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Neutrosophic estimation of population variance using two auxiliary variables in simple random sampling

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Abstract. In survey sampling, the efficient estimation of population variance plays an important role in drawing inference, particularly when handling uncertainty and imprecise data. This article develops a generalized class of Searls type power ratio estimators for the population variance under a neutrosophic framework, incorporating two auxiliary variables (TAV) in simple random sampling (SRS). The proposed approach accounts for indeterminacy and imprecision that frequently arise in real-world data collection processes by utilizing neutrosophic representation of both study and auxiliary variables. By simultaneously employing two auxiliary variables, the efficiency of the proposed variance estimators is significantly improved over some basic adapted neutrosophic population variance estimators. Theoretical properties of the proposed estimators, including bias and mean square error (MSE), are derived under the neutrosophic setup. A comparative performance evaluation is conducted using both simulated and real-world datasets, demonstrating the superiority of the proposed estimators in terms of efficiency, especially in uncertain data.

Keywords: Mean square error; Neutrosophic framework; Variance estimation; Auxiliary variables; Efficiency.

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1. **Introduction**

In survey sampling, variance estimation is a key aspect of statistical analysis, as it provides a measure of the dispersion or variability within a population. Accurate estimation of population variance is essential for evaluating the precision of the estimators. The need for reliable estimates of population variance becomes even more significant when allocating resource or making policy decisions based on survey data. When auxiliary variables are highly associated with the study variable, they help reduce the sampling error and improve the precision of the variance estimates. Estimation methods that utilize such auxiliary information, including ratio, regression, logarithmic, or combined estimators, have shown to outperform the traditional estimators by minimizing bias and MSE. Therefore, using auxiliary information in the estimation process not only enhances the inference drawn but also optimizes survey designs, especially under constraints of cost or sample size. Many authors recommended various estimation procedures to estimate the population variance using auxiliary information in survey sampling. [1] estimated the variance of a finite population using SRS. Some fundamental variance estimators were proposed by [2] in SRS. An improved exponential estimator for population variance using two auxiliary variables was proposed by [3]. [4] introduced a new family of variance estimators using known population variance of an auxiliary variable. [5] introduced a family of estimators for the population variance utilizing TAV. [6] introduced an elevated estimator of population variance utilizing TAV under SRS. The traditional variance estimation methods typically assume that data are accurate and complete. However, indeterminate data are often encountered in real-world surveys due to poor data collection practices or inconsistent respondent answers. Although challenging, such data can be effectively addressed using advanced approaches like fuzzy logic or neutrosophic theory, which incorporate uncertainty into the estimation process, leading to more flexible and realistic outcomes.

1.1. *Neutrosophic statistics and neutrosophic data*

Neutrosophic statistics is an emerging field of statistical science that extends the classical and fuzzy statistical frameworks to better handle uncertain and inconsistent data often encountered in real-world. Based on the neutrosophic logic introduced by [7], neutrosophic statistics incorporates the degrees of truth, indeterminacy, and falsity simultaneously, allowing for a more flexible and comprehensive representation of uncertainty. Unlike traditional statistical approaches that assume complete information, neutrosophic statistics recognizes the presence of ambiguity in data collection. This makes it particularly useful in areas like social sciences, environmental studies, medicine, and survey sampling, where data may be incomplete. By using neutrosophic numbers, this framework enhances the robustness and adaptability of the estimators, decision-making models, and hypothesis testing procedures under uncertain conditions. As a result, neutrosophic statistics provides a powerful and realistic tool for analyzing complex systems where conventional methods fall short.

Neutrosophic data is a generalized form of data representation that accounts for uncertainty, indeterminacy, and inconsistency, which are often present in real-world. Unlike classical or fuzzy data, neutrosophic data is characterized by three independent components: truth (T), indeterminacy (I), and falsity (F), each ranging in the interval [0, 1]. By accommodating uncertainty in data, neutrosophic approaches allow for more robust and realistic analyses in complex or uncertain environments. For instance, in climate studies, the prediction of rainfall for a specific area may have certain probability from different models (T), contradictory prediction from other models (F), and unknown factors like sudden atmospheric changes (I).

Similarly, in customer feedback analysis, a product review may contain clear positive remarks (T), clear negative remarks (F), and neutral sentiments that cannot be classified confidently (I). In stock market analysis, expert opinions about a future performance of the company can be supportive (T), opposing (F), and uncertain due to unforeseen economic events (I). Such examples show how neutrosophic data express the real-world complexity better than purely crisp or fuzzy representations.

1.2. *Current status of work in survey sampling*

In survey sampling literature, only a limited number of estimation methods of population mean have been proposed for handling indeterminate data using some sampling designs. [8] introduced neutrosophic ratio-type estimators for estimating the population mean under SRS. A generalized neutrosophic sampling framework was proposed by [9] to improve the estimation of the population mean. Using auxiliary variable, [10] developed the neutrosophic factor-type exponential estimators for more accurate mean estimation. [11] proposed the robust neutrosophic exponential estimators of population mean under uncertainty. [12] introduced a new comprehensive mean estimator utilizing the regression-cum-exponential type estimator with the application of neutrosophic data. [13] suggested an optimal neutrosophic framework for population mean estimation under SRS by employing two auxiliary variables. [14] combined the two auxiliary variables for the enhanced estimation of finite population mean under neutrosophy. [15] estimated the population mean utilizing generalized neutrosophic exponential estimators. [16] estimated the population mean using two neutrosophic auxiliary variables with imprecise information. [17] developed an improved estimator for the population mean utilizing known medians of two auxiliary variables under neutrosophic framework. [18] constructed an almost unbiased estimator for the population median using neutrosophic information. [19] pioneered the neutrosophic imputation methods of the population mean under uncertainty.

1.3. *Scope of the research*

Although some authors have addressed the population mean estimation under neutrosophic statistics using single and two auxiliary variables but there is no existing work on the variance estimation under the neutrosophic framework based on TAV. The current scope of the study focuses on developing efficient and reliable estimators for the population variance using TAV under uncertainty and indeterminacy, as modeled by the neutrosophic framework. By incorporating two auxiliary variables, the article aims to enhance the precision of the variance estimation under SRS, especially when data are uncertain and incomplete. The article investigates how the joint use of TAV under neutrosophic setup can lead to improved estimator's performance in terms of MSE. This article also evaluates the theoretical properties and empirical efficiency of the proposed estimators, contributing to the broader application of the neutrosophic theory in modern survey sampling where classical statistical approaches may fall short.

1.4. *Notations under neutrosophic setup*

The statistical literature presents many types of neutrosophic data, including quantitative neutrosophic data, which is characterized by values lying within an unknown interval $[r,\ s]$. This interval, based on neutrosophic numbers, can be represented in multiple forms. In the current article, we represent the neutrosophic interval values as $W_N = W_L + W_U I_N$, where $I_N \in [I_L,\ I_U]$. This formulation implies that the neutrosophic data are expressed in interval form $W_N \in [r,\ s]$, with r and s denoting the lower and upper bounds, respectively. For a more detailed discussion on neutrosophic notations, readers are suggested to follow [20].

Let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N)$ denote a finite population of N identifiable units. For each unit ζ_{iN}, where $i = 1, 2, \ldots, N$, let y_{iN}, x_{iN}, and z_{iN} represent the neutrosophic study variable y_N and the neutrosophic auxiliary variables x_N and z_N, respectively. Let n_N units be selected from the population ζ by means of simple random sampling without replacement (SRSWOR). Let $\mu_{x_N} = \sum_{i=1}^{N} x_{iN}/N$, $\mu_{z_N} = \sum_{i=1}^{N} z_{iN}/N$ and $\mu_{y_N} = \sum_{i=1}^{N} y_{iN}/N$ are respectively the population means corresponding to the sample means $\bar{x}_N = \sum_{i=1}^{n} x_{iN}/n_N$, $\bar{z}_N = \sum_{i=1}^{n} z_{iN}/n_N$ and $\bar{y}_N = \sum_{i=1}^{n} y_{iN}/n_N$ of variables x_N, z_N, and y_N. Let $S^2_{x_N} = \sum_{i=1}^{N}(x_{iN} - \mu_{x_N})^2/N$, $S^2_{z_N} = \sum_{i=1}^{N}(z_{iN} - \mu_{z_N})^2/N$, and $S^2_{y_N} = \sum_{i=1}^{N}(y_{iN} - \mu_{y_N})^2/N$ be respectively the population variances corresponding to the sample variances $s^2_{x_N} = \sum_{i=1}^{n}(x_{iN} - \bar{x}_N)^2/(n_N - 1)$, $s^2_{z_N} = \sum_{i=1}^{n}(z_{iN} - \bar{z}_N)^2/(n_N - 1)$, and $s^2_{y_N} = \sum_{i=1}^{n}(y_{iN} - \bar{y}_N)^2/(n_N - 1)$ of variables x_N, z_N, and y_N. Let $C_{x_N} = S_{x_N}/\mu_{x_N}$, $C_{z_N} = S_{z_N}/\mu_{z_N}$, and $C_{y_N} = S_{y_N}/\mu_{y_N}$ be the population coefficient of variations corresponding to the sample coefficient of variations $c_{x_N} = s_{x_N}/\bar{x}_N$, $c_{z_N} = s_{z_N}/\bar{z}_N$, and $c_{y_N} = s_{y_N}/\bar{y}_N$ of variables x_N, z_N, and y_N. Moreover, $\beta_2(x_N)$, $\beta_2(z_N)$, and $\beta_2(y_N)$ are the population coefficient of kurtosis of variables x_N, z_N, and y_N, respectively.

To derive the properties of the neutrosophic estimators, let $s^2_{y_N} = S^2_{y_N}(1 + \epsilon_{0N})$, $s^2_{x_N} = S^2_{x_N}(1 + \epsilon_{1N})$ and $s^2_{z_N} = S^2_{z_N}(1 + \epsilon_{2N})$ where ϵ_{0N}, ϵ_{1N}, and ϵ_{2N} are error terms such that

$$\left.\begin{aligned}
E(\epsilon_{0N}) = E(\epsilon_{1N}) = E(\epsilon_{2N}) = 0;\ E(\epsilon^2_{0N}) = \frac{(\beta_2(y_N) - 1)}{n_N} = \Delta_0;\\
E(\epsilon^2_{1N}) = \frac{(\beta_2(x_N) - 1)}{n_N} = \Delta_1;\ E(\epsilon^2_{2N}) = \frac{(\beta_2(z_N) - 1)}{n_N} = \Delta_2,\\
E(\epsilon_{0N}\epsilon_{1N}) = \frac{(\lambda_{yx} - 1)}{n_N} = \Delta_{01};\ E(\epsilon_{0N}\epsilon_{2N}) = \frac{(\lambda_{yz} - 1)}{n_N} = \Delta_{02};\\
E(\epsilon_{1N}\epsilon_{2N}) = \frac{(\lambda_{xz} - 1)}{n_N} = \Delta_{12};
\end{aligned}\right\} \quad (1)$$

where, $\lambda_{yx} = \mu_{yx}/\mu_{2y}\mu_{2x}$, $\lambda_{yz} = \mu_{yz}/\mu_{2y}\mu_{2z}$, and $\lambda_{xz} = \mu_{xz}/\mu_{2x}\mu_{2z}$. Here, $\mu_{yx} = \sum_{i=1}^{N}(y_{iN} - \mu_{y_N})^2(x_{iN} - \mu_{x_N})^2/N$, $\mu_{yz} = \sum_{i=1}^{N}(y_{iN} - \mu_{y_N})^2(z_{iN} - \mu_{z_N})^2/N$, and $\mu_{xz} = \sum_{i=1}^{N}(x_{iN} - \mu_{x_N})^2(z_{iN} - \mu_{z_N})^2/N$ are the population product moments, while $\mu_{2y} = \sum_{i=1}^{N}(y_{iN} - \mu_{y_N})^2/N$, $\mu_{2x} = \sum_{i=1}^{N}(x_{iN} - \mu_{x_N})^2/N$ and $\mu_{2z} = \sum_{i=1}^{N}(z_{iN} - \mu_{z_N})^2/N$ are the population moments.

The subsequent section introduces a class of adapted neutrosophic estimators of population variance based on TAV, along with a discussion of their key properties. Section 3 presents the formulation of the proposed neutrosophic estimators of population variance and explores their properties, including the conditions under which they outperform the adapted neutrosophic estimators. Section 4 provides the comparative study of the proposed estimators with the adapted estimators. Section 5 details a simulation study conducted using a hypothetically generated population. In Section 6, both the proposed and adapted neutrosophic estimators are applied over the real-world temperature data to evaluate their practical performance. Finally, Section 7 summarizes the main findings and concludes the article.

2. **Adapted neutrosophic estimators**

This section focuses on adapting the classical estimators into neutrosophic estimators for estimating the population variance in SRS, using information from two auxiliary variables.

If the auxiliary variable is not available then the usual neutrosophic unbiased estimator is the only choice to estimate the population variance $S_{y_N}^2$ which is defined as

$$\kappa_m = s_{y_N}^2$$

Theorem 2.1. *The estimator κ_m has the variance as*

$$V(\kappa_m) = S_{y_N}^4 \Delta_0.$$

Proof. The estimator κ_m is given as

$$\kappa_m = s_{y_N}^2.$$

Employing notations of (1), the estimator κ_m can be presented as

$$\kappa_m = S_{y_N}^2(1 + \epsilon_{0N})$$

$$\kappa_m - S_{y_N}^2 = S_{y_N}^2 \epsilon_{0N} \tag{2}$$

Applying the expectation operator to both sides of (2), results in:

$$Bias(\kappa_m) = 0$$

This indicates that the estimator κ_m is unbiased. Again, applying square and expectation operator to both sides of (2), results in:

$$V(\kappa_m) = S_{y_N}^4 \Delta_0$$

□

The generalized neutrosophic ratio estimator of population variance $S_{y_N}^2$ under SRS, incorporating TAV, is expressed as

$$\kappa_{gr} = s_{y_N}^2 \left(\frac{a_N S_{x_N}^2 + b_N}{a_N s_{x_N}^2 + b_N} \right) \left(\frac{c_N S_{z_N}^2 + d_N}{c_N s_{z_N}^2 + d_N} \right).$$

Here, a_N, b_N, c_N, and d_N represent either real constants or known parameters determined from the neutrosophic auxiliary variables x_N and z_N. These parameters may include the neutrosophic mean, standard deviation, correlation coefficient, coefficient of variation, skewness, kurtosis, and other related measures. Some particular forms of the generalized neutrosophic ratio estimator κ_{gr}, designed using TAV, are presented in Table 1 for illustrative purposes and ease of reference.

Table 1. Some particular forms of the generalized neutrosophic ratio estimator κ_{gr} based on TAV

a_N	b_N	c_N	d_N	TAV based estimators
1	0	1	0	$\kappa_{gr}^1 = s_{y_N}^2 \left(\frac{S_{x_N}^2}{s_{x_N}^2} \right) \left(\frac{S_{z_N}^2}{s_{z_N}^2} \right)$
1	C_{x_N}	1	C_{z_N}	$\kappa_{gr}^2 = s_{y_N}^2 \left(\frac{S_{x_N}^2 + C_{x_N}}{s_{x_N}^2 + C_{x_N}} \right) \left(\frac{S_{z_N}^2 + C_{z_N}}{s_{z_N}^2 + C_{z_N}} \right)$
C_{x_N}	$\beta_2(x_N)$	C_{z_N}	$\beta_2(z_N)$	$\kappa_{gr}^3 = s_{y_N}^2 \left(\frac{C_{x_N} S_{x_N}^2 + \beta_2(x_N)}{C_{x_N} s_{x_N}^2 + \beta_2(x_N)} \right) \left(\frac{C_{z_N} S_{z_N}^2 + \beta_2(z_N)}{C_{z_N} s_{z_N}^2 + \beta_2(z_N)} \right)$
$\beta_2(x_N)$	C_{x_N}	$\beta_2(z_N)$	C_{z_N}	$\kappa_{gr}^4 = s_{y_N}^2 \left(\frac{\beta_2(x_N) S_{x_N}^2 + C_{x_N}}{\beta_2(x_N) s_{x_N}^2 + C_{x_N}} \right) \left(\frac{\beta_2(z_N) S_{z_N}^2 + C_{z_N}}{\beta_2(z_N) s_{z_N}^2 + C_{z_N}} \right)$
1	ρ_{xy_N}	1	ρ_{yz_N}	$\kappa_{gr}^5 = s_{y_N}^2 \left(\frac{S_{x_N}^2 + \rho_{xy_N}}{s_{x_N}^2 + \rho_{xy_N}} \right) \left(\frac{S_{z_N}^2 + \rho_{yz_N}}{s_{z_N}^2 + \rho_{yz_N}} \right)$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$\kappa_{gr}^6 = s_{y_N}^2 \left(\frac{S_{x_N}^2 + \beta_2(x_N)}{s_{x_N}^2 + \beta_2(x_N)} \right) \left(\frac{S_{z_N}^2 + \beta_2(z_N)}{s_{z_N}^2 + \beta_2(z_N)} \right)$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$\kappa_{gr}^7 = s_{y_N}^2 \left(\frac{S_{x_N}^2 + \beta_1(x_N)}{s_{x_N}^2 + \beta_1(x_N)} \right) \left(\frac{S_{z_N}^2 + \beta_1(z_N)}{s_{z_N}^2 + \beta_1(z_N)} \right)$

Theorem 2.2. *The bias and MSE of the generalized neutrosophic ratio estimator κ_{gr} utilizing TAV are given as*

$$Bias(\kappa_{gr}) = S_{y_N}^2 \left(\Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - \Psi_N \Delta_{01} - \Pi_N \Delta_{02} + \Psi_N \Pi_N \Delta_{12} \right)$$

$$MSE(\kappa_{gr}) = S_{y_N}^4 \left(\Delta_0 + \Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - 2\Psi_N \Delta_{01} - 2\Pi_N \Delta_{02} + 2\Psi_N \Pi_N \Delta_{12} \right)$$

where

$$\Psi_N = \left(\frac{a_N S_{x_N}^2}{a_N S_{x_N}^2 + b_N} \right) \text{ and } \Pi_N = \left(\frac{c_N S_{z_N}^2}{c_N S_{z_N}^2 + d_N} \right)$$

Proof. The estimator κ_{gr} is given as

$$\kappa_{gr} = s^2_{y_N}\left(\frac{a_N S^2_{x_N} + b_N}{a_N s^2_{x_N} + b_N}\right)\left(\frac{c_N S^2_{z_N} + d_N}{c_N s^2_{z_N} + d_N}\right).$$

Employing notations of (1), the estimator κ_{gr} can be presented as

$$\begin{aligned}\kappa_{gr} =& S^2_{y_N}(1+\epsilon_{0N})\left(\frac{a_N S^2_{x_N} + b_N}{a_N S^2_{x_N}(1+\epsilon_{1N}) + b_N}\right)\left(\frac{c_N S^2_{z_N} + d_N}{c_N S^2_{z_N}(1+\epsilon_{2N}) + d_N}\right)\\ =& S^2_{y_N}(1+\epsilon_{0N})\left(\frac{a_N S^2_{x_N} + b_N}{a_N S^2_{x_N} + a_N S^2_{x_N}\epsilon_{1N} + b_N}\right)\left(\frac{c_N S^2_{z_N} + d_N}{c_N S^2_{z_N} + c_N S^2_{z_N}\epsilon_{2N} + d_N}\right)\\ =& S^2_{y_N}(1+\epsilon_{0N})(1+\Psi_N\epsilon_{1N})^{-1}(1+\Pi_N\epsilon_{2N})^{-1}\end{aligned}$$

Applying a Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_{gr} = S^2_{y_N}\left(\begin{array}{l} 1+\epsilon_{0N} - \Psi_N\epsilon_{1N} - \Pi_N\epsilon_{2N} + \Psi^2_N\epsilon^2_{1N} + \Pi^2_N\epsilon^2_{2N} - \Psi_N\epsilon_{0N}\epsilon_{1N} - \Pi_N\epsilon_{0N}\epsilon_{2N}\\ +\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N}\end{array}\right)$$

Subtracting $S^2_{y_N}$ both side to the above expression, results in:

$$\kappa_{gr} - S^2_{y_N} = S^2_{y_N}\left(\begin{array}{l} \epsilon_{0N} - \Psi_N\epsilon_{1N} - \Pi_N\epsilon_{2N} + \Psi^2_N\epsilon^2_{1N} + \Pi^2_N\epsilon^2_{2N} - \Psi_N\epsilon_{0N}\epsilon_{1N}\\ -\Pi_N\epsilon_{0N}\epsilon_{2N} + \Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N}\end{array}\right) \tag{3}$$

Applying expectation operator both sides of (3), results in:

$$Bias(\kappa_{gr}) = S^2_{y_N}\left(\Psi^2_N\Delta_1 + \Pi^2_N\Delta_2 - \Psi_N\Delta_{01} - \Pi_N\Delta_{02} + \Psi_N\Pi_N\Delta_{12}\right)$$

Applying square and expectation both sides of (3), results in:

$$MSE(\kappa_{gr}) = S^4_{y_N}\left(\Delta_0 + \Psi^2_N\Delta_1 + \Pi^2_N\Delta_2 - 2\Psi_N\Delta_{01} - 2\Pi_N\Delta_{02} + 2\Psi_N\Pi_N\Delta_{12}\right)$$

□

The generalized neutrosophic power ratio estimator κ_{pr} for the population variance employing TAV under SRS, is expressed as

$$\kappa_{pr} = s^2_{y_N}\left(\frac{a_N S^2_{x_N} + b_N}{a_N s^2_{x_N} + b_N}\right)^{\theta_{1N}}\left(\frac{c_N S^2_{z_N} + d_N}{c_N s^2_{z_N} + d_N}\right)^{\theta_{2N}}$$

where θ_{1N} and θ_{2N} are suitably chosen scalars. Some particular forms of the estimator κ_{pr} are presented in Table 2 for illustrative purposes and ease of reference.

TABLE 2. Some particular forms of the generalized neutrosophic power ratio estimator t_{pr} based on TAV

a_N	b_N	c_N	d_N	TAV based estimators
1	0	1	0	$\kappa_{pr}^1 = s_{y_N}^2 \left(\frac{S_{x_N}^2}{s_{x_N}^2}\right)^{\theta_{1N}} \left(\frac{S_{z_N}^2}{s_{z_N}^2}\right)^{\theta_{2N}}$
1	C_{x_N}	1	C_{z_N}	$\kappa_{pr}^2 = s_{y_N}^2 \left(\frac{S_{x_N}^2+C_{x_N}}{s_{x_N}^2+C_{x_N}}\right)^{\theta_{1N}} \left(\frac{S_{z_N}^2+C_{z_N}}{s_{z_N}^2+C_{z_N}}\right)^{\theta_{2N}}$
C_{x_N}	$\beta_2(x_N)$	C_{z_N}	$\beta_2(z_N)$	$\kappa_{pr}^3 = s_{y_N}^2 \left(\frac{C_{x_N}S_{x_N}^2+\beta_2(x_N)}{C_{x_N}s_{x_N}^2+\beta_2(x_N)}\right)^{\theta_{1N}} \left(\frac{C_{z_N}S_{z_N}^2+\beta_2(z_N)}{C_{z_N}s_{z_N}^2+\beta_2(z_N)}\right)^{\theta_{2N}}$
$\beta_2(x_N)$	C_{x_N}	$\beta_2(z_N)$	C_{z_N}	$\kappa_{pr}^4 = s_{y_N}^2 \left(\frac{\beta_2(x_N)S_{x_N}^2+C_{x_N}}{\beta_2(x_N)s_{x_N}^2+C_{x_N}}\right)^{\theta_{1N}} \left(\frac{\beta_2(z_N)S_{z_N}^2+C_{z_N}}{\beta_2(z_N)s_{z_N}^2+C_{z_N}}\right)^{\theta_{2N}}$
1	ρ_{xy_N}	1	ρ_{yz_N}	$\kappa_{pr}^5 = s_{y_N}^2 \left(\frac{S_{x_N}^2+\rho_{xy_N}}{s_{x_N}^2+\rho_{xy_N}}\right)^{\theta_{1N}} \left(\frac{S_{z_N}^2+\rho_{yz_N}}{s_{z_N}^2+\rho_{yz_N}}\right)^{\theta_{2N}}$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$\kappa_{pr}^6 = s_{y_N}^2 \left(\frac{S_{x_N}^2+\beta_2(x_N)}{s_{x_N}^2+\beta_2(x_N)}\right)^{\theta_{1N}} \left(\frac{S_{z_N}^2+\beta_2(z_N)}{s_{z_N}^2+\beta_2(z_N)}\right)^{\theta_{2N}}$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$\kappa_{pr}^7 = s_{y_N}^2 \left(\frac{S_{x_N}^2+\beta_1(x_N)}{s_{x_N}^2+\beta_1(x_N)}\right)^{\theta_{1N}} \left(\frac{S_{z_N}^2+\beta_1(z_N)}{s_{z_N}^2+\beta_1(z_N)}\right)^{\theta_{2N}}$

Theorem 2.3. *The bias, MSE, and minimum MSE of the neutrosophic generalized power ratio estimator κ_{pr} are given by*

$$\begin{aligned}
Bias(\kappa_{pr}) &= S_{y_N}^2 \begin{pmatrix} \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2 - \theta_{1N}\Psi_N\Delta_{01} \\ -\theta_{2N}\Pi_N\Delta_{02} + \Psi_N\Pi_N\theta_{1N}\theta_{2N}\Delta_{12} \end{pmatrix} \\
MSE(\kappa_{pr}) &= S_{y_N}^4 \begin{pmatrix} \Delta_0 + \theta_{1N}^2\Psi_N^2\Delta_1 + \theta_{2N}^2\Pi_N^2\Delta_2 - 2\theta_{1N}\Psi_N\Delta_{01} \\ -2\theta_{2N}\Pi_N\Delta_{02} + 2\theta_{1N}\theta_{2N}\Psi_N\Pi_N\Delta_{12} \end{pmatrix} \\
min.MSE(\kappa_{pr}) &= S_{y_N}^4 \left(\Delta_0 - \frac{\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2 - \Delta_{12}^2}\right)
\end{aligned}$$

Proof. The estimator κ_{pr} is given as

$$\kappa_{pr} = s_{y_N}^2 \left(\frac{a_N S_{x_N}^2 + b_N}{a_N s_{x_N}^2 + b_N}\right)^{\theta_{1N}} \left(\frac{c_N S_{z_N}^2 + d_N}{c_N s_{z_N}^2 + d_N}\right)^{\theta_{2N}}$$

Employing notations of (1), the estimator κ_{pr} can be presented as

$$\begin{aligned}
\kappa_{pr} &= S_{y_N}^2(1+\epsilon_{0N}) \left(\frac{a_N S_{x_N}^2 + b_N}{a_N S_{x_N}^2(1+\epsilon_{1N}) + b_N}\right)^{\theta_{1N}} \left(\frac{c_N S_{z_N}^2 + d_N}{c_N S_{z_N}^2(1+\epsilon_{2N}) + d_N}\right)^{\theta_{2N}} \\
&= S_{y_N}^2(1+\epsilon_{0N})(1+\Psi_N\epsilon_{1N})^{-\theta_{1N}}(1+\Pi_N\epsilon_{2N})^{-\theta_{2N}}
\end{aligned}$$

Applying Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_{pr} = S^2_{y_N}\begin{pmatrix} 1+\epsilon_{0N}-\theta_{1N}\Psi_N\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{2N}+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{2N}+\theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N}\end{pmatrix} \tag{4}$$

Subtracting $S^2_{y_N}$ both sides of (4), results in:

$$\kappa_{pr}-S^2_{y_N} = S^2_{y_N}\begin{pmatrix} \epsilon_{0N}-\theta_{1N}\Psi_N\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{2N}+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{2N}+\theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N}\end{pmatrix} \tag{5}$$

Applying expectation operator both sides of (5), results in:

$$Bias(\kappa_{pr}) = S^2_{y_N}\begin{pmatrix} \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2-\theta_{1N}\Psi_N\Delta_{01}-\theta_{2N}\Pi_N\Delta_{02} \\ +\Psi_N\Pi_N\theta_{1N}\theta_{2N}\Delta_{12}\end{pmatrix}$$

Applying square and expectation both sides of (5), results in:

$$MSE(\kappa_{pr}) = S^4_{y_N}\begin{pmatrix} \Delta_0+\theta_{1N}^2\Psi_N^2\Delta_1+\theta_{2N}^2\Pi_N^2\Delta_2-2\theta_{1N}\Psi_N\Delta_{01}-2\theta_{2N}\Pi_N\Delta_{02} \\ +2\theta_{1N}\theta_{2N}\Psi_N\Pi_N\Delta_{12}\end{pmatrix} \tag{6}$$

The optimum values of θ_{1N} and θ_{2N} can be determined by minimizing (6) as

$$\theta_{1N} = \frac{\Delta_{01}\Delta_2-\Delta_{02}\Delta_{12}}{\Psi_N(\Delta_1\Delta_2-\Delta_{12}^2)} \text{ and } \theta_{2N} = \frac{\Delta_1\Delta_{02}-\Delta_{01}\Delta_{12}}{\Pi_N(\Delta_1\Delta_2-\Delta_{12}^2)}$$

Using optimum values of θ_{1N} and θ_{2N} in (6), we get minimum MSE of κ_{pr} as

$$min.MSE(\kappa_{pr}) = S^4_{y_N}\left(\Delta_0-\frac{\Delta_2\Delta_{01}^2+\Delta_1\Delta_{02}^2-2\Delta_{01}\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2-\Delta_{12}^2}\right)$$

□

Remark 2.4. Note that the minimum MSE of the estimator κ_{pr} is independent of the values Ψ_N and Π_N which depend on different known parameters of auxiliary variables x_N and z_N. This shows that the MSE of the members $\kappa^i_{pr},\ i=1,2,...,7$ will be same.

To enhance the estimation of finite population variance under SRS, we examined the work of Singh and Gupta (2025) and presented an exponential ratio-cum-product type estimator κ_{ps} by utilizing TAV as

$$\kappa_{ps} = s^2_{y_N}\exp\left(\frac{S^2_{x_N}-\bar{t}_{1N}}{S^2_{x_N}+\bar{t}_{1N}}\right)\exp\left(\frac{\bar{t}_{2N}-S^2_{z_N}}{\bar{t}_{2N}+S^2_{z_N}}\right)$$

In the estimator κ_{ps}, $\bar{t}_{1N}=s^2_{x_N}+\Gamma_N(S^2_{x_N}-s^2_{x_N})$ and $\bar{t}_{2N}=s^2_{z_N}+\Omega_N(S^2_{z_N}-s^2_{z_N})$, where Γ_N and Ω_N are duly chosen scalars.

Theorem 2.5. *The bias and minimum MSE of the exponential ratio-cum-product type estimator κ_{ps}, employing TAV, are presented as*

$$Bias(\kappa_{ps}) = S_{y_N}^2 \left\{ \tfrac{3(1-\Gamma_N)^2\Delta_1}{8} - \tfrac{(1-\Omega_N)^2\Delta_2}{8} - \tfrac{(1-\Gamma_N)\Delta_{01}}{2} + \tfrac{(1-\Omega_N)\Delta_{02}}{2} - \tfrac{(1-\Gamma_N)(1-\Omega_N)\Delta_{12}}{4} \right\}$$

$$min.MSE(\kappa_{ps}) = S_{y_N}^4 \left\{ \begin{array}{l} \Delta_0 + \frac{(1-\Gamma_N^*)^2\Delta_1}{4} + \frac{(1-\Omega_N^*)^2\Delta_2}{4} - (1-\Gamma_N^*)\Delta_{01} + (1-\Omega_N^*)\Delta_{02} \\ -\frac{(1-\Gamma_N^*)(1-\Omega_N^*)\Delta_{12}}{2} \end{array} \right\}$$

where

$$\Gamma_N = \left(\frac{\Delta_1\Delta_2 - 2\Delta_{01}\Delta_2 - \Delta_{12}^2 + 2\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2 - \Delta_{12}^2} \right) \text{ and } \Omega_N = \left(\frac{\Delta_1\Delta_2 - \Delta_{12}^2 - 2\Delta_{01}\Delta_{12} + 2\Delta_{02}\Delta_1}{\Delta_1\Delta_2 - \Delta_{12}^2} \right)$$

Proof. Consider the estimator κ_{ps} as

$$\kappa_{ps} = s_{y_N}^2 \exp\left(\frac{S_{x_N}^2 - \bar{t}_{1N}}{S_{x_N}^2 + \bar{t}_{1N}} \right) \exp\left(\frac{\bar{t}_{2N} - S_{z_N}^2}{\bar{t}_{2N} + S_{z_N}^2} \right)$$

Employing notations of (1), the estimator κ_{ps} can be presented as

$$\begin{aligned} \kappa_{ps} =& S_{y_N}^2(1+\varepsilon_{0N}) \left\{ \begin{array}{l} \exp\left[\frac{S_{x_N}^2 - S_{x_N}^2(1+\varepsilon_{1N}) - \Gamma_N\left(S_{x_N}^2 - S_{x_N}^2(1+\varepsilon_{1N})\right)}{S_{x_N}^2 + S_{x_N}^2(1+\varepsilon_{1N}) + \Gamma_N\left(S_{x_N}^2 - S_{x_N}^2(1+\varepsilon_{1N})\right)} \right] \\ \exp\left[\frac{S_{z_N}^2(1+\varepsilon_{2N}) + \Omega_N\left(S_{z_N}^2 - S_{z_N}^2(1+\varepsilon_{2N})\right) - S_{z_N}^2}{S_{z_N}^2(1+\varepsilon_{2N}) + \Omega_N\left(S_{z_N}^2 - S_{z_N}^2(1+\varepsilon_{2N})\right) + S_{z_N}^2} \right] \end{array} \right\} \\ =& S_{y_N}^2 (1+\varepsilon_{0N}) \exp\left[\frac{-\varepsilon_{1N}(1-\Gamma_N)}{2\left(1 + \frac{\varepsilon_{1N}(1-\Gamma_N)}{2}\right)} \right] \exp\left[\frac{\varepsilon_{2N}(1-\Omega_N)}{2\left(1 + \frac{\varepsilon_{2N}(1-\Omega_N)}{2}\right)} \right] \end{aligned}$$

Applying a Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_{ps} = S_{y_N}^2 \left[\begin{array}{l} 1 + \epsilon_{0N} - \frac{\epsilon_{1N}(1-\Gamma_N)}{2} + \frac{3\epsilon_{1N}^2(1-\Gamma_N)^2}{8} + \frac{\epsilon_{2N}(1-\Omega_N)}{2} - \frac{\epsilon_{1N}\epsilon_{2N}(1-\Gamma_N)(1-\Omega_N)}{4} \\ -\frac{\epsilon_{2N}^2(1-\Omega_N)^2}{8} - \frac{\epsilon_{0N}\epsilon_{1N}(1-\Gamma_N)}{2} + \frac{\epsilon_{0N}\epsilon_{2N}(1-\Omega_N)}{2} \end{array} \right]$$

Subtracting $S_{y_N}^2$ both sides of the above expression, results in:

$$\kappa_{ps} - S_{y_N}^2 = S_{y_N}^2 \left[\begin{array}{l} \epsilon_{0N} - \frac{\epsilon_{1N}(1-\Gamma_N)}{2} + \frac{3\epsilon_{1N}^2(1-\Gamma_N)^2}{8} + \frac{\epsilon_{2N}(1-\Omega_N)}{2} - \frac{\epsilon_{1N}\epsilon_{2N}(1-\Gamma_N)(1-\Omega_N)}{4} \\ -\frac{\epsilon_{2N}^2(1-\Omega_N)^2}{8} - \frac{\epsilon_{0N}\epsilon_{1N}(1-\Gamma_N)}{2} + \frac{\epsilon_{0N}\epsilon_{2N}(1-\Omega_N)}{2} \end{array} \right] \tag{7}$$

Appying expectation operator both sides of (7), results in:

$$Bias(\kappa_{ps}) = S_{y_N}^2 \left[\begin{array}{l} \frac{3(1-\Gamma_N)^2\Delta_1}{8} - \frac{(1-\Omega_N)^2\Delta_2}{8} - \frac{(1-\Gamma_N)\Delta_{01}}{2} + \frac{(1-\Omega_N)\Delta_{02}}{2} \\ -\frac{(1-\Gamma_N)(1-\Omega_N)\Delta_{12}}{4} \end{array} \right]$$

Appying square and expectation both sides of (7), results in:

$$MSE(\kappa_{ps}) = S_{y_N}^4 \left[\begin{array}{l} \Delta_0 + \frac{(1-\Gamma_N)^2\Delta_1}{4} + \frac{(1-\Omega_N)^2\Delta_2}{4} - (1-\Gamma_N)\Delta_{01} + (1-\Omega_N)\Delta_{02} \\ -\frac{(1-\Gamma_N)(1-\Omega_N)\Delta_{12}}{2} \end{array} \right] \tag{8}$$

The optimum values of $\mathit{\Gamma}_N$ and $\mathit{\Omega}_N$ can be determined by minimizing (8) as

$$\mathit{\Gamma}_{N(opt)} = \left(\frac{\Delta_1\Delta_2 - 2\Delta_{01}\Delta_2 - \Delta_{12}^2 + 2\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2 - \Delta_{12}^2}\right) = \mathit{\Gamma}_N^* \text{ (say)}$$

$$\mathit{\Omega}_{N(opt)} = \left(\frac{\Delta_1\Delta_2 - \Delta_{12}^2 - 2\Delta_{01}\Delta_{12} + 2\Delta_{02}\Delta_1}{\Delta_1\Delta_2 - \Delta_{12}^2}\right) = \mathit{\Omega}_N^* \text{ (say)}$$

Using optimum values of $\mathit{\Gamma}_{N(opt)}$ and $\mathit{\Omega}_{N(opt)}$ in (8), we get minimum MSE of κ_{ps} as

$$min.MSE(\kappa_{ps}) = S_{y_N}^4 \left\{ \begin{array}{l} \Delta_0 + \frac{(1-\mathit{\Gamma}_N^*)^2\Delta_1}{4} + \frac{(1-\mathit{\Omega}_N^*)^2\Delta_2}{4} - (1-\mathit{\Gamma}_N^*)\Delta_{01} + (1-\mathit{\Omega}_N^*)\Delta_{02} \\ -\frac{(1-\mathit{\Gamma}_N^*)(1-\mathit{\Omega}_N^*)\Delta_{12}}{2} \end{array} \right\}$$

□

3. Proposed optimal neutrosophic estimators

Ref. [21] proposed a significant contribution to survey sampling by developing an innovative method to improve the estimation of the population mean when the population variance is known. He proposed to pre-multiply a constant in the estimator to reduce the MSE of the estimator. This approach demonstrated that using known population parameters could enhance efficiency of the estimation, laying the groundwork for later developments in auxiliary information-based estimators in sampling theory. Further, in the literature of neutrosophic survey sampling, a limited work is available for the estimation of population mean under uncertainty using different sampling frameworks. However, there is no work conducted under neutrosophic framework for the estimation of population variance in uncertainty, employing TAV. This inspired us to propose a class of neutrosophic estimators for the population variance using TAV under SRS. Therefore, following [21], we propose Searls type generalized neutrosophic power ratio estimator for population variance as follows:

$$\kappa_a = \alpha_N s_{y_N}^2 \left(\frac{a_N S_{x_N}^2 + b_N}{a_N s_{x_N}^2 + b_N}\right)^{\theta_{1N}} \left(\frac{c_N S_{z_N}^2 + d_N}{c_N s_{z_N}^2 + d_N}\right)^{\theta_{2N}}.$$

Here α_N, θ_{1N}, and θ_{2N} are duly opted scalars. Some particular forms of the proposed estimator κ_a are reported in Table 3 for ready reference.

TABLE 3. Some particular forms of the neutrosophic estimator κ_a based on TAV

a_N	b_N	c_N	d_N	Particular forms of estimator κ_a based on TAV
1	0	1	0	$\kappa_a^1 = \alpha_N s^2_{y_N}\left(\frac{S^2_{x_N}}{s^2_{x_N}}\right)^{\theta_{1N}}\left(\frac{S^2_{z_N}}{s^2_{z_N}}\right)^{\theta_{2N}}$
1	C_{x_N}	1	C_{z_N}	$\kappa_a^2 = \alpha_N s^2_{y_N}\left(\frac{S^2_{x_N}+C_{x_N}}{s^2_{x_N}+C_{x_N}}\right)^{\theta_{1N}}\left(\frac{S^2_{z_N}+C_{z_N}}{s^2_{z_N}+C_{z_N}}\right)^{\theta_{2N}}$
C_{x_N}	$\beta_2(x_N)$	C_{z_N}	$\beta_2(z_N)$	$\kappa_a^3 = \alpha_N s^2_{y_N}\left(\frac{C_{x_N}S^2_{x_N}+\beta_2(x_N)}{C_{x_N}s^2_{x_N}+\beta_2(x_N)}\right)^{\theta_{1N}}\left(\frac{C_{z_N}S^2_{z_N}+\beta_2(z_N)}{C_{z_N}s^2_{z_N}+\beta_2(z_N)}\right)^{\theta_{2N}}$
$\beta_2(x_N)$	C_{x_N}	$\beta_2(z_N)$	C_{z_N}	$\kappa_a^4 = \alpha_N s^2_{y_N}\left(\frac{\beta_2(x_N)S^2_{x_N}+C_{x_N}}{\beta_2(x_N)s^2_{x_N}+C_{x_N}}\right)^{\theta_{1N}}\left(\frac{\beta_2(z_N)S^2_{z_N}+C_{z_N}}{\beta_2(z_N)s^2_{z_N}+C_{z_N}}\right)^{\theta_{2N}}$
1	ρ_{xy_N}	1	ρ_{yz_N}	$\kappa_a^5 = \alpha_N s^2_{y_N}\left(\frac{S^2_{x_N}+\rho_{xy_N}}{s^2_{x_N}+\rho_{xy_N}}\right)^{\theta_{1N}}\left(\frac{S^2_{z_N}+\rho_{yz_N}}{s^2_{z_N}+\rho_{yz_N}}\right)^{\theta_{2N}}$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$\kappa_a^6 = \alpha_N s^2_{y_N}\left(\frac{S^2_{x_N}+\beta_2(x_N)}{s^2_{x_N}+\beta_2(x_N)}\right)^{\theta_{1N}}\left(\frac{S^2_{z_N}+\beta_2(z_N)}{s^2_{z_N}+\beta_2(z_N)}\right)^{\theta_{2N}}$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$\kappa_a^7 = \alpha_N s^2_{y_N}\left(\frac{S^2_{x_N}+\beta_1(x_N)}{s^2_{x_N}+\beta_1(x_N)}\right)^{\theta_{1N}}\left(\frac{S^2_{z_N}+\beta_1(z_N)}{s^2_{z_N}+\beta_1(z_N)}\right)^{\theta_{2N}}$

Theorem 3.1. *The bias, MSE, and minimum MSE of the proposed neutrosophic variance estimator κ_a are derived as*

$$\begin{aligned}
Bias(\kappa_a) =& S^2_{y_N}\left[\alpha_N\left\{\begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2-\theta_{1N}\Psi_N\Delta_{01} \\ -\theta_{2N}\Pi_N\Delta_{02}+\theta_{1N}\theta_{2N}\Pi_N\Psi_N\Delta_{12}\end{array}\right\}-1\right] \\
MSE(\kappa_a) =& S^4_{y_N}\left[\begin{array}{l} 1+\alpha_N^2\left(\begin{array}{l} 1+\Delta_0+2\theta_{1N}^2\Psi_N^2\Delta_1+2\theta_{2N}^2\Pi_N^2\Delta_2+\theta_{1N}\Psi_N^2\Delta_1 \\ +\theta_{2N}\Pi_N^2\Delta_2-4\theta_{1N}\Psi_N\Delta_{01}-4\theta_{2N}\Pi_N\Delta_{02} \\ +4\theta_{1N}\theta_{2N}\Psi_N\Pi_N\Delta_{12}\end{array}\right) \\ -2\alpha_N\left\{\begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2-\theta_{1N}\Psi_N\Delta_{01} \\ -\theta_{2N}\Pi_N\Delta_{02}+\theta_{1N}\theta_{2N}\Pi_N\Psi_N\Delta_{12}\end{array}\right\}\end{array}\right] \\
min.MSE(\kappa_a) =& S^4_{y_N}\left(1-\frac{B_N^2}{A_N}\right)
\end{aligned}$$

Proof. The estimator κ_a is given as

$$\kappa_a = \alpha_N s^2_{y_N}\left(\frac{a_N S^2_{x_N}+b_N}{a_N s^2_{x_N}+b_N}\right)^{\theta_{1N}}\left(\frac{c_N S^2_{z_N}+d_N}{c_N s^2_{z_N}+d_N}\right)^{\theta_{2N}}$$

Employing notations of (1), the estimator κ_a can be presented as

$$\begin{aligned}
\kappa_a =& \alpha_N S^2_{y_N}(1+\epsilon_{0N})\left(\frac{a_N S^2_{x_N}+b_N}{a_N S^2_{x_N}(1+\epsilon_{1N})+b_N}\right)^{\theta_{1N}}\left(\frac{c_N S^2_{z_N}+d_N}{c_N S^2_{z_N}(1+\epsilon_{2N})+d_N}\right)^{\theta_{2N}} \\
=& \alpha_N S^2_{y_N}(1+\epsilon_{0N})(1+\Psi_N\epsilon_{1N})^{-\theta_{1N}}(1+\Pi_N\epsilon_{2N})^{-\theta_{2N}}
\end{aligned}$$

Applyin Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_a = \alpha_N S_{y_N}^2 \begin{pmatrix} 1+\epsilon_{0N}-\theta_{1N}\Psi_N\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{2N}+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\Psi_N\theta_{1N}\epsilon_{0N}\epsilon_{1N}-\Pi_N\theta_{2N}\epsilon_{0N}\epsilon_{2N}+\Psi_N\Pi_N\theta_{1N}\theta_{2N}\epsilon_{1N}\epsilon_{2N} \end{pmatrix}$$

Subtracting $S_{y_N}^2$ both sides of the above expression, results in:

$$\kappa_a - S_{y_N}^2 = S_{y_N}^2 \left[\alpha_N \begin{pmatrix} 1+\epsilon_{0N}-\theta_{1N}\Psi_N\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{2N}+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2 \\ +\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2-\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N}-\theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{2N} \\ +\theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N} \end{pmatrix} - 1 \right] \tag{9}$$

Appying expectation operator both sides of (9), results in:

$$Bias(\kappa_a) = S_{y_N}^2 \left[\alpha_N \left\{ \begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2-\theta_{1N}\Psi_N\Delta_{01} \\ -\theta_{2N}\Pi_N\Delta_{02}+\theta_{1N}\theta_{2N}\Pi_N\Psi_N\Delta_{12} \end{array} \right\} - 1 \right]$$

Appying square and expectation operator both sides of (9), results in:

$$\begin{aligned} MSE(\kappa_a) &= S_{y_N}^4 \left[\begin{array}{l} 1+\alpha_N^2 \begin{pmatrix} 1+\Delta_0+2\theta_{1N}^2\Psi_N^2\Delta_1+2\theta_{2N}^2\Pi_N^2\Delta_2+\theta_{1N}\Psi_N^2\Delta_1 \\ +\theta_{2N}\Pi_N^2\Delta_2-4\theta_{1N}\Psi_N\Delta_{01}-4\theta_{2N}\Pi_N\Delta_{02} \\ +4\theta_{1N}\theta_{2N}\Psi_N\Pi_N\Delta_{12} \end{pmatrix} \\ -2\alpha_N \left\{ \begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2-\theta_{1N}\Psi_N\Delta_{01} \\ -\theta_{2N}\Pi_N\Delta_{02}+\theta_{1N}\theta_{2N}\Pi_N\Psi_N\Delta_{12} \end{array} \right\} \end{array} \right] \\ &= S_{y_N}^4(1+\alpha_N^2A_N-2\alpha_NB_N) \end{aligned} \tag{10}$$

where,

$$A_N = \begin{pmatrix} 1+\Delta_0+2\theta_{1N}^2\Psi_N^2\Delta_1+2\theta_{2N}^2\Pi_N^2\Delta_2+\theta_{1N}\Psi_N^2\Delta_1+\theta_{2N}\Pi_N^2\Delta_2-4\theta_{1N}\Psi_N\Delta_{01} \\ -4\theta_{2N}\Pi_N\Delta_{02}+4\theta_{1N}\theta_{2N}\Psi_N\Pi_N\Delta_{12} \end{pmatrix}$$

$$B_N = \left\{ \begin{array}{l} 1+\frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1+\frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2-\theta_{1N}\Psi_N\Delta_{01}-\theta_{2N}\Pi_N\Delta_{02} \\ +\theta_{1N}\theta_{2N}\Pi_N\Psi_N\Delta_{12} \end{array} \right\}$$

The minimization of (10) regarding α_N, provides the optimum value of scalar as follows:

$$\alpha_{N(opt)} = \frac{B_N}{A_N}$$

Use of $\alpha_{N(opt)}$ in (10), provides:

$$min.MSE(\kappa_a) = S_{y_N}^4\left(1-\frac{B_N^2}{A_N}\right)$$

It is crucial to note that simultaneous optimization of α_N, θ_{1N}, and θ_{2N} is not feasible. Therefore, to provide the minimization of the MSE of the estimator κ_a, we fix $\alpha_N = 1$. Under this condition, the optimal values of θ_{1N} and θ_{2N} can be derived and given as follows:

$$\theta_{1N} = \frac{\Delta_{01}\Delta_2-\Delta_{02}\Delta_{12}}{\Psi_N(\Delta_1\Delta_2-\Delta_{12}^2)} \text{ and } \theta_{2N} = \frac{\Delta_1\Delta_{02}-\Delta_{01}\Delta_{12}}{\Pi_N(\Delta_1\Delta_2-\Delta_{12}^2)}.$$

□

4. Comparative study

In this section, a comparative study is carried out to assess the performance of the proposed estimator against the existing ones in terms of MSE. The conditions are obtained under which the proposed estimator consistently outperforms the adapted estimators.

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic sample variance $s^2_{y_N}$ in terms of MSE/variance.

$$\begin{aligned} min.MSE(\kappa_a) &< V(\kappa_m) \\ S^4_{y_N}\left(1-\frac{B_N^2}{A_N}\right) &< S^4_{y_N}\Delta_0 \\ 1-\frac{B_N^2}{A_N} &< \Delta_0 \\ \frac{B_N^2}{A_N} &> 1-\Delta_0 \end{aligned}$$

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic generalized ratio estimator κ_{gr} in terms of MSE.

$$\begin{aligned} &min.MSE(\kappa_a) < MSE(\kappa_{gr}) \\ &S^4_{y_N}\left(1-\frac{B_N^2}{A_N}\right) < S^4_{y_N}\left(\ \Delta_0+\Psi_N^2\Delta_1+\Pi_N^2\Delta_2-2\Psi_N\Delta_{01}-2\Pi_N\Delta_{02}+2\Psi_N\Pi_N\Delta_{12}\ \right) \\ &\qquad\quad \frac{B_N^2}{A_N} > 1-\left(\ \Delta_0+\Psi_N^2\Delta_1+\Pi_N^2\Delta_2-2\Psi_N\Delta_{01}-2\Pi_N\Delta_{02}+2\Psi_N\Pi_N\Delta_{12}\ \right) \end{aligned}$$

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic generalized power ratio estimator κ_{pr} in terms of MSE.

$$\begin{aligned} min.MSE(\kappa_a) &< min.MSE(\kappa_{pr}) \\ S^4_{y_N}\left(1-\frac{B_N^2}{A_N}\right) &< S^4_{y_N}\left[\Delta_0-\frac{\Delta_2\Delta_{01}^2+\Delta_1\Delta_{02}^2-2\Delta_{01}\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2-\Delta_{12}^2}\right] \\ \frac{B_N^2}{A_N} &> 1-\Delta_0+\frac{\Delta_2\Delta_{01}^2+\Delta_1\Delta_{02}^2-2\Delta_{01}\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2-\Delta_{12}^2} \end{aligned}$$

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic ratio-cum-product-exponential type estimator κ_{ps} in terms of MSE.

$$\begin{aligned} &min.MSE(\kappa_a) < MSE(\kappa_{ps}) \\ &S^4_{y_N}\left(1-\frac{B_N^2}{A_N}\right) < S^4_{y_N}\left\{\begin{array}{l} \Delta_0+\frac{(1-\Gamma_N^*)^2\Delta_1}{4}+\frac{(1-\Omega_N^*)^2\Delta_2}{4}-(1-\Gamma_N^*)\Delta_{01}+(1-\Omega_N^*)\Delta_{02} \\ -\frac{(1-\Gamma_N^*)(1-\Omega_N^*)\Delta_{12}}{2} \end{array}\right\} \\ &\qquad\quad \frac{B_N^2}{A_N} > 1-\left\{\begin{array}{l} \Delta_0+\frac{(1-\Gamma_N^*)^2\Delta_1}{4}+\frac{(1-\Omega_N^*)^2\Delta_2}{4}-(1-\Gamma_N^*)\Delta_{01}+(1-\Omega_N^*)\Delta_{02} \\ -\frac{(1-\Gamma_N^*)(1-\Omega_N^*)\Delta_{12}}{2} \end{array}\right\} \end{aligned}$$

5. Simulation study

To assess the performance of the adapted and newly proposed neutrosophic estimators, we conduct a simulation study on an artificially generated population. Employing R software, a trivariate normal population with the size of N=500, is created using the set parameters $\mu_{y_N} \sim [11,\ 12]$, $\mu_{x_N} \sim [16,\ 17]$, $\mu_{z_N} \sim [21,\ 22]$, $\sigma^2_{y_N} \sim [25,\ 26]$, $\sigma^2_{x_N} \sim [26,\ 28]$, $\sigma^2_{z_N} \sim [27,\ 30]$, and different values of correlation coefficients ρ_{xy_N}, ρ_{yz_N}, and ρ_{xz_N}. Based on 15,000 repeatition, the bias, MSE, and RE are computed using the following formulas:

$$Bias(\kappa^*) = \frac{1}{15,000}\sum_{i=1}^{15,000}(\kappa_i^* - S^2_{y_N}),\ \text{where } \kappa^* = \kappa_m,\ \kappa_{gr},\ \kappa_{pr}, \kappa_{ps},\ \text{and } \kappa_a \tag{11}$$

$$MSE(\kappa^*) = \frac{1}{15,000}\sum_{i=1}^{15,000}(\kappa_i^* - S^2_{y_N})^2 \tag{12}$$

$$RE(\kappa_m,\ \kappa^*) = \frac{V(\kappa_m)}{MSE(\kappa^*)} \tag{13}$$

The bias, MSE, and RE are calculated and presented in Table 4, Table 5, and Table 6, respectively. After carefully analyzing these findings, we draw a comprehensive interpretation.

Table 4 shows the bias intervals $[\text{Bias}_L, \text{Bias}_U]$ for each estimator under different correlation setups $((\rho_{xy_N}, \rho_{yz_N}, \rho_{xz_N})$, which shows that:

- The usual sample variance estimator $\kappa_m = s^2_{y_N}$ has zero bias, as expected for an unbiased estimator.
- The adapted generalized ratio estimators κ_{gr} demonstrate high positive biases ranging from approximately 24 to over 47. This makes these estimators less desirable.
- The generalized power ratio estimator κ_{pr} demonstrates small positive biases.
- The ratio-cum-product exponential estimator κ_{ps} and the proposed estimators $\kappa_i^a,\ i = 1, 2, ..., 7$ show negligibly small negative biases, approximately between -0.004 and -0.0009. This exhibits that the proposed estimators κ_i^a are slightly negatively biased but far more stable and significantly less biased than the existing ones.

Table 5 shows the MSE intervals $[\text{MSE}_L, \text{MSE}_U]$ for each estimator under different correlation setups $(\rho_{xy_N}, \rho_{yz_N}, \rho_{xz_N})$, which shows that:

- The adapted ratio estimators κ_{gr} have extremely high MSEs, often exceeding 60,000, indicating very poor efficiency.
- The generalized power ratio estimator κ_{pr} and exponential estimator κ_{ps} show much lower MSEs, especially when correlation increases.
- The proposed estimators κ_i^a consistently show the lowest MSEs across all scenarios, with values dropping from 23,900 to as low as 4538. This shows that the proposed

class of estimators κ_i^a outperform all the existing methods in terms of MSE, especially as the auxiliary variables become more strongly correlated with the study variable.

Table 6 shows the RE intervals $[\mathrm{RE}_L, \mathrm{RE}_U]$ for each estimator under different correlation setups $(\rho_{xy_N}, \rho_{yz_N}, \rho_{xz_N})$, which shows that:

- The adapted ratio estimators κ_{gr} have low RE values, indicating worse performance.
- Generalized power ratio estimator κ_{pr} and exponential estimator κ_{ps} are more efficient than κ_m.
- The proposed estimators κ_i^a consistently achieve the highest RE values, reaching up to 4.12, meaning they are over 4 times more efficient than the basic estimator. This offers the substantial improvement in efficiency of the proposed estimators, particularly in high-correlation setups.

TABLE 4. $Bias_N \in [Bias_L,\ Bias_U]$ of neutrosophic estimators using neutrosophic normal population

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
κ_m	(0, 0)	(0, 0)	(0, 0)	(0, 0)
κ_{gr}^1	(41.715, 47.556)	(37.173, 42.890)	(30.903, 35.840)	(24.150, 28.027)
κ_{gr}^2	(41.398, 47.225)	(36.861, 42.983)	(30.638, 35.571)	(23.846, 27.685)
κ_{gr}^3	(43.142, 49.529)	(35.604, 41.187)	(23.656, 27.460)	(7.960, 9.246)
κ_{gr}^4	(42.399, 63.648)	(40.637, 44.443)	(42.219, 37.792)	(44.918, 36.505)
κ_{gr}^5	(41.699, 47.542)	(37.147, 42.867)	(30.866, 35.806)	(24.099, 27.979)
κ_{gr}^6	(41.750, 47.591)	(37.208, 42.926)	(30.939, 35.876)	(24.185, 28.061)
κ_{gr}^7	(41.717, 47.557)	(37.172, 42.890)	(30.901, 35.838)	(24.147, 28.024)
κ_{pr}	(1.395, 1.738)	(2.947, 3.401)	(3.156, 3.641)	(1.209, 1.407)
κ_{ps}	(-0.000, 0.000)	(-0.000, -0.000)	(-0.000, -0.000)	(-0.001, -0.001)
κ_a^1	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^2	(-0.004, 0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^3	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^4	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^5	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^6	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^7	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)

6. **Real data applications**

In climate analysis, daily temperature data provide exact numbers for the highest, lowest, and average temperatures for each observation period. But the elements that cause temperature changes are sometimes hard to pin down. There are several things that might cause

TABLE 5. $MSE_N \in [MSE_L,\ MSE_U]$ of neutrosophic estimators using simulated data

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
κ_m	(21307.320, 27891.470)	(21192.980, 28049.700)	(20414.710, 27200.470)	(18625.230, 24955.050)
κ_{gr}^1	(47683.520, 61986.730)	(41024.880, 54279.600)	(31392.520, 41912.320)	(20180.450, 27080.550)
κ_{gr}^2	(47462.930, 61723.450)	(40815.430, 54448.640)	(31219.840, 41721.940)	(19980.060, 26817.980)
κ_{gr}^3	(47698.680, 62002.16)	(41040.160, 54295.200)	(31407.710, 41927.940)	(20194.310, 27094.95)
κ_{gr}^4	(51782.280, 113630.400)	(53302.470, 144726.400)	(59381.850, 68545.530)	(37659.860, 218591.700)
κ_{gr}^5	(47656.080, 61955.650)	(40979.120, 54226.230)	(31329.910, 41838.610)	(20100.410, 26986.440)
κ_{gr}^6	(47708.300, 62014.920)	(41050.280, 54308.440)	(31418.610, 41942.190)	(20205.300, 27109.340)
κ_{gr}^7	(47683.650, 61986.280)	(41024.070, 54278.120)	(31391.130, 41910.410)	(20178.280, 27077.990)
κ_{pr}	(19297.470, 25040.840)	(17153.890, 22517.050)	(12270.470, 16238.470)	(4609.050, 6145.796)
κ_{ps}	(19297.470, 25040.840)	(17153.890, 22517.050)	(12270.470, 16238.470)	(4609.050, 6145.796)
κ_a^1	(18410.260, 23908.170)	(16365.530, 21498.700)	(11811.570, 15638.000)	(4538.730, 6052.475)
κ_a^2	(18410.790, 23908.990)	(16366.630, 21499.210)	(11812.850, 15639.170)	(4538.926, 6053.161)
κ_a^3	(18410.620, 23908.590)	(16366.200, 21499.450)	(11812.850, 15638.880)	(4539.137, 6052.927)
κ_a^4	(18410.480, 23908.470)	(16365.900, 21499.100)	(11811.810, 15638.47 0)	(4538.939, 6052.730)
κ_a^5	(18410.360, 23908.290)	(16365.760, 21498.970)	(11811.930, 15638.420)	(4538.963, 6052.744)
κ_a^6	(18410.930, 23909.030)	(16366.790, 21500.190)	(11813.050, 15639.730)	(4539.476, 6053.340)
κ_a^7	(18410.230, 23908.120)	(16365.490, 21498.650)	(11811.55, 15637.980)	(4538.732, 6052.476)

TABLE 6. $RE_N \in [RE_L,\ RE_U]$ of neutrosophic estimators using simulated data

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
κ_m	(1.000, 1.000)	(1.000, 1.000)	(1.000, 1.000)	(1.000, 1.000)
κ_{gr}^1	(0.446, 0.449)	(0.516, 0.516)	(0.650, 0.648)	(0.922, 0.921)
κ_{gr}^2	(0.448, 0.451)	(0.519, 0.515)	(0.653, 0.651)	(0.932, 0.930)
κ_{gr}^3	(0.446, 0.459)	(0.516, 0.516)	(0.649, 0.648)	(0.922, 0.921)
κ_{gr}^4	(0.411, 0.245)	(0.397, 0.193)	(0.343, 0.396)	(0.494, 0.114)
κ_{gr}^5	(0.447, 0.450)	(0.517, 0.517)	(0.651, 0.650)	(0.926, 0.924)
κ_{gr}^6	(0.446, 0.449)	(0.516, 0.516)	(0.649, 0.648)	(0.921, 0.920)
κ_{gr}^7	(0.446, 0.449)	(0.516, 0.516)	(0.650, 0.649)	(0.923, 0.921)
κ_{pr}	(1.104, 1.113)	(1.235, 1.245)	(1.663, 1.675)	(4.041, 4.060)
κ_{ps}	(1.104, 1.113)	(1.235, 1.245)	(1.663, 1.675)	(4.041, 4.060)
κ_a^1	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.123)
κ_a^2	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^3	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^4	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^5	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^6	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.102, 4.122)
κ_a^7	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.123)

temperature patterns to alter, include rapid changes in the weather, disturbances in the atmosphere, geographical factors, and larger climate changes. These factors introduce imprecision and incompleteness into the analytical process, making it challenging to model or forecast temperature behavior with complete accuracy. Neutrosophic estimators provide a strong foundation for capturing the uncertainty and partial truth that come with these kinds of influences. These estimators provide a more nuanced understanding and enhanced prediction of temperature dynamics, particularly in situations where causal information is incomplete or uncertain.

In January 2023, the Indian Meteorological Department (IMD) published information based on the seasonal and annual minimum/maximum temperature series from 1901 to 2021. The publicly accessible website `https://www.data.gov.in/resource/seasonal-and-\annual-minimum-maximum-temperatu\re-series-period-1901-2021` offers this data for evaluation. In this data, the neutrosophic study variable y_N is the temperature for the months of October to December in 1901-2021, and the neutrosophic auxiliary variables x_N and z_N are the temperatures for the months of June-September and March-May in 1901-2021. Table 7 contains the required parameters of this data. The bias, MSE, and RE of the neutrosophic estimators are computed for this data and compiled in Table 8.

The results show that the proposed estimators $\kappa_a^i,\ i = 1, 2, ..., 7$ dominate the existing neutrosophic estimators in terms of least MSE values and highest RE values.

TABLE 7. Descriptive values of the real dataset

Neutrosophic parameters	Neutrosophic values
N	121
n_N	(40, 40)
μ_{y_N}	(17.431, 28.478)
μ_{x_N}	(24.092, 31.684)
μ_{z_N}	(21.747, 33.209)
C_{y_N}	(0.028, 0.019)
C_{x_N}	(0.009, 0.013)
C_{z_N}	(0.019, 0.017)
ρ_{xy_N}	(0.452, 0.694)
ρ_{yz_N}	(0.247, 0.405)
ρ_{xz_N}	(0.427, 0.357)
$\beta_1(x_N)$	(-0.122, 0.074)
$\beta_1(z_N)$	(0.183, -0.163)
$\beta_2(x_N)$	(3.121, 2.997)
$\beta_2(z_N)$	(2.828, 3.136)

TABLE 8. Bias, MSE, and RE of the neutrosophic estimators for real data

Estimators	$(Bias_L,\, Bias_U)$	$(MSE_L,\, MSE_U)$	$(RE_L,\, RE_U)$
κ_m	(0, 0)	(0.003, 0.003)	(1.00, 1.00)
κ_{gr}^1	(0.022, 0.027)	(0.008, 0.010)	(0.378, 0.350)
κ_{gr}^2	(0.016, 0.023)	(0.006, 0.009)	(0.454, 0.389)
κ_{gr}^3	(-0.000, -0.000)	(0.003, 0.003)	(1.000, 1.001)
κ_{gr}^4	(0.021, 0.026)	(0.007, 0.010)	(0.404, 0.363)
κ_{gr}^5	(0.014, 0.017)	(0.003, 0.003)	(0.910, 1.006)
κ_{gr}^6	(-0.000, -0.000)	(0.003, 0.003)	(1.009, 1.079)
κ_{gr}^7	(0.012, 0.071)	(0.006, 0.024)	(0.473, 0.149)
κ_{pr}	(0.001, 0.002)	(0.002, 0.003)	(1.073, 1.193)
κ_{ps}	(0.000, 0.000)	(0.002, 0.003)	(1.073, 1.193)
κ_a^1	(-0.011, -0.009)	(0.002, 0.002)	(1.135, 1.251)
κ_a^2	(-0.011, -0.009)	(0.002, 0.002)	(1.133, 1.248)
κ_a^3	(-0.011, -0.009)	(0.002, 0.003)	(1.120, 1.222)
κ_a^4	(-0.011, -0.009)	(0.002, 0.002)	(1.134, 1.250)
κ_a^5	(-0.011, -0.009)	(0.002, 0.002)	(1.122, 1.229)
κ_a^6	(-0.011, -0.009)	(0.002, 0.003)	(1.120, 1.224)
κ_a^7	(-0.011, -0.009)	(0.002, 0.002)	(1.110, 1.254)

7. Conclusions

In this study, we proposed Searls type generalized neutrosophic power ratio estimators for the population variance by incorporating TAV under SRS. Identifying the inherent uncertainty and indeterminacy often available in real-world data, the proposed estimators utilizes neutrosophic theory to effectively handle the imprecise and uncertain information. The proposed estimators were constructed by using the correlation structure between the study variable and two auxiliary variables, thereby improving the efficiency and robustness of the variance estimation.

The theoretical properties of the proposed estimators, such as bias and MSE, were derived under the neutrosophic setup. A comparative study was conducted by comparing the proposed neutrosophic estimators with the adapted neutrosophic estimators in terms of MSE. The efficiency conditions obtained under comparative study were futher assessed by both simulation study and real data illustrations. The empirical results exhibited that the proposed neutrosophic estimators outperformed the adapted estimators based on TAV, in terms of least MSE and maximum RE under indeterminate and uncertain data.

Conclusively, the theoretical and empirical findings highlighted the potential of employing the TAV within the neutrosophic framework to produce more efficient and reliable estimates of the population variance. The proposed approach provided a valuable extension to the survey sampling and opens new domain for further research in uncertain data environment, including the development of modified neutrosophic estimators and their application in other sampling designs.

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