l1}(\mathbf{w}^*_{l1},r^*) & f_{l2}(\mathbf{w}^*_{l1},r^*) & \dots & f_{lt_l}(\mathbf{w}^*_{l1},r^*) \\ (\mathbf{w}^*_{l2},r^*) & f_{l1}(\mathbf{w}^*_{l2},r^*) & f_{l2}(\mathbf{w}^*_{l2},r^*) & \dots & f_{lt_l}(\mathbf{w}^*_{l2},r^*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\mathbf{w}^*_{lt_l},r^*) & f_{l1}(\mathbf{w}^*_{lt_l},r^*) & f_{l1}(\mathbf{w}^*_{lt_l},r^*) & \dots & f_{lt_l}(\mathbf{w}^*_{lt_l},r^*) \end{array} \tag{18}$$

The value of $U^{\mu}_{lk} = \max\{f_{lk}(\mathbf{w}^*_{l1},r^*), f_{lk}(\mathbf{w}^*_{l2},r^*),\dots, f_{lk}(\mathbf{w}^*_{lt_l},r^*)\}$ gives aspired-level or upper relaxation limit, where as, the value of $L^{\mu}_{lk} = \min\{f_{lk}(\mathbf{w}^*_{l1},r^*), f_{lk}(\mathbf{w}^*_{l2},r^*),\dots, f_{lk}(\mathbf{w}^*_{lt_l},r^*)\}$ gives lower acceptable relaxation limit for $\mu_{lk}(f_{lk}(\mathbf{w},r))$. Similar, the upper and lower-relaxation limit for $\nu_{lk}(f_{lk}(\mathbf{w},r))$ are determined by $U^{\nu}_{lk} = U_{lk} - \epsilon_{lk}(U_{lk} - L_{lk})$ and $L^{\nu}_{lk} = L_{lk}$, where $\epsilon_{lk} \in (0,1)$ respectively. The arithmetic weighted value of $\epsilon_{lk} \in (0,1)$ first set by DMs.

The geometrical definition of membership, $\mu_{lk}(f_{lk}(\mathbf{w},r))$ and non membership, $\nu_{lk}(f_{lk}(\mathbf{w},r))$ functions for lkth, $k = 1,2,\dots,t_l$ objective functions of $DM_l -$ MOLO problem are shown in the figure:1-(1) and formulated mathematically as:

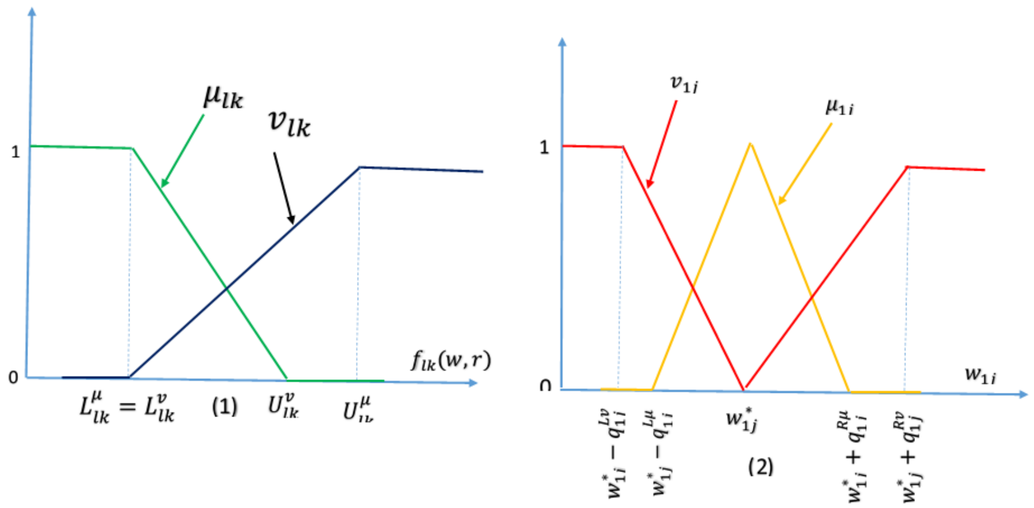

Figure 1: Membership and non membership function for each $f_{lk}(\mathbf{w},r),(a)$ and $w_{1j},(b)$

$$\mu_{lk}(f_{lk}(\mathbf{w},r)) = \begin{cases} 1, & f_{lk}(\mathbf{w},r) \geq U^{\mu}_{lk} \\ \frac{f_{lk}(\mathbf{w},r) - L^{\mu}_{lk}}{U^{\mu}_{lk} - L^{\mu}_{lk}}, & L^{\mu}_{lk} \leq f_{lk}(\mathbf{w},r) \leq U^{\mu}_{lk} \\ 0, & f_{lk}(\mathbf{w},r) \leq L^{\mu}_{lk} \end{cases} \tag{19}$$

$$\nu_{lk}(f_{lk}(\mathbf{w},r)) = \begin{cases} 0, & f_{lk}(\mathbf{w},r) \geq U^{\nu}_{lk} \\ \frac{U^{\nu}_{lk} - f_{lk}(\mathbf{w},r)}{U^{\nu}_{lk} - L^{\nu}_{lk}}, & L^{\nu}_{lk} \leq f_{lk}(\mathbf{w},r) \leq U^{\nu}_{lk} \\ 1, & f_{lk}(\mathbf{w},r) \leq L^{\nu}_{lk} \end{cases} \tag{20}$$

4.2 Goal Programming (GP) method

GP is an optimization technique designed to handle decision-making situation where a number of conditions characterized as goals are to be met as closely as possible. In an intuitionistic fuzzy decision environment, the goal of each decision-maker is to achieve the highest level of achievement. Therefore, each decision-maker should try to maximize and minimize his/her linear membership function (MF) and non-membership function (NMF) in order to attain their aspired-level 1(unity) and 0(zero), respectively by minimize their unwanted deviation variables with the help of interactive GP method. Therefore, the concept of GP model can be employed to formulate the flexible membership and non membership goals with the aspired level 1 and 0 for MF ($\mu_{lk}(f_{lk}(\mathbf{w}, r))$) and NMF ($\nu_{lk}(f_{lk}(\mathbf{w}, r))$) of objective functions defined in the Eqs. (11 accordingly:

$$\begin{aligned} &\mu_{lk}(f_{lk}(\mathbf{w}, r)) + D_{lk}^{-\mu} - D_{lk}^{+\mu} = 1, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l. \\ &\nu_{lk}(f_{lk}(\mathbf{w}, r)) + D_{lk}^{-\nu} - D_{lk}^{+\nu} = 0, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l. \end{aligned} \tag{21}$$

Here, the variables $D_{lk}^{-\mu}, d_{1j}^{-\mu}, D_{lk}^{-\nu}, d_{1j}^{-\nu}$ are used to represents negative-deviation and $D_{lk}^{+\mu}, d_{1j}^{+\mu}, D_{lk}^{+\nu}, d_{1j}^{+\nu}$ are positive-deviation from aspiration levels. Since the over-deviation, $D_{lk}^{+\mu}, d_{1j}^{+\mu}$ and under-deviation, $D_{lk}^{-\nu}, d_{1j}^{-\nu}$ from an IFGs are illustrates highest and lowest-achievement of MF and NMFs respectively [35,39], those variables have been fully-accepted and hence, it removed from the Eqs. (12). Therefore, the Eqs. (12) is reduced to:

$$\begin{aligned} &\mu_{lk}(f_{lk}(\mathbf{w}, r)) + D_{lk}^{-\mu} \geq 1, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l. \\ &\nu_{lk}(f_{lk}(\mathbf{w}, r)) - D_{lk}^{+\nu} \leq 0, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l. \end{aligned} \tag{22}$$

Now, using the summarized goal programming in the Eqs. (13), we introduced the IFO plus GP method called as IFGP method. Therefore, the proposed IFGP model for relaxed upper-level decision-maker (DM_1) of problem (1) is formulated as:

IFGP Model-I for ULDM problem:

$$\begin{aligned} &\min \textstyle\sum_{k=1}^{t_1} (W_{1k}^{\mu} D_{1k}^{-\mu} + W_{1k}^{\nu} D_{1k}^{+\nu}) \\ &\text{Subject to: } \begin{cases} \mu_{1k}(f_{1k}(\mathbf{w}, r)) + D_{1k}^{-\mu} \geq 1; \quad \nu_{1k}(f_{1k}(\mathbf{w}, r)) - D_{1k}^{+\nu} \leq 0, \quad \forall k = 1, 2, \dots, t_1. \\ (\mathbf{w}, r) \in S^{wr} \text{ (system constraint from Eqs.(8))}; \quad D_{1k}^{-\mu}, D_{1k}^{+\nu} \geq 0. \end{cases} \end{aligned} \tag{23}$$

4.3 Negotiation between Bilevel Decision-Makers

The benefit of DM_l for $l = 2, 3, \dots, p$ (follower-level decision-maker) depends on an imprecise-tolerance of the decision control-variable of a DM_1. The decision-control variable at the ULDM is set within a tolerance range in order to avoid a decision lock for the follower-level, i.e,

$$w_{1j} \approx w_{1j}^*, \quad j = 1, 2, \dots, n_1.$$

Thus, the decision-control variable range for $w_{1j}, j = 1, 2, \ldots, n_1$ should be "nearly w^*_{1j}, with maximum acceptable right relaxation values $q^{R\mu}_{1j} > 0$ and maximum acceptable left relaxation values $q^{L\mu}_{1j} > 0$. This values are specified by the ULDM. Those parameters $q^{R\mu}_{1j}$ and $q^{L\mu}_{1j}$ haven't necessarily the same values.

The level of optimality for the decision-control variable monitored by DM_1 is fully satisfactory at w^*_{1j} monotonically increasing in the $[w^*_{1j} - q^{L\mu}_{1j}, w^*_{1j}]$, monotonically decreasing in the $[w^*_{1j}, w^*_{1j} + q^{R\mu}_{1j}]$, otherwise is zero-satisfactory by ULDM. But due to inaccurate decisions, zero-acceptance by ULDM does not imply complete-rejection, thus the DM may interpret rejection in an optimistic manner. As a result, regarding rejection over the interval $[w^*_{1j} - q^{L\mu}_{1j}, w^*_{1j} - q^{L\mu}_{1j}]$ and $[w^*_{1j} + q^{R\mu}_{1j}, w^*_{1j} + q^{R\mu}_{1j}]$ where, $q^{L\mu}_{1j} \geq q^{L\mu}_{1j} > 0$, and/or $0 < q^{R\mu}_{1j} \leq q^{R\mu}_{1j}$, the DM is open-minded.

With the help of this discussion, we can built the membership, $\mu_{1j}(w_{1j})$ and non membership functions, $\nu_{1j}(w_{1j})$ for the decision-control variable w_{1j} monitored by DM_1 in the intuitionistic fuzzy environment, as illustrated in Figure1- (2). These functions represent the acceptance and rejection levels for the variable w_{1j} by ULDM and defined mathematically as:

$$\mu_{1j}(w_{1j}) = \begin{cases} \frac{w_{1j} - (w^*_{1j} - q^{L\mu}_{1j})}{q^{L\mu}_{1j}}, & \text{if } w^*_{1j} - q^{L\mu}_{1j} \leq w_{1j} \leq w^*_{1j}, \\ \frac{w^*_{1j} + q^{R\mu}_{1j} - w_{1j}}{q^{R\mu}_{1j}}, & \text{if } w^*_{1j} \leq w_{1j} \leq w^*_{1j} + q^{R\mu}_{1j}, \quad j = 1, 2, \ldots, n_1. \end{cases} \tag{24}$$

$$\nu_{1j}(w_{1j}) = \begin{cases} \frac{w^*_{1j} - w_{1j}}{q^{L\nu}_{1j}}, & \text{if } w^*_{1j} - q^{L\nu}_{1j} \leq w_{1j} \leq w^*_{1j}, \\ \frac{w^*_{1j} - w_{1j}}{q^{R\nu}_{1j}}, & \text{if } w^*_{1j} \leq w_{1j} \leq w^*_{1j} + q^{R\nu}_{1j}, \quad j = 1, 2, \ldots, n_1. \\ 1, & \text{otherwise.} \end{cases} \tag{25}$$

To formulate the next IFGP model- II (17), the vector of decision-variables $w_{1j}, j = 1, 2, \ldots, n_1$ monitored by ULDM used as a binding additional constraints for follower-level $DM_l -$MOLDM, Eqs.(8) where, $l = 2, 3, .., p$ problem. Therefore, the IFGP model- II for NBL-MOLFCCo problem (1) given as follows:

IFGP Model- II for LLDM problem:

$$\begin{aligned} &\min \textstyle\sum_{l=1}^{p} \sum_{k=1}^{t_l} (W^{\mu}_{lk} D^{-\mu}_{lk} + W^{\nu}_{lk} D^{+\nu}_{lk}) + \sum_{j=1}^{n_1} [w^{L\mu}_{1j} d^{-L\mu}_{1j} + w^{R\mu}_{1j} d^{-R\mu}_{1j} + w^{L\nu}_{1j} d^{+L\nu}_{1j} + w^{R\nu}_{1j} d^{+R\nu}_{1j}] \\ &\text{Subject to: } \begin{cases} \mu_{1k}(f_{1k}(\mathbf{w}, r)) + D^{-\mu}_{1k} \geq 1; \quad \nu_{1k}(f_{1k}(\mathbf{w}, r)) - D^{+\nu}_{1k} \leq 0, \quad \forall k = 1, 2, \ldots, t_1, \\ \mu^{L}_{1j}(w_{1j}) + d^{-L\mu}_{1j} \geq 1; \quad \nu^{L}_{1j}(w_{1j}) - d^{+L\nu}_{1j} \leq 0, \quad \forall j = 1, 2, \ldots, n_1, \\ \mu^{R}_{1j}(w_{1j}) + d^{-R\mu}_{1j} \geq 1; \quad \nu^{R}_{1j}(w_{1j}) - d^{+R\nu}_{1j} \leq 0, \quad \forall j = 1, 2, \ldots, n_1, \\ (\mathbf{w}, r) \in S^{wr} \text{ (see Eqs. 8)}; \quad D^{-\mu}_{1k}, D^{+\nu}_{1k}, d^{-L\mu}_{1j}, d^{+L\nu}_{1j}, d^{-R\mu}_{1j}, d^{+L\nu}_{1j} \geq 0. \end{cases} \end{aligned} \tag{26}$$

Here, $W^{\mu}_{lk}, W^{\nu}_{lk}, w^{L\mu}_{1j}, w^{R\mu}_{1j}, w^{L\nu}_{1j}$, and $w^{R\nu}_{1j}$ are numerical weights indicates the relative importance of IFG (9). These parametric values are calculated using the weighting scheme introduced by [35] as follows:

$$W^{\mu}_{lk} = \frac{1}{U^{\mu}_{lk} - L^{\mu}_{lk}}, \quad \text{and} \quad W^{\nu}_{lk} = \frac{1}{U^{\nu}_{lk} - L^{\nu}_{lk}} \tag{27}$$

$$w^{L\mu}_{1j} = \frac{1}{q^{L\mu}_{1j}}; \quad w^{R\mu}_{1j} = \frac{1}{q^{R\mu}_{1j}}, \quad w^{L\nu}_{1j} = \frac{1}{q^{L\nu}_{1j}} \quad \text{and} \quad w^{R\nu}_{1j} = \frac{1}{q^{R\nu}_{1j}}. \tag{28}$$

Definition 4.1. Let w^* with corresponding r^* be an intuitionistic fuzzy (IF)-efficient solution to proposed IFGP model-II (17). Then there does not exist another vector $(\mathbf{w}, r) \in S^{wr}$ such that

$$\mu_{lk}(f_{lk}(\mathbf{w}, r)) \geq \mu_{lk}(f_{lk}(\mathbf{w}^*, r^*)) \wedge \nu_{lk}(f_{lk}(\mathbf{w}, r)) \leq \nu_{lk}(f_{lk}(\mathbf{w}^*, r^*)), \quad \forall l, k, \text{ and}$$

$$\mu_{lk}(f_{lk}(\mathbf{w}, r)) > \mu_{lk}(f_{lk}(\mathbf{w}^*, r^*)) \wedge \nu_{lk}(f_{lk}(\mathbf{w}, r)) < \nu_{lk}(f_{lk}(\mathbf{w}^*, r^*)) \text{ for some } l, k.$$

Theorem 4.1. A unique IF-efficient solution to proposed IFGP model-II (17) is also known as a Pareto-optimal (efficient) solution to the NBL-MOLFCCO problem (1).

Proof. Suppose that $(\mathbf{w}^*, r^*)$ is unique IF-efficient solution to IFGP model-II (17). Thus, we have:

$$\mu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) > \mu_{1k}(f_{1k}(\mathbf{w}, r)) \wedge \nu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) < \nu_{1k}(f_{1k}(\mathbf{w}, r)), \quad \forall k = 1, 2, \ldots, t_1.$$

Assume that $(\mathbf{w}^*, r) = (\mathbf{w}_1^*, \mathbf{w}_2^*, \ldots, \mathbf{w}_p^*, r^*)$ is not efficient solution to BL-MOLO (3) problem. Then there exist a new vector $(\mathbf{w}, r) \in S^{wr}$ such that

$$f_{1k}(\mathbf{w}^*, r^*) \leq f_{1k}(\mathbf{w}, r), \quad \forall k = 1, 2, \ldots, t_1 \quad \text{and} \quad f_{1k}(\mathbf{w}^*, r^*) < f_{1k}(\mathbf{w}, r) \text{ for some } k.$$

Following that, we obtain:

$$f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^{\mu} \leq f_{1k}(\mathbf{w}, r) - L_{1k}^{\mu}, \quad \forall k \quad \text{and} \quad f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^{\mu} < f_{1k}(\mathbf{w}, r) - L_{1k}^{\mu} \text{ for some } k.$$

$$U_{1k}^{\nu} - f_{1k}(\mathbf{w}^*, r^*) \geq U_{1k}^{\nu} - f_{1k}(\mathbf{w}, r), \quad \forall k \quad \text{and} \quad U_{1k}^{\nu} - f_{1k}(\mathbf{w}^*, r^*) > U_{1k}^{\nu} - f_{1k}(\mathbf{w}, r), \text{ for some } k.$$

Divide the above inequality using positive quantity $U_{1k}^{\mu} - L_{1k}^{\mu}$ and $U_{1k}^{\nu} - L_{1k}^{\nu}$ respectively, to obtain:

$$\frac{f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^{\mu}}{U_{1k}^{\mu} - L_{1k}^{\mu}} \leq \frac{f_{1k}(\mathbf{w}, r) - L_{1k}^{\mu}}{U_{1k}^{\mu} - L_{1k}^{\mu}}, \quad \forall k \quad \text{and} \quad \frac{f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^{\mu}}{U_{1k}^{\mu} - L_{1k}^{\mu}} < \frac{f_{1k}(\mathbf{w}, r) - L_{1k}^{\mu}}{U_{1k}^{\mu} - L_{1k}^{\mu}}, \text{ for some } k.$$

$$\frac{U_{1k}^{\nu} - f_{1k}(\mathbf{w}^*, r^*)}{U_{1k}^{\nu} - L_{1k}^{\nu}} \geq \frac{U_{1k}^{\nu} - f_{1k}(\mathbf{w}, r)}{U_{1k}^{\nu} - L_{1k}^{\nu}}, \quad \forall k \quad \text{and} \quad \frac{U_{1k}^{\nu} - f_{1k}(\mathbf{w}^*, r^*)}{U_{1k}^{\nu} - L_{1k}^{\nu}} > \frac{U_{1k}^{\nu} - f_{1k}(\mathbf{w}, r)}{U_{1k}^{\nu} - L_{1k}^{\nu}}, \text{ for some } k.$$

This implies that

$$\mu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) \leq \mu_{1k}(f_{1k}(\mathbf{w}, r)), \quad \forall k \quad \text{and} \quad \mu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) < \mu_{1k}(f_{1k}(\mathbf{w}, r)), \text{ for some } k.$$

$$\nu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) \geq \nu_{1k}(f_{1k}(\mathbf{w}, r)), \quad \forall k \quad \text{and} \quad \nu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) > \nu_{1k}(f_{1k}(\mathbf{w}, r)), \text{ for some } k.$$

This implies that $(\mathbf{w}, r)$ is also an IF-efficient solution to (17) problem. Thus, we have reached at contradiction to $(\mathbf{w}^*, r^*)$ is unique IF-efficient solution to (17) problem. Therefore, the $\mathbf{x}^* = \frac{\mathbf{w}^*}{r^*}$ is Pareto-optimal (efficient) solution to NBL-MOLFCCO (1) problem. □

5 Algorithm of proposed methodology for NBL-MOLFCCo problem

The algorithm of the proposed efficient method, which combined the benefits of the interactive goal programming approach and the intuitionistic fuzzy decision set for solving the introduced NBL-MOLFCCo (1) problem, is developed as follows:

1. Based on discussion given in subsection 3.2, the crisp-parametric model of NBL-MOLFCCo problem (1) is formulated according to problem (3).

2. Following the chance-constraint optimization strategy discussed in subsection 3.3, the deterministic constraint of problem (3) is developed as illustrated in Eqs. 6.

3. Using introduced linearization techniques, convert a problem (3) into crisp BL-MOLO problem (8).

4. Solve each linear objective function $f_{lk}(\mathbf{w}, r)$ independently under the convex constraint set S^{wr} and then construct a payoff-matrix (10) to obtain the upper $U^{\mu}_{lk}, U^{\nu}_{lk}$ and lower $L^{\mu}_{lk}, L^{\nu}_{lk}$ relaxation limits for each $f_{lk}(\mathbf{w}, r), l = 1, 2, \ldots, p, k = 1, 2, \ldots, t_l$, based on the obtained individual ideal solution.

5. Calculate the weights W^{μ}_{lk} and W^{ν}_{lk} as defined in Eqs. (18) for each IFG of objective functions $f_{lk}(\mathbf{w}, r), k = 1, 2, \ldots, t_l, l = 1, 2, \ldots, p$

6. Construct both the membership function $\mu_{lk}(f_{lk}(\mathbf{w}, r))$ and non membership function $\nu_{lk}(f_{lk}(\mathbf{w}, r))$ based on Eqs. (11) for each of the objective $f_{lk}(\mathbf{w}, r)$ function in the BL-MOLO problem (8).

7. Formulate the proposed IFGP Model- I (14) for ULDM problem and solve it to generate decision-control solution of DM_1, set as: $\mathbf{w}^*_1 = (w^*_{11}, w^*_{12}, \ldots, w^*_{1n_1})$

8. The leader-level decision-maker (DM_1) set the maximum acceptance for membership and minimum rejection for non membership as right $q^{\mathrm{R}\nu}_{1j}, q^{\mathrm{R}\mu}_{1j}$, and left $q^{L\nu}_{1j}, q^{L\mu}_{1j}$, relaxation limit on its decision-control variables $\mathbf{w}_1 = (w_{11}, w_{12}, \ldots, w_{1n_1})$.

9. Build the linear-membership $\mu_{1j}(w_{1j})$ see Eqs (15) and linear non membership $\nu_{1j}(w_{1j})$ see Eqs.(16) functions for binding-constraints $w_{1j} \approx w^*_{1j}, j = 1, 2, \ldots, n_1$.

10. Assign the weight of importance relation for imprecise aspiration-level of decision-control variables using Eqs.(19).

11. Formulate the next proposed IFGP Model- II (17) for NBL-MOLFCCo problem (1) and solve it to get a preferable compromise IF-efficient solution.

12. If all decision-makers ($DM_l, l = 1, 2, \ldots, p$) are satisfied for the preferable IF-efficient solution in Step 11, stop and the Pareto-optimal solution to NBL-MOLFCCo problem (1) is obtained by backward substitution $\mathbf{x}^* = (x^*_1, x^*_2, \ldots, x^*_p) = (\frac{w^*_1}{r}, \frac{w^*_2}{r}, \ldots, \frac{w^*_p}{r})$. Else, modify the upper and lower-relaxation limit for objective function, $f_{lk}(\mathbf{w}, r), k = 1, 2, \ldots, t_l, l = 1, 2, \ldots, p$ and/ or decision-control variable $w_{1j}, j = 1, 2, \ldots, n_1$, then go to Step 5.

We developed the distance-metric function, $\mathcal{D}(\mathbf{x})$, to analysis the performance of different algorithms available in the literature study for solving the NBL-MOLFCCo problem (1).

Definition 5.1. Let f^I_{lk} is Ideal value of crisp linear-fractional objective function $f_{lk}(\mathbf{x}), l = 1, \ldots, p, k = 1, 2, \ldots, t_l$ obtained by solving $f^I_{lk} = \max_{\mathbf{x} \in X} f_{lk}(\mathbf{x})$ and f^*_{lk} is the objective/ optimal values at the IF-efficient solution $\mathbf{x}^* \in X$ obtained using available approaches in the literature. Then the distance-metric value $\mathcal{D}(\mathbf{x})$ between f^I_{lk} and $f^*_{lk} = f^*_{lk}(\mathbf{x}^*)$ can be calculated as follows [35, 39]:

$$\mathcal{D}(\mathbf{x}^*) = \frac{\sqrt{\sum_{l=1}^{p} \sum_{k=1}^{t_l} (f^I_{lk} - f^*_{lk})^2}}{2 \times \sum_{l=1}^{p} t_l} \tag{29}$$

Where $t_l, l = 1, 2, \ldots, p$ is presents the total number of neutrosophic linear-fractional objective function under l^{th} DM in the NBL-MOLFCCo problem (1). The smaller value of $\mathcal{D}(\mathbf{x}^*)$ delivered the most preferable IF-efficient solution is $\mathbf{x}^*$ among all existing solutions.

6 Numerical Illustrations

To illustrate and assess the suggested efficient method, two distinct numerical-examples and real-life application problem are examined in this section, where the example 6.2 is directly lifted from Baky [8] and Saraj and Safaraei [51] papers.

6.1 Numerical Example 1:

Consider NBL-MOLFCCO problem with neutrosophic linear parameters and right-hand-side quantity follow Cauchy random distribution having mean (θ) and standard division (σ) mathematically formulated as:

$$\text{ULDM}\begin{cases}\max_{x_1}(f_{11}(x), f_{12}(x)),\\ \text{where,}\\ f_{11}(x)=\frac{(2+3[0.2,0.4])x_1+(-4+0.5[0.2,0.5])x_2+(3+3[0.15,0.25])x_3+(1+0.3[0,1])}{(0.3+2[0.3,0.5])x_1+(3+1.5[0.1,0.5])x_2+(1+5[0.1,0.5])x_3},\\ f_{12}(x)=\frac{3[0.1,0.6]x_1+(0.3+4[0.3,0.5])x_2+(-3.3+2[0.4,0.6])x_3+(3+3[0.15,0.25])}{3[0.3,0.7]x_1+(2+1.7[0.2,0.5])x_2+(1+1.1[0.4,1])x_3}.\end{cases}$$

$$\text{LLDM}\begin{cases}\text{Use } x_1 \text{ and } x_2, x_3 \text{ solves,}\\ \max_{x_2}(f_{21}(x), f_{22}(x)),\\ \text{where,}\\ f_{21}(x)=\frac{(0.3+2[0.5,0.9])x_1+(1.19+2[0.1,0.3])x_2+(0.3+2.9[0.5,1])}{(0.4+3[0.4,0.6])x_1+(1.4+1[0.5,0.7])x_2+(0.1+2[0.6,0.8])x_3},\\ f_{22}(x)=\frac{(3.9+6[0.2,0.5])x_1+(0.5+1.06[0.5,1])x_2-(0.2+1.9[0.5,1])x_3}{(1.3+2[0.2,0.4])x_1+(0.7+2[0.8,0.9])x_2+(0.8+2[0.6,0.8])x_3}.\\ \max_{x_3}(f_{31}(x), f_{32}(x)),\\ \text{where,}\\ f_{31}(x)=\frac{(4.3+5[0.2,0.6])x_1-(2+3[0.2,0.4])x_2+(2+3[0.2,0.4])x_3+(0.36+3[0.2,0.4])}{(0.5+3[0.12,0.2])x_1+(1.45+4[0.5,0.7])x_2},\\ f_{32}(x)=\frac{(3.2+3[0.5,1])x_1-(1.3+2[0.2,0.5])x_3}{(0.3+2[0.3,0.4])x_1+(2.1+1.5[0.3,0.5])x_2+(2.1+2.4[0.1,0.3])x_3}.\end{cases} \tag{30}$$

Subject to:

$$\begin{cases}(1.5+1[0.2,0.5])x_1+(1.2+0.9[0.1,0.2])x_2 \geq (2.1+2[0.6,0.7]),\\ Pr[(1.5+2[0.6,0.8])x_1+(0.44+2[0.5,0.7])x_2+(0.7+2[0.8,0.9])x_3 \leq b_1] \geq 0.95,\\ Pr[(0.24+2[0.6,0.8])x_1-(1.5+2[0.3,0.5])x_2+(2+0.6[0.6,0.9])x_3 \leq b_2] \geq 0.95,\\ x_1, x_2, x_3 \geq 0.\end{cases}$$

where, the random variables b_1, b_2 having mean $\theta_1 = 11.002, \theta_2 = 5.767$ and standard division $\sigma_1 = 7.4$, and $\sigma_2 = 3.9$.

Based on interval-ratio and chance-constraint optimization method developed in the proposed methodology, the analogous crisp-parametric and deterministic constraint model of problem (21) formulated as:

$$\begin{aligned}&\text{ULDM}\left\{\max_{x_1}\{f_{11}(x)=\frac{3.2x_1-3.75x_2+3.75x_3+1.3}{0.9x_1+3.15x_2+1.5x_3}, \quad f_{12}(x)=\frac{1.8x_1+2.3x_2-2.1x_3+3.75}{0.9x_1+2.34x_2+1.44x_3}\}\right.\\ &\text{LLDM}\begin{cases}\text{Use } x_1 \text{ and } x_2, x_3 \text{ solves:}\\ \max_{x_2}\{f_{21}(x)=\frac{2.1x_1+1.79x_2+3.2}{1.6x_1+1.9x_2+1.3x_3}, \quad f_{22}(x)=\frac{6.9x_1+1.56x_2-2.1x_3}{1.7x_1+2.3x_2+2x_3}\}\\ \max_{x_3}\{f_{31}(x)=\frac{7.3x_1-3.2x_2+3.2x_3+1.56}{0.86x_1+3.45x_2}, \quad f_{32}(x)=\frac{6.2x_1-2.3x_3}{0.9x_1+2.55x_2+2.34x_3}\}\end{cases}\\ &\text{Subject to: } S=\begin{cases}2x_1+1.38x_2 \geq 3.3, \quad 2.7x_1+1.44x_2+2.3x_3 \leq 10.95,\\ 1.44x_1-2.1x_2+2.36x_3 \leq 5.74, \quad x_1, x_2, x_3 \geq 0.\end{cases}\end{aligned} \tag{31}$$

The individual maximum i.e., solve $\max_{x\in S} f_{lk}(x)$ and minimum i.e., solve $\min_{x\in S} f_{lk}(x)$ value of NBL-MOLFCCO (22) model presented in the Table 2.

Table 1: Individual min. and max. value for model (22)

	f_{11}	f_{12}	f_{21}	f_{22}	f_{31}	f_{32}
$\max f_{lk}(x)$	4.431	4.526	2.525	4.059	12.802	6.889
$\min f_{lk}(x)$	-1.136	0.016	0.067	-0.26	-0.868	-0.547

Following the introduced linearized techniques, the NBL-MOLFCCO (22) model can be reformulated according to:

$$\text{ULDM}\begin{cases}\max\{f_{11}(\mathbf{w},r)=3.2w_1-3.75w_2+3.75w_3+1.3r,\quad f_{12}(\mathbf{w},r)=1.8w_1+2.3w_2-2.1w_3+3.75r\}\\ \text{Use } w_1 \text{ and } w_2,w_3 \text{ solves:}\\ \max\{f_{21}(\mathbf{w},r)=2.1w_1+1.79w_2+3.2r,\quad f_{22}(\mathbf{w},r)=6.9w_1+1.56w_2-2.1w_3\}\\ \max\{f_{31}(\mathbf{w},r)=7.3w_1-3.2w_2+3.2w_3+1.56r,\quad f_{32}(\mathbf{w},r)=6.2w_1-2.3w_3\}\\ \text{Subject to: } S^{wr}=\begin{cases}0.9w_1+3.15w_2+1.5w_3\leq 1,\quad 0.9w_1+2.34w_2+1.44w_3\leq 1,\\ 1.6w_1+1.9w_2+1.3w_3\leq 1,\quad 1.7w_1+2.3w_2+2w_3\leq 1,\\ 0.86w_1+3.45w_2\leq 1,\quad 0.9w_1+2.55w_2+2.34w_3\leq 1,\\ 2w_1+1.38w_2-3.3r\geq 0,\quad 2.7w_1+1.44w_2+2.3w_3-10.95r\leq 0,\\ 1.44w_1-2.1w_2+2.36w_3-5.74r\leq 0,\quad w_1,w_2,w_3\geq 0,r>0.\end{cases}\end{cases} \tag{32}$$

Formulate the IFGP Model for DM_1-MOLDM (23) problem and solve it.

IFGP Model- I for DM$_1$-MOLDM problem:

$$\min 0.295D_{11}^{-\mu}+0.299D_{11}^{+\mu}+0.421D_{12}^{-\mu}+0.434D_{11}^{+\mu}$$

$$\text{Subject to: }\begin{cases}3.2w_1-3.75w_2+3.75w_3+1.3r+3.383D_{11}^{-\mu}\geq 2.346,\\ -3.2w_1+3.75w_2-3.75w_3-1.3r-3.337D_{11}^{+\mu}\leq -2.3,\\ 1.8w_1+2.3w_2-2.1w_3+3.75r+2.3798D_{12}^{-\mu}\geq 2.396,\\ -1.8w_1-2.3w_2+2.1w_3-3.75r-2.3038D_{12}^{+\mu}\leq -2.32,\\ S^{wr}\text{ (Use the system constraint from Eqs. (23)),}\quad D_{11}^{-\mu},D_{11}^{+\mu},D_{12}^{-\mu},D_{12}^{+\mu}\geq 0.\end{cases} \tag{33}$$

Thus, the optimal solution of Model (24) is $w_1^*=0.5882353, w_2^*=0, w_3^*=0, r=0.3565062$.

For the sake of avoiding decision-lock, the control variable of upper-level w_1 set as binding constraint for the next-level with given acceptable tolerance $q_{11}^{R\mu}=q_{11}^{L\mu}=0.025$ and rejected tolerance $q_{11}^{L\nu}=q_{11}^{R\nu}=0.05$. Therefore, the suggested IFGP Model- II for NBL-MOLFCCCO (21) problem formulated and solved as:

$$
\begin{aligned}
&\min 0.295D_{11}^{-\mu} + 0.299D_{11}^{+\mu} + 0.421D_{12}^{-\mu} + 0.434D_{11}^{+\mu} + 0.433D_{21}^{-\mu} + 0.446D_{21}^{+\mu}\\
&\quad +0.233D_{22}^{-\mu} + 0.236D_{22}^{+\mu} + 0.175D_{31}^{-\mu} + 0.195D_{31}^{+\mu} + 0.238D_{32}^{-\mu} + 0.241D_{32}^{+\mu}\\
&\quad +0.404D_{11}^{-\mu} + 0.404D_{11}^{+\mu} + 0.204D_{11}^{-\mu} + 0.204D_{11}^{+\mu}\\
&\text{Subject to:}\\
&2.1w_1 + 1.79w_2 + 3.2r + 2.31D_{21}^{-\mu} \geq 2.376,\\
&-2.1w_1 - 1.79w_2 - 3.2r - 2.24D_{21}^{+\mu} \leq -2.31,\\
&6.9w_1 + 1.56w_2 - 2.1w_3 + 3.831D_{22}^{-\mu} \geq 4.058,\\
&-6.9w_1 - 1.56w_2 + 2.1w_3 - 3.783D_{22}^{+\mu} \leq -4.01,\\
&7.3w_1 - 3.2w_2 + 3.2w_3 + 1.56r + 5.718D_{31}^{-\mu} \geq 4.85,\\
&-7.3w_1 + 3.2w_2 - 3.2w_3 - 1.56r - 5.118D_{31}^{+\mu} \leq -4.25,\\
&6.2w_1 - 2.3w_3 + 4.193D_{32}^{-\mu} \geq 3.647,\\
&-6.2w_1 + 2.3w_3 - 4.146D_{32}^{+\mu} \leq -3.6,\\
&S^{wr} \text{ (Use the system constraint from Eqs.(24))},\\
&D_{11}^{-\mu}, D_{11}^{+\mu}, D_{12}^{-\mu}, D_{12}^{+\mu}, D_{21}^{-\mu}, D_{21}^{+\mu}, D_{22}^{-\mu}, D_{22}^{+\mu},\\
&D_{31}^{-\mu}, D_{31}^{+\mu}, D_{32}^{-\mu}, D_{32}^{+\mu}, d_{11}^{-L\mu}, d_{11}^{-R\mu}, d_{11}^{+L\nu}, d_{11}^{+R\nu} \geq 0.
\end{aligned}
\tag{34}
$$

Using Lingo-19 software, the compromise IF-efficient solution of the BL-MOLO problem is generated as: $w_1 = 0.5882353$, $w_2 = 0$, $w_3 = 0$, $r = 0.3565062$ and its corresponding efficient solution of NBL-MOLFCC0 (21) problem using backward technique obtained as: $\mathbf{x}^* = (x_1 = 1.650, x_2 = 0, x_3 = 0)$. The objective values, degree of satisfaction and dissatisfaction at these efficient solution are illustrated in Table-3.

Table 2: Objective, satisfaction and dissatisfaction value at $\mathbf{x}^*$ for model (22)

	f_{11}	f_{12}	f_{21}	f_{22}	f_{31}	f_{32}
Objective value ($f_{lk}(\mathbf{x}^*)$)	4.431	4.525	2.525	4.059	9.589	6.889
Satisfaction level ($\mu_{lk}(\mathbf{x}^*)$)	1	1	1	1	0.94	1
Dissatisfaction level ($\nu_{lk}(\mathbf{x}^*)$)	0	0	0	0	0.04	0

6.2 Numerical Example 2:

The proposed method cannot be directly compared with existing methodologies, as the introduced NBL-MOLFCC0 problem is a novel formulation not previously addressed in the literature. While the NBL-MOLFCC0 problem represents a general framework applicable to real-world decision-making scenarios involving uncertain parameters and chance constraints and can be reduced to fuzzy, intuitionistic fuzzy, or crisp-type problems, a direct comparison is not feasible due to its unique characteristics. Therefore, we will use a crisp-type problem adapted from Baky [8] and Saraj and Safaraei [51] as a benchmark to evaluate the efficiency of the proposed method.

$$\begin{array}{l} \text{ULDM}\left\{\ \max(f_{11}(\mathbf{x}) = \frac{-x_1-4x_2+x_3+1}{2x_1+3x_2+x_3+2}; \quad f_{12}(\mathbf{x}) = \frac{-2x_1+x_2+3x_3+4}{2x_1-x_2+x_3+5}) \quad [DM_1]\right. \\ \text{Use } x_1, \text{ where } x_2 \text{ and } x_3 \text{ solves;} \\ \text{LLDM}\left\{\begin{array}{l} \max(f_{21}(\mathbf{x}) = \frac{3x_1-2x_2+2x_3}{x_1+x_2+x_3+3}; \quad f_{22}(\mathbf{x}) = \frac{-7x_1-2x_2+x_3+1}{5x_1+2x_2+x_3+1}) \quad [DM_2] \\ \max(f_{31}(\mathbf{x}) = \frac{x_1+x_2+x_3-4}{x_1-2x_2+10x_3+6}; \quad f_{32}(\mathbf{x}) = \frac{2x_1-x_2+x_3+4}{-x_1+x_2+x_3+10}) \quad [DM_3] \end{array}\right. \end{array} \tag{35}$$

Since the given problem (26) formulated as crisp-parametric and deterministic-constraint model, directly go to the linearization procedure to obtain the following BL-MOLO problem:

$$\text{ULDM}\left\{\begin{array}{l} \max(f_{11}(\mathbf{w},r) = -w_1 - 4w_2 + w_3 + r; \quad f_{12}(\mathbf{w},r) = -2w_1 + w_2 + 3w_3 + 4r) \\ \text{Use } w_1, \text{ where } w_2 \text{ and } w_3 \text{ solves} \\ \text{LLDM}\left\{\begin{array}{l} \max(f_{21}(\mathbf{w},r) = 3w_1 - 2w_2 + 2w_3; \quad f_{22}(\mathbf{w},r) = -7w_1 - 2w_2 + w_3 + r) \\ \max(f_{31}(\mathbf{w},r) = w_1 + w_2 + w_3 - 4r; \quad f_{32}(\mathbf{w},r) = 2w_1 - w_2 + w_3 + 4r) \end{array}\right. \end{array}\right. \tag{36}$$

Subject to:

$$\begin{aligned} &2w_1 + 3w_2 + w_3 \le 1, \\ &2w_1 - w_2 + w_3 \le 1, \\ &w_1 + w_2 + w_3 + 3r \le 1, \\ &5w_1 + 2w_2 + w_3 + r \le 1, \\ &w_1 - 2w_2 + 10w_3 + 6r \le 1, \\ &-w_1 + w_2 + w_3 + 10r \le 1, \\ &w_1, w_2, w_3 \ge 0, r > 0. \end{aligned} \tag{37}$$

Following the next steps of our proposed method, we generated different results as given in Table 4. Based on information given in Table 4, we formulate IFGP Model-I for the upper-level decision-making problem as shown below:

Table 3: The numerical values $U^{\mu}_{lk}, U^{\nu}_{lk}, L^{\mu}_{lk} = L^{\nu}_{lk}, W^{\mu}_{lk}$ and W^{ν}_{lk} for each $f_{lk}(\mathbf{x})$

	$f_{11}(\omega,r)$	$f_{12}(\omega,r)$	$f_{21}(\omega,r)$	$f_{22}(\omega,r)$	$f_{31}(\omega,r)$	$f_{32}(\omega,r)$
$U^{\mu}_{lk} = \max_{(\omega,r)\in S^{wr}} f_{lk}(\omega,r)$	0.125	0.5606	0.6164	0.125	0.0204	0.8258
$L^{\mu}_{lk} = L^{\nu}_{lk} = \min_{(\omega,r)\in S^{wr}} f_{lk}(\omega,r)$	-0.50	-0.1818	-1.1818	-0.33	0.003	-0.56
U^{ν}_{lk}	0.1	0.5	0.6	0.1	0.8	-0.5
$W^{\mu}_{lk} = \frac{1}{U^{\mu}_{lk} - L^{\mu}_{lk}}$	1.6	1.78	1.25	0.765	2.85	1.215
$W^{\nu}_{lk} = \frac{1}{U^{\nu}_{lk} - L^{\nu}_{lk}}$	1.61	1.786	1.28	0.768	2.857	1.25

IFGP Model- I for DM$_1$-MOLDM problem

$$\begin{aligned}
&\min 1.6D_{11}^{-\mu} + 1.61D_{11}^{+\nu} + 1.78D_{12}^{-\mu} + 1.786D_{12}^{+\nu}\\
&\text{Subject to:}\\
&-w_1 - 4w_2 + w_3 + r + 0.625D_{11}^{-\mu} \geq 0.125,\\
&w_1 + 4w_2 - w_3 - r - 0.62D_{11}^{+\nu} \leq -0.1,\\
&-2w_1 + w_2 + 3w_3 + 4r + 0.5606D_{12}^{-\mu} \geq 0.5606,\\
&2w_1 - w_2 - 3w_3 - 4r - 0.56D_{12}^{+\nu} \leq -0.5,\\
&S^{wr} \text{ (Use the system constraint from Eqs. (27)).}
\end{aligned} \tag{38}$$

Solving the above problem (28) via LINGO-19 software, we obtained the solution ($w_1^* = 0, w_2^* = 0.0192, w_3^* = 0.0577, r = 0.0769$) to the ULDM problem. Thus, the decision control variable of $ULDM$ is $w_1 = 0$.

Now, the ULDM set acceptable and rejected relaxation on its decision-control variable $w_1 = 0$ to avoid decision-lock for follower-level DMs, as: $q_{11}^{R\mu} = 0.155$ and $q_{11}^{R\nu} = 1$. The degree of membership and nonmembership for IFG of the decision control variable w_1 are measured by $\mu_{11}(w_{11}) = \frac{0.155 - w_1}{0.155}$ and $\nu_{11}(w_{11}) = \frac{w_1}{1}$, respectively. Also, the weight for IFG of the decision control variable w_1 using the formula set in Eqs. (19), are: $w_1^{R\mu} = 6.45, w_1^{R\nu} = 1$. Therefore, the final proposed IFGP Model- II for the given problem (26) formulated as:

IFGP Model- II for problem (26)

$$
\begin{aligned}
&\min[1.6D_{11}^{-\mu}+1.61D_{11}^{+\nu}+1.78D_{12}^{-\mu}+1.786D_{12}^{+\nu}+1.25D_{21}^{-\mu}+1.28D_{21}^{+\nu}\\
&\quad+0.765D_{22}^{-\mu}+0.768D_{22}^{+\nu}+2.85D_{31}^{-\mu}+2.857D_{31}^{+\nu}+1.215D_{32}^{-\mu}+1.25D_{32}^{+\nu}+6.45d_{11}^{-\mu}+d_{11}^{+\nu}]\\
&\text{Subject to:}\\
&3w_1-2w_2+2w_3+0.7982D_{21}^{-\mu}\geq 0.6164,\\
&-3w_1+2w_2-2w_3-0.7818D_{21}^{+\nu}\leq -0.6,\\
&-7w_1-2w_2+w_3+r+1.3068D_{22}^{-\mu}\geq 0.125,\\
&7w_1+2w_2-w_3-r-1.3018D_{22}^{+\nu}\leq -0.1,\\
&w_1+w_2+w_3-4r+0.3504D_{31}^{-\mu}\geq 0.0204,\\
&-w_1-w_2-w_3+4r-0.35D_{31}^{+\nu}\leq 0,\\
&2w_1-w_2+w_3+4r+0.8228D_{32}^{-\mu}\geq 0.8258,\\
&-2w_1+w_2-w_3-4r-0.797D_{32}^{+\nu}\leq -0.8,\\
&-w_1+0.155d_{11}^{-\mu}\geq 0,\quad w_1-d_{11}^{+\nu}\leq 0,\\
&S^{wr}\ \text{(Use system constraint from Eqs. (27))},\\
&-w_1-4w_2+w_3+r+0.625D_{11}^{-\mu}\geq 0.125,\\
&w_1+4w_2-w_3-r-0.62D_{11}^{+\nu}\leq -0.1,\\
&-2w_1+w_2+3w_3+4r+0.5606D_{12}^{-\mu}\geq 0.5606,\\
&2w_1-w_2-3w_3-4r-0.56D_{12}^{+\nu}\leq -0.5,\\
&D_{lk}^{-\mu},D_{lk}^{+\nu}\geq 0,\forall l,k,\quad d_{11}^{-\mu},d_{11}^{+\nu}\geq 0, r>0.
\end{aligned}
\tag{39}
$$

Finally, solving the above problem (29) using LINGO-19 software, we generate an IF-efficient solution to the given problem (26) as $w_1^*=0.00348983, w_2^*=0, w_3^*=0.06365932, r=0.06016949$. Therefore, the preferable solution obtained with backward substitution are $x_1^*=0.058, x_2^*=0, x_3^*=1.058$.

In literature review, Saraj and Safaraei [51] proposed a Taylor series method using fuzzy decision sets, while Baky [8] presented the FGP approach for the same numerical example (2(26)), yielding the result $\hat{x}=(0,0,1)$. To compare the priority of the proposed solution $\mathbf{x}^*=(0.058,0,1.058)$ with the existing solution $\hat{x}=(0,0,1)$, we employed a distance-metric function. As shown in Table 5, the distance-metric value for $\mathbf{x}^*=(0.058,0,1.058)$ is less than that for $\hat{x}=(0,0,1)$ from Saraj and Safaraei [51] and Baky [8]. Thus, the proposed IFGP approach yields the most satisfactory Pareto-optimal solution.

6.3 Application: Industrial production planning problem

This subsection explores the applicability of the proposed efficient method to the real-world NBL-MOLFCC problem. We focus on six machine types—lathes, drill presses, grinders, jigsaws, band saws, and milling machines—used in industrial production planning [16, 37]. These machines will utilize their weekly capacities to produce three items. Table 6 presents each machine's total neutrosophic linear available time, along with parameters b_1, b_2 and b_3, which follow a Cauchy random distribution with known mean $\theta_1=100, \theta_2=90.6, \theta_3=150.27$ and standard deviation $\sigma_1=1443.6, \sigma_2=1000.29, \sigma_3=1328.86$.

Table 4: Comparison of IF-efficient solution for numerical example 2(26) obtained by IFGP method and the fuzzy-based approach by Saraj and Safaraei (2015), and Baky (2009).

Methods	$\mathcal{D}(\mathbf{x})$	$f_{01}(\mathbf{x})$	$f_{02}(\mathbf{x})$	$f_{11}(\mathbf{x})$	$f_{12}(\mathbf{x})$	$f_{21}(\mathbf{x})$	$f_{22}(\mathbf{x})$
$\max f_{lk}(\mathbf{x})$		0	0.67	1.25	1.47	1	0.02
1.25							
Proposed	0.10443	0.63	1.15	0.56	0.9	1	-0.17
0.47							
(Saraj and Safaraei, 2015; Baky, 2009)	0.10644	0.67	1.17	0.51	-0.19	0.45	

Table 5: Portfolio of available capacities with NLN parameters

Machine	Machine time	Product-1 (w_1)	Product-2 (w_2)	Product-3 (w_3)
Milling	b_1	$(2+20[0.5,0.9])$	$(15+4[0.5,0.9])$	0
Lathe	b_2	$(2+2[0.5,0.8])$	$(0+18[0.5,0.7])$	$(6+4[0.5,0.9])$
Grinder	b_3	$(5+10[0.5,1])$	$(7+12[0.5,0.9])$	$(13.5+5[0.3,0.6])$
Jigsaw	$(1300+50.2[0.3,0.5])$	$(4+4[0.5,0.8])$	0	$(14+4[0.5,0.9])$
Drill press	$(890.925+18[0.3,0.5])$	0	$(6+12[0.5,0.7])$	$(9+16[0.5,0.8])$
Band saw	$(1070+10[0.2,0.5])$	$(5.5+8[0.5,0.9])$	$(5.5+8[0.5,0.9])$	$(2+4[0.5,1])$
Profit		$(40+20[0.1,0.5])$	$(80+40[0.1,0.5])$	$(16+2[0.4,0.8])$
Product liability		$(0.9[0.6,0.8])$	$(0.8+0.05[0.5,1])$	$(0.7+0.2[0.1,0.4])$
Quality		$(90+4[0.2,0.5])$	$(70+10[0.3,0.5])$	$(45+10[0.2,0.5])$
Worker satisfaction		$(20+10[0.2,0.5])$	$(60+80[0.1,0.5])$	$(65+20[0.3,0.5])$

Let w_1, w_2, w_3 stand for the three products. Then, with the data in Table-6 and the formulation of the crisp model that was covered in the preceding subsection, the crisp model is provided as follows:

$$
\begin{aligned}
&\text{ULDM}\begin{cases} \max_{w_1,w_2}(z_1(w) = 50w_1 + 100w_2 + 17.6w_3 \rightarrow \text{profit}, \\ z_2(w) = 0.72w_1 + 0.85w_2 + 0.78w_3 \rightarrow \text{product liability}), \end{cases}\\
&\text{LLDM}\begin{cases} \max_{w_3}(z_{21}(w) = 92w_1 + 75w_2 + 50w_3 \rightarrow \text{Quality}, \\ z_{22}(w) = 25w_1 + 100w_2 + 75w_3 \rightarrow \text{Worker's Satisfaction}), \end{cases}\\
&\text{Subject to:}\\
&\begin{cases} 12w_1 + 17w_2 \leq 1411 \rightarrow \text{Milling Machine}, \\ 3w_1 + 9w_2 + 8w_3 \leq 998.025 \rightarrow \text{Lathe}, \\ 10w_1 + 13w_2 + 15w_3 \leq 1753 \rightarrow \text{Grinder}, \\ 6w_1 + 16w_3 \leq 1325.1 \rightarrow \text{Jigsaw}, \\ 12w_2 + 17w_3 \leq 899.925 \rightarrow \text{Drill Press}, \\ 9.5w_1 + 9.5w_2 + 4w_3 \leq 1075 \rightarrow \text{Band Saw}, \\ w_1, w_2, w_3 \geq 0. \end{cases}
\end{aligned}
\tag{40}
$$

Now, according to the proposed method, the IFGP model of ULDM is formulated as:

$$\begin{aligned}
&\min 0.00315D_{11}^{-\mu} + 0.0044D_{11}^{+\nu} + 0.022D_{12}^{-\mu} + 0.033D_{12}^{+\nu}\\
&\text{Subject to:}\\
&50w_1 + 100w_2 + 17.5w_3 + 317.15D_{11}^{-\mu} \geq 7767.995,\\
&-50w_1 - 100w_2 - 17.5w_3 - 225D_{11}^{+\nu} \leq -5767.995,\\
&0.72w_1 + 0.85w_2 + 0.78w_3 + 45D_{12}^{-\mu} \geq 105.824,\\
&-0.72w_1 - 0.85w_2 - 0.78w_3 - 30D_{12}^{+\nu} \leq -90.824,\\
&\text{Include system of constraint from Eqs. (30).}
\end{aligned} \tag{41}$$

The optimal solution of the IFGP (31) model of DM1 is $w^* = (w_1 = 64.60414, w_2 = 37.3694, w_3 = 26.5627)$.

Consider the decision control-variables of DM1, $w_1 = 64.604$, and $w_2 = 37.36$ as binding constraints for the subsequent LLDMs. Based on suggested solution method algorithm, the IFGP model of a neutrosophic bi-level multi-objective production planning problem is formulated as follows:

$$\begin{aligned}
&\min 0.00315D_{11}^{-\mu} + 0.0044D_{11}^{+\nu} + 0.022D_{12}^{-\mu} + 0.033D_{12}^{+\nu} + 0.00033D_{21}^{-\mu} + 0.0005D_{21}^{+\nu}\\
&\quad +0.00033D_{22}^{-\mu} + 0.0005D_{22}^{+\nu} + 100d_{12}^{-L\mu} + 100d_{12}^{-R\mu} + 50d_{12}^{+L\nu} + 50d_{12}^{+R\nu}\\
&\quad +50d_{11}^{-L\mu} + 50d_{11}^{-R\mu} + 20d_{11}^{+L\nu} + 20d_{11}^{+R\nu}\\
&\text{Subject to:}\\
&92w_1 + 75w_2 + 50w_3 + 3000D_{21}^{-\mu} \geq 10950.67,\\
&-92w_1 - 75w_2 - 50w_3 - 2000D_{21}^{+\nu} \leq -9950.67,\\
&25w_1 + 100w_2 + 75w_3 + 3000D_{22}^{-\mu} \geq 7783.33,\\
&-25w_1 - 100w_2 - 75w_3 - 2000D_{22}^{+\nu} \leq -6783.33,\\
&w_1 - 0.02d_{11}^{-L\mu} \geq 64.604, \quad -w_1 + 0.02d_{11}^{-R\mu} \geq -64.604,\\
&-w_1 - 0.05d_{11}^{+L\nu} \leq -64.604, \quad w_1 - 0.05d_{11}^{+R\nu} \leq 64.604,\\
&w_2 + 0.01d_{12}^{-L\mu} \geq 37.36, \quad -w_2 + 0.01d_{12}^{-R\mu} \geq -37.36,\\
&-w_2 - 0.02d_{12}^{+L\nu} \leq -37.36, \quad w_2 - 0.02d_{12}^{+R\nu} \leq 37.36,\\
&\text{Include system of constraint from Eqs. (31).}
\end{aligned} \tag{42}$$

Using LINGO-19 software, we obtained an optimal solution to the above model: $w_1 = 64.604, w_2 = 37.36, w_3 = 26.5691$.

The optimal solutions obtained from the proposed approach for the real-life production planning problem (31) match those achieved by current methods, such as TOPSIS given by El Sayed and Baky [16] and new method suggested by Moges and Wordofa [37].

7 Conclusion

This study introduces an efficient methodology to address a contemporary NBL-MOLFCCCO problem within an uncertain optimization, wherein each input data is characterized as a neutrosophic linear parameter, and certain right-hand-side quantities adhere to a Cauchy random distribution defined by their mean and standard deviation. The primary objective of this endeavor is to devise a solution method that is both practical and accessible for implementation, while adequately reflecting real-world decision-making contexts. In these scenarios, multiple DMs collaboratively within a hierarchical structure and engaging in various decision-making processes such as agreement, disagreement, and uncertainty. In this framework, all unknown coefficients pertaining to constraints, objective functions, and resources are treated as neutrosophic linear numbers, alongside chance constraints, thereby accommodating real-world complexities. The standard formulations are subsequently derived utilizing modified interval and chance-constraint methodologies. Thereafter, through a linearization approach, the resultant model is reformulated into a bi-level multi-objective linear mathematical model. Ultimately, the formulated model is resolved through the application of the proposed method. The main advantage of the proposed algorithm is its ability to eliminate the frequent rejection of solutions by the ULDM, thereby avoiding the need for reassessment of the problem through a redefinition of the elicited membership and non-membership functions necessary for achieving an adequate decision. To illustrate the efficacy and applicability of this method, two numerical examples along with a real-world application problem have been presented. The distinctiveness of this method is evident in its adeptness at managing uncertain data while maintaining a straightforward execution.

The proposed methodology will undergo further refinement for its implementation in a real-world context, specifically addressing multi-objective linear fractional hierarchical decision-making within an uncertain environment.

Author contributions

All the authors have equal contribution in preparing and writing the paper.

Acknowledgments

The author thank the editors-in-chief and anonymous reviewers for their detailed comments and suggestions, which improved the manuscript's clarity and readability.

Conflict of Interest

The corresponding author declares that there is no conflict of interest on account of all authors.

Funding

The authors did not receive any funding for this research.

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Received: Jan 9, 2026. Accepted: June 10, 202