University of New Mexico

Neutrosophic Selective Separability

Giorgio Nordo[1,*], Lorenzo Affè[1] and Florentin Smarandache[2]
[1]MIFT Department – Mathematical and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, 98166 Sant'Agata, Messina, Italy; Emails: giorgio.nordo@unime.it; lorenzo.affe@studenti.unime.it
[2]Emeritus Professor, Mathematics, Physical and Natural Sciences Division, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; Email: smarand@unm.edu; ORCID: 0000-0002-5560-5926
[*]Correspondence: giorgio.nordo@unime.it; ORCID: 0000-0002-9585-9680

Abstract. We develop a confidence-sensitive neutrosophic framework for selective separability. Single-valued neutrosophic sets are evaluated through an admissible aggregation operator $\Phi(T, 1-I, 1-F)$, so that the conservative minimum is only one possible choice. We first study uncertain dense data over an ordinary topology and prove operator-independent transfer results for M-, H-, R-, and S-separability, while showing that the admissible witnesses remain genuinely confidence-sensitive. We then pass to single-valued neutrosophic topological spaces and distinguish strong intrinsic density, which reaches the full confidence height of every effective open set, from weak intrinsic density, which only requires positive confidence overlap. The weak and strong notions differ for individual neutrosophic sets, although both recover the expected classical space classes on crisp-induced topologies. To keep countable network weight meaningful, we introduce countably calibrated neutrosophic networks based on a dense rational set of point profiles. We also formulate both trace-based and genuinely graded selection principles controlled by single-valued neutrosophic filters on the index set. Finally, we compare the minimum and product confidence aggregators, discuss numerical resolution thresholds for intrinsic selection, and derive consequences for Fréchet–Urysohn and function spaces.

Keywords: Single-valued neutrosophic set; neutrosophic topology; selective separability; M-separability; S-separability; strong neutrosophic density; weak neutrosophic density; neutrosophic network; single-valued neutrosophic filter; Fréchet–Urysohn space.

2020 Mathematics Subject Classification. Primary 54D65, 54A20, 54A05; Secondary 03B52, 03E35.

1. **Introduction**

Selection principles provide a common language for many covering and density properties in General Topology. In the density setting, M-separability asks for finite selections from a sequence of dense sets whose union remains dense, while R- and H-separability impose, respectively, one-point selections and an eventual hitting condition on non-empty open sets; see Scheepers [16], Gruenhage–Sakai [6], and Repovš–Zdomskyy [13–15].

Two recent papers involving Maesano are especially relevant here. Bardyla, Maesano and Zdomskyy [2] study selective separability in Fréchet–Urysohn spaces and their products, obtaining in particular a ZFC example of a countable Hausdorff Fréchet–Urysohn space which is not H-separable. Aurichi, Maesano and Zdomskyy [1] introduce S-separability as a density counterpart of the Scheepers covering property and prove, among other results, that S-separability is equivalent to a finite-power property called pM-separability and that every countable Hausdorff Fréchet–Urysohn space is S-separable.

Neutrosophic sets, originating in Smarandache's theory of neutrosophy [17], represent truth, indeterminacy and falsity as independent components. For applications on an ordinary universe, the single-valued model of Wang, Smarandache, Zhang and Sunderraman [18] is especially useful. Early developments of neutrosophic topology include the contributions of Gallego Lupiáñez [5] and Salama and Alblowi [12]. A computational framework for single-valued neutrosophic sets and mappings was developed in [8] and extended to neutrosophic topologies in [9]. Neutrosophic points and induced neutrosophic topologies were studied by Jafari, Nordo and Thakur [7]. Earlier notions of neutrosophic filters were considered by Salama and Alagamy [11], while single-valued neutrosophic filters were introduced and investigated in [10].

The purpose of this paper is to distinguish two levels of neutrosophication. At the first level the ambient topology is ordinary and only the dense input data are neutrosophic. This layer is deliberately conservative and is useful as a calibration device: it shows exactly which classical properties are recovered and which additional conditions are imposed on their witnesses. At the second level the topology itself is single-valued neutrosophic. Here density is no longer determined by ordinary non-empty intersections; it is measured by confidence overlap with neutrosophic open sets. This second layer produces a genuinely graded topological theory.

A further aim is to remove the dependence on one particular scalarization of neutrosophic information. Instead of fixing the minimum from the outset, we work with an admissible aggregation operator Φ. The principal transfer theorem is invariant under this choice. We also isolate the stronger compatibility condition needed when a confidence product is required.

Our main results can be summarized as follows. We give a self-contained presentation of the single-valued neutrosophic operations, points and topologies used later. We characterize confidence-density by high-confidence cuts and prove an operator-independent transfer principle between confidence-sensitive and classical selective separability. We then exhibit a classical M-selection witness which fails as a neutrosophic witness, showing that the equivalence of classes does not identify the witnesses. In the intrinsic layer we distinguish strong and weak neutrosophic density, explain the exact crisp reductions of both notions, and introduce a resolution parameter that prevents unstable normalization on very low-confidence open sets. We refine the network construction by testing only a fixed countable dense calibration of neutrosophic point profiles, and we supplement the classical index-filter transfer theorem with preliminary trace-based and graded definitions using single-valued neutrosophic filters on ω. Finally, we compare non-equivalent confidence aggregators and recover recent consequences concerning Fréchet–Urysohn and function spaces.

The paper is organized as follows. Section 2 collects the classical and neutrosophic notions needed throughout the paper. Sections 3 and 4 develop the confidence-sensitive theory over ordinary topological spaces, while Section 5 passes to genuinely neutrosophic topological spaces. Sections 6 and 7 are devoted to neutrosophic networks and filter-parametrized selection principles, respectively. Section 8 discusses consequences for Fréchet–Urysohn and function spaces, and Section 9 concludes with some open problems and directions for further research.

2. **Preliminaries**

Throughout, ω denotes the set of non-negative integers. We first recall the classical selective properties and then fix all neutrosophic conventions required in the sequel.

2.1. *Classical selective separability*

Let (X, τ) be a topological space. A subset $D \subseteq X$ is dense if $D \cap U \neq \varnothing$ for every non-empty $U \in \tau$.

Definition 2.1. A topological space X is

(a) *M-separable* if for every sequence $(D_n)_{n\in\omega}$ of dense subsets of X there are finite sets $F_n \subseteq D_n$ such that $\bigcup_{n\in\omega} F_n$ is dense in X;

(b) *R-separable* if for every sequence $(D_n)_{n\in\omega}$ of dense subsets of X there are points $x_n \in D_n$ such that $\{x_n : n \in \omega\}$ is dense in X;

(c) *H-separable* if for every sequence $(D_n)_{n\in\omega}$ of dense subsets of X there are finite sets $F_n \subseteq D_n$ such that every non-empty open set meets all but finitely many F_n;

(d) *S-separable* if for every sequence $(D_n)_{n\in\omega}$ of dense subsets of X there are finite sets $F_n \subseteq D_n$ such that, for every finite family $U_1, \ldots, U_m$ of non-empty open subsets of X, there exists $n \in \omega$ with $F_n \cap U_j \neq \varnothing$ for all $j \leq m$.

The last property is due to Aurichi, Maesano and Zdomskyy [1]. We shall use the elementary implications $H \Rightarrow S \Rightarrow M$ and $R \Rightarrow M$.

2.2. *Single-valued neutrosophic sets*

Definition 2.2. Let X be a non-empty set. A *single-valued neutrosophic set* (SVNS) A on X is a triple $A = (T_A, I_A, F_A)$, where $T_A, I_A, F_A : X \to [0,1]$ are the truth-membership, indeterminacy-membership and falsity-membership functions, respectively.

The truth-membership, indeterminacy-membership, and falsity-membership components are mutually independent. Consequently, no constraint is imposed on their sum, and in general $T_A(x) + I_A(x) + F_A(x)$ need not be equal to 1. We denote by $\mathcal{SVN}(X)$ the family of all SVNSs on X.

There are different but equivalent coordinate conventions in the neutrosophic-topology literature. Because our confidence decreases when indeterminacy increases, we adopt the following *evidence order*, used in smooth single-valued neutrosophic topology and particularly convenient for confidence-based arguments [4].

Definition 2.3. For $A, B \in \mathcal{SVN}(X)$ put $A \preceq_N B$ if, for every $x \in X$, $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$ and $F_A(x) \geq F_B(x)$. Define

$$\mathbf{0}_N = (0,1,1), \qquad \mathbf{1}_N = (1,0,0),$$

where constant triples are understood pointwise. The neutrosophic union and intersection are

$$A \sqcup_N B = (\max\{T_A, T_B\}, \min\{I_A, I_B\}, \min\{F_A, F_B\}),$$
$$A \sqcap_N B = (\min\{T_A, T_B\}, \max\{I_A, I_B\}, \max\{F_A, F_B\}),$$

and the complement is $A^c = (F_A, 1 - I_A, T_A)$.

Arbitrary unions and intersections are obtained by taking the corresponding pointwise suprema and infima. Notice that $\mathbf{0}_N \preceq_N A \preceq_N \mathbf{1}_N$. The convention used in [9, 10] orders the indeterminacy coordinate in the opposite direction and uses the absolute set $(1,1,0)$. The two descriptions are order-isomorphic under the coordinate change $I \mapsto 1 - I$. Thus Definition 2.3 is a confidence-compatible reparametrization rather than a competing theory.

Definition 2.4 (Crisp embedding)**.** For $D \subseteq X$, define its crisp neutrosophic embedding $\widehat{D}$ by $\widehat{D}(x) = \mathbf{1}_N$ for $x \in D$ and $\widehat{D}(x) = \mathbf{0}_N$ for $x \notin D$.

Then $\widehat{D \cup E} = \widehat{D} \sqcup_N \widehat{E}$, $\widehat{D \cap E} = \widehat{D} \sqcap_N \widehat{E}$ and $(\widehat{D})^c = \widehat{X \setminus D}$.

2.3. *Neutrosophic topologies and points*

Definition 2.5. A *single-valued neutrosophic topology* on X is a family $\mathcal{T} \subseteq \mathcal{SVN}(X)$ such that $\mathbf{0}_N, \mathbf{1}_N \in \mathcal{T}$, $\mathcal{T}$ is closed under arbitrary neutrosophic unions and under finite neutrosophic intersections. The pair $(X, \mathcal{T})$ is a *single-valued neutrosophic topological space* (NTS). Members of $\mathcal{T}$ are neutrosophic open sets; their complements are neutrosophic closed sets.

Neutrosophic topological structures have appeared in several formulations; see, among others, [4, 5, 7, 9, 12]. The present formulation is the SVNS lattice version determined by the evidence order fixed in Definition 2.3. Every ordinary topology τ induces the crisp NTS $\widehat{\tau} = \{\widehat{U} : U \in \tau\}$.

Definition 2.6. Let $x \in X$ and $(t, i, f) \in [0, 1]^3 \setminus \{(0, 1, 1)\}$. The *neutrosophic point* $p = x_{t,i,f}$ is the SVNS equal to (t, i, f) at x and to $\mathbf{0}_N$ at all other points. We say that p belongs to A, and write $p \in_N A$, when $p \preceq_N A$.

Neutrosophic points and the topology induced on their collection were studied in [7]. In the evidence convention, $p = x_{t,i,f} \in_N A$ means $t \le T_A(x)$, $i \ge I_A(x)$ and $f \ge F_A(x)$.

2.4. *Confidence aggregators*

The scalar profile used to decide whether a selected point is reliable should not be tied to one aggregation rule.

Definition 2.7. An *admissible confidence aggregator* is a continuous mapping $\Phi : [0, 1]^3 \to [0, 1]$ which is non-decreasing in each variable and satisfies $\Phi(0, 0, 0) = 0$ and $\Phi(1, 1, 1) = 1$. For $A \in \mathcal{SVN}(X)$ define

$$\rho^{\Phi}_A(x) = \Phi\big(T_A(x), 1 - I_A(x), 1 - F_A(x)\big).$$

When no ambiguity is possible, the superscript Φ is omitted.

The conservative choice used most often below is $\Phi_{\min}(a, b, c) = \min\{a, b, c\}$. Other admissible choices include the product abc and weighted arithmetic means with positive weights summing to one. The next condition is relevant only when products of SVNSs are interpreted componentwise.

Definition 2.8. An admissible Φ is *meet-compatible* if for every finite family $u_1, \dots, u_m \in [0, 1]^3$ one has

$$\Phi\Big(\bigwedge_{j=1}^{m} u_j\Big) = \min_{1 \le j \le m} \Phi(u_j),$$

where the meet on the left is coordinatewise minimum.

The minimum aggregator is meet-compatible. Product and weighted arithmetic mean aggregators are generally not. This distinction will let us state precisely which results are operator-independent.

3. **Confidence-sensitive selection over a crisp topology**

In this section (X, τ) is an ordinary topological space and Φ is an arbitrary admissible confidence aggregator.

3.1. *Neutrosophic density*

For $0 \le \alpha < 1$, put $A_\Phi^{>\alpha} = \{x \in X : \rho_A^\Phi(x) > \alpha\}$ and $A_\Phi^+ = A_\Phi^{>0}$.

Definition 3.1. An SVNS A on (X, τ) is Φ-*dense* (or simply N-*dense*) if $\sup_{x\in U} \rho_A^\Phi(x) = 1$ for every non-empty $U \in \tau$.

Lemma 3.2 (Cut characterization)**.** *For $A \in \mathcal{SVN}(X)$, the following are equivalent: A is N-dense; and $A_\Phi^{>\alpha}$ is dense in X for every $\alpha < 1$.*

Proof. If A is N-dense, then for every non-empty open U and every $\alpha < 1$ there is $x \in U$ with $\rho_A^\Phi(x) > \alpha$, hence $U \cap A_\Phi^{>\alpha} \neq \varnothing$. Conversely, if every confidence cut is dense, then every non-empty open U contains points of confidence larger than every $\alpha < 1$, and therefore its confidence supremum is one. □

For the crisp embedding $\widehat{D}$, admissibility gives $\rho_{\widehat{D}}^\Phi = \chi_D$. Hence $\widehat{D}$ is N-dense if and only if D is dense, independently of the chosen aggregator.

3.2. *Neutrosophic M-, R-, H-, and S-separability*

For a finite set $F \subseteq X$ and non-empty open U, define

$$c_\Phi(A, F; U) = \max\big(\{\rho_A^\Phi(x) : x \in F \cap U\} \cup \{0\}\big).$$

Definition 3.3. Let $(A_n)_{n\in\omega}$ range over sequences of N-dense SVNSs on X. The space X is

(a) NM-*separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every non-empty open U, $\sup_n c_\Phi(A_n, F_n; U) = 1$;

(b) NR-*separable* if there are $x_n \in A_{n,\Phi}^+$ such that, for every non-empty open U,
$$\sup\big(\{\rho_{A_n}^\Phi(x_n) : x_n \in U\} \cup \{0\}\big) = 1;$$

(c) NH-*separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that $\lim_{n\to\infty} c_\Phi(A_n, F_n; U) = 1$ for every non-empty open U;

(d) NS-*separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every finite family $U_1, \ldots, U_m$ of non-empty open sets and every $\varepsilon > 0$, some n satisfies $c_\Phi(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \le m$.

Theorem 3.4 (Operator-independent transfer principle)**.** *For every admissible Φ and every topological space X,* NM-*,* NH-*,* NR-*, and* NS-*separability are respectively equivalent to classical M-, H-, R-, and S-separability.*

Proof. We first prove the classical-to-neutrosophic implications. Fix a sequence $(A_n)_{n\in\omega}$ of N-dense SVNSs.

Assume that X is M-separable. Partition ω into pairwise disjoint infinite blocks $(P_k)_{k\in\omega}$ and put $\alpha_k = 1 - 2^{-k}$. For $n \in P_k$, let $D_n = A^{>\alpha_k}_{n,\Phi}$. By Lemma 3.2, every D_n is dense. Applying M-separability to the subsequence indexed by P_k, we obtain finite sets $F_n \subseteq D_n$ $(n \in P_k)$ such that $\bigcup_{n\in P_k} F_n$ is dense. Since the blocks form a partition, these choices define one finite set F_n for every n. Let U be non-empty open and let $\varepsilon > 0$. Choose k with $\alpha_k > 1 - \varepsilon$. Density of $\bigcup_{n\in P_k} F_n$ yields $n \in P_k$ and $x \in F_n \cap U$. Then $c_\Phi(A_n, F_n; U) \geq \rho^\Phi_{A_n}(x) > \alpha_k > 1 - \varepsilon$. As ε is arbitrary, $\sup_n c_\Phi(A_n, F_n; U) = 1$, and hence X is NM-separable.

Assume next that X is H-separable. Put $\alpha_n = 1 - (n + 2)^{-1}$ and set $D_n = A^{>\alpha_n}_{n,\Phi}$. Choose finite $F_n \subseteq D_n$ witnessing H-separability. For every non-empty open U, there is n_0 such that $F_n \cap U \neq \varnothing$ whenever $n \geq n_0$. Hence, for $n \geq n_0$, $c_\Phi(A_n, F_n; U) > \alpha_n$. Since also $c_\Phi(A_n, F_n; U) \leq 1$ and $\alpha_n \to 1$, it follows that $c_\Phi(A_n, F_n; U) \to 1$. Thus X is NH-separable.

If X is R-separable, use again the blocks P_k and the cuts $D_n = A^{>\alpha_k}_{n,\Phi}$ for $n \in P_k$. On each block choose points $x_n \in D_n$ so that $\{x_n : n \in P_k\}$ is dense. Given a non-empty open U and $\varepsilon > 0$, choose k with $\alpha_k > 1 - \varepsilon$. Some $n \in P_k$ satisfies $x_n \in U$, and then $\rho^\Phi_{A_n}(x_n) > 1 - \varepsilon$. Hence the supremum required in the definition of NR-separability is one.

Finally, assume that X is S-separable. On each block P_k, apply S-separability to the dense cuts $D_n = A^{>\alpha_k}_{n,\Phi}$ $(n \in P_k)$, obtaining finite $F_n \subseteq D_n$. Given non-empty open sets $U_1, \dots, U_m$ and $\varepsilon > 0$, choose k with $\alpha_k > 1 - \varepsilon$. The S-selection on P_k provides some $n \in P_k$ such that $F_n \cap U_j \neq \varnothing$ for every $j \leq m$. Consequently $c_\Phi(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$, so X is NS-separable.

For the converse implications, let (D_n) be a sequence of ordinary dense sets and put $A_n = \widehat{D_n}$. Then $A^+_{n,\Phi} = D_n$ and every confidence value involved in Definition 3.3 is either 0 or 1. Thus an NM witness has dense union, an NR witness is a dense set of selected points, and convergence to 1 in the NH condition means that every non-empty open set meets all but finitely many F_n. For NS, taking $\varepsilon = 1/2$ gives exactly the classical simultaneous-hitting condition. Hence the four neutrosophic properties imply their classical counterparts. □

Corollary 3.5. *For every admissible* Φ*,* N$H \Rightarrow$ N$S \Rightarrow$ NM *and* N$R \Rightarrow$ NM*.*

Theorem 3.4 is a calibration result: it says that the first layer does not manufacture a new class of underlying spaces. It does not, however, identify the witnesses. The next example makes this distinction explicit.

Example 3.6 (A classical witness rejected by confidence)**.** Let $X = \mathbb{Q}$ with its usual topology and choose a partition $\mathbb{Q} = D \cup E$ into two disjoint dense sets. Such a partition can be constructed by enumerating a countable base and recursively choosing two new rational points in each basic open set. Fix $\Phi = \Phi_{\min}$ and let $A_n = A$ for all n, where A has confidence 1 on D and confidence $1/2$ on E; for instance set $A(x) = (1, 0, 0)$ on D and $A(x) = (1/2, 0, 0)$ on E. Then A is N-dense because D is dense. Enumerate $E = \{e_n : n \in \omega\}$ and put $F_n = \{e_n\}$. The union $\bigcup_n F_n = E$ is dense, so (F_n) is a perfectly valid classical M-selection witness for the support sequence. Nevertheless, for every non-empty open U one has $\sup_n c_\Phi(A, F_n; U) \leq 1/2$, and hence the same finite selections do not witness NM-separability. The space class is unchanged, but the admissible witness class is strictly refined.

4. Finite powers and the NpM property

Aurichi, Maesano and Zdomskyy prove that classical S-separability is equivalent to pM-separability [1, Proposition 4.2]. We now lift the finite power argument at the level of confidence profiles.

Lemma 4.1 (Tail strengthening)**.** *Let $(A_n)_{n\in\omega}$ be a sequence of N-dense SVNSs.*

(i) If X is NM-separable, there are finite $F_n \subseteq A^+_{n,\Phi}$ such that, for every non-empty open U and every $N \in \omega$,

$$\sup_{n\geq N} c_\Phi(A_n, F_n; U) = 1.$$

(ii) If X is NS-separable, there are finite $F_n \subseteq A^+_{n,\Phi}$ such that, for every finite family $U_1, \ldots, U_m$ of non-empty open sets, every $\varepsilon > 0$, and every $N \in \omega$, some $n \geq N$ satisfies $c_\Phi(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$.

Proof. Partition ω into pairwise disjoint infinite blocks $(P_r)_{r\in\omega}$. On each block, reindex the restricted sequence $(A_n)_{n\in P_r}$ by ω, apply the relevant selective property, and transport the resulting finite selections back to the original indices. Because the blocks are disjoint, this defines a single sequence (F_n).

Fix one of the requirements appearing in (i) or (ii). The selection on every block P_r supplies at least one index in that block satisfying the requirement (with any prescribed confidence tolerance in case (i)). These successful indices are distinct for distinct r; hence there are infinitely many of them and therefore they are unbounded in ω. This proves the tail assertions. □

For an SVNS A and $m \geq 1$, we use the *confidence product*

$$\rho_A^{[m]}(x_1, \ldots, x_m) = \min_{1\leq j\leq m} \rho_A^\Phi(x_j).$$

This definition is meaningful for every admissible Φ. If Φ is meet-compatible, it is exactly the confidence induced by the usual componentwise neutrosophic product obtained by taking minimum truth and maximum indeterminacy and falsity. Thus the scalar construction and the SVNS product construction agree in the most important conservative case $\Phi_{\min}$.

Definition 4.2. A space X is NpM*-separable* if, for every sequence (A_k) of N-dense SVNSs and every fixed integer $m \geq 1$, there are finite sets $F_k \subseteq A^+_{k,\Phi}$ such that, for every non-empty open $W \subseteq X^m$ and every $N \in \omega$,

$$\sup\Big(\{\rho^{[m]}_{A_k}(x_1, \dots, x_m) : k \geq N,\ (x_1, \dots, x_m) \in F_k^m \cap W\} \cup \{0\}\Big) = 1.$$

The finite selections are allowed to depend on m. The explicit tail requirement is the confidence-sensitive form convenient for the diagonal argument below; Lemma 4.1 shows why such tail witnesses arise naturally from the corresponding selective property.

Theorem 4.3 (NS–NpM equivalence)**.** *For every admissible Φ, a topological space X is* NS*-separable if and only if it is* NpM*-separable.*

Proof. Assume first that X is NS-separable and fix a sequence (A_k) of N-dense SVNSs. By Lemma 4.1(ii), choose finite $F_k \subseteq A^+_{k,\Phi}$ with the strengthened tail property. Fix $m \geq 1$, a non-empty open set $W \subseteq X^m$, and $N \in \omega$. Choose a non-empty basic rectangle $U_1 \times \cdots \times U_m \subseteq W$. For every $\varepsilon > 0$, Lemma 4.1(ii) gives $k \geq N$ such that $c_\Phi(A_k, F_k; U_j) > 1 - \varepsilon$ for all $j \leq m$. Hence, for each j, one can choose $x_j \in F_k \cap U_j$ with $\rho^\Phi_{A_k}(x_j) > 1 - \varepsilon$. The tuple $(x_1, \dots, x_m)$ belongs to $F_k^m \cap W$ and satisfies

$$\rho^{[m]}_{A_k}(x_1, \dots, x_m) > 1 - \varepsilon.$$

Since ε is arbitrary, the supremum in Definition 4.2 is one. Thus X is NpM-separable.

Conversely, assume that X is NpM-separable and fix a sequence (A_k) of N-dense SVNSs. For each $m \geq 1$, choose a witness $(F_{m,k})_{k\in\omega}$ for the m-th power, and define

$$F_k = \bigcup_{1 \leq r \leq k} F_{r,k}$$

(with $F_0 = \varnothing$). Each F_k is finite and is contained in $A^+_{k,\Phi}$. Let $U_1, \dots, U_m$ be non-empty open subsets of X and let $\varepsilon > 0$. Apply the m-th witness with $W = U_1 \times \cdots \times U_m$ and with tail parameter $N = m$. Since the corresponding supremum equals one, there exist $k \geq m$ and $(x_1, \dots, x_m) \in F^m_{m,k} \cap W$ such that

$$\rho^{[m]}_{A_k}(x_1, \dots, x_m) > 1 - \varepsilon.$$

By the definition of the minimum confidence product, $\rho^\Phi_{A_k}(x_j) > 1 - \varepsilon$ for every $j \leq m$. Moreover, $F_{m,k} \subseteq F_k$ because $m \leq k$. Hence $c_\Phi(A_k, F_k; U_j) > 1 - \varepsilon$ for all $j \leq m$, which is exactly the NS condition. □

The theorem is operator-independent at the scalar confidence level. The extra meet-compatibility hypothesis is needed only if one insists that the confidence product arise from a particular componentwise product of SVNSs. This separates the robust selection statement from the algebraic choice of neutrosophic product.

5. Intrinsic selective separability on neutrosophic topological spaces

We now leave the conservative crisp-topological setting. Let $(X, \mathcal{T})$ be an NTS and let Φ be admissible. For a neutrosophic open set U define its *confidence height*

$$h_\Phi(U) = \sup_{x \in X} \rho_U^\Phi(x).$$

For a numerical or observational tolerance $\delta \in [0, 1)$, we call U *δ-effective* if $h_\Phi(U) > \delta$. The word *effective* without qualification means 0-effective. For $A \in \mathcal{SVN}(X)$ and $U \in \mathcal{T}$ put

$$\mathrm{ov}_\Phi(A, U) = \sup_{x \in X} \min\{\rho_A^\Phi(x), \rho_U^\Phi(x)\}.$$

Thus $\mathrm{ov}_\Phi(A, U)$ measures the greatest confidence at which A and U are simultaneously supported. Equivalently, A and U are *α-quasi-coincident in confidence* when this overlap exceeds α. This confidence-based quasi-coincidence avoids the need to choose among several nonequivalent quasi-coincidence conventions while retaining the local viewpoint used in smooth neutrosophic topology [4].

Definition 5.1 (Strong and weak intrinsic density)**.** An SVNS A is

(a) *strongly intrinsically N-dense* if $\mathrm{ov}_\Phi(A, U) = h_\Phi(U)$ for every effective $U \in \mathcal{T}$;

(b) *weakly intrinsically N-dense* if $\mathrm{ov}_\Phi(A, U) > 0$ for every effective $U \in \mathcal{T}$.

Strong intrinsic density asks the data to reach the full confidence height made available by each open set. Weak intrinsic density only asks for a positive confidence contact. The distinction is useful because the former is a genuinely high-confidence requirement, whereas the latter is the direct graded analogue of non-empty intersection.

Proposition 5.2 (Crisp reduction of strong and weak density)**.** *Let (X, τ) be an ordinary topological space and $(X, \widehat{\tau})$ its crisp induced NTS. Then an SVNS A is strongly intrinsically N-dense if and only if it is N-dense in the sense of Definition 3.1. Moreover, A is weakly intrinsically N-dense if and only if its positive support A_Φ^+ is classically dense in X.*

Proof. For every non-empty $U \in \tau$ one has $h_\Phi(\widehat{U}) = 1$ and $\mathrm{ov}_\Phi(A, \widehat{U}) = \sup_{x \in U} \rho_A^\Phi(x)$. Equality with the height is therefore exactly Definition 3.1. Positivity of the overlap is equivalent to the existence of $x \in U$ with $\rho_A^\Phi(x) > 0$, which is equivalent to $U \cap A_\Phi^+ \neq \varnothing$. □

Example 5.3 (Weak density need not be strong)**.** Let $X = \mathbb{Q}$ with its usual topology and let A have constant confidence $1/2$; for instance take $A(x) = (1/2, 0, 0)$ for every $x \in X$ and $\Phi = \Phi_{\min}$. Then $A^+_\Phi = X$, so A is weakly intrinsically N-dense in the crisp-induced NTS. It is not strongly intrinsically N-dense, since its overlap with every non-empty crisp open set is only $1/2$ rather than one.

For a finite $F \subseteq X$ and effective U, define the normalized local selection score

$$\gamma_\Phi(A, F; U) = \frac{1}{h_\Phi(U)} \max\Big(\{\min(\rho^\Phi_A(x), \rho^\Phi_U(x)) : x \in F\} \cup \{0\}\Big)$$

and the unnormalized overlap score

$$\eta_\Phi(A, F; U) = \max\Big(\{\min(\rho^\Phi_A(x), \rho^\Phi_U(x)) : x \in F\} \cup \{0\}\Big).$$

Then $0 \leq \gamma_\Phi \leq 1$. Mathematically the quotient is well-defined for every effective open set. For numerical work, however, division by a very small height can amplify rounding error. We therefore also use the *resolution-δ* convention, with a fixed $\delta > 0$, in which the *selection requirements* are tested only on δ-effective open sets. The input sequences are still required to be strongly or weakly intrinsically N-dense in the sense of Definition 5.1; thus the resolution parameter changes only the family of opens tested by the selection condition, not the admissible input class. The denominator is then bounded away from zero. We write a superscript δ on an intrinsic selection property when this restriction is imposed. With this convention, if $0 \leq \delta_1 \leq \delta_2 < 1$, any property at resolution δ_1 implies the corresponding property at resolution δ_2. On crisp-induced topologies all non-empty opens have height one, so every choice $\delta < 1$ gives the same theory.

Definition 5.4 (Strong intrinsic selective separability)**.** Let (A_n) range over sequences of strongly intrinsically N-dense SVNSs. The NTS $(X, \mathcal{T})$ is

(a) iNM*-separable* if there are finite $F_n \subseteq A^+_{n,\Phi}$ such that $\sup_n \gamma_\Phi(A_n, F_n; U) = 1$ for every effective open U;

(b) iNR*-separable* if one can choose $x_n \in A^+_{n,\Phi}$ such that, for every effective open U, the supremum of the normalized values $\min(\rho^\Phi_{A_n}(x_n), \rho^\Phi_U(x_n))/h_\Phi(U)$ over n is one;

(c) iNH*-separable* if there are finite $F_n \subseteq A^+_{n,\Phi}$ such that $\gamma_\Phi(A_n, F_n; U) \to 1$ for every effective open U;

(d) iNS*-separable* if there are finite $F_n \subseteq A^+_{n,\Phi}$ such that, for every finite family of effective opens $U_1, \ldots, U_m$ and every $\varepsilon > 0$, some n satisfies $\gamma_\Phi(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$.

Definition 5.5 (Weak intrinsic selective separability)**.** Let (A_n) range over sequences of weakly intrinsically N-dense SVNSs. The NTS $(X, \mathcal{T})$ is

(a) wiNM*-separable* if there are finite $F_n \subseteq A^+_{n,\Phi}$ such that $\sup_n \eta_\Phi(A_n, F_n; U) > 0$ for every effective open U;

(b) wiNR-*separable* if there are points $x_n \in A^+_{n,\Phi}$ such that, for every effective open U,

$$\sup\Big(\{\min(\rho^\Phi_{A_n}(x_n), \rho^\Phi_U(x_n)) : n \in \omega\} \cup \{0\}\Big) > 0;$$

(c) wiNH-*separable* if there are finite $F_n \subseteq A^+_{n,\Phi}$ such that $\eta_\Phi(A_n, F_n; U) > 0$ for all but finitely many n, for every effective open U;

(d) wiNS-*separable* if there are finite $F_n \subseteq A^+_{n,\Phi}$ such that, for every finite family $U_1, \dots, U_m$ of effective opens, some n satisfies $\eta_\Phi(A_n, F_n; U_j) > 0$ for every $j \leq m$.

Proposition 5.6 (Crisp reduction of intrinsic selection)**.** *Let (X, τ) be an ordinary topological space and let $(X, \widehat{\tau})$ be its induced crisp NTS. Then* iNM*,* iNR*,* iNH*, and* iNS *reduce respectively to* NM*,* NR*,* NH*, and* NS*. Consequently, by Theorem 3.4, they determine exactly the classical M-, R-, H-, and S-separable space classes. The weak properties* wiNM*,* wiNR*,* wiNH*, and* wiNS *also determine exactly those same classical space classes.*

Proof. Let $U \in \tau$ be non-empty. Since $\rho^\Phi_{\widehat{U}} = \chi_U$ and $h_\Phi(\widehat{U}) = 1$, for every finite $F \subseteq X$ we have

$$\gamma_\Phi(A, F; \widehat{U}) = \max\big(\{\rho^\Phi_A(x) : x \in F \cap U\} \cup \{0\}\big) = c_\Phi(A, F; U).$$

The analogous one-point formula holds for iNR. Hence the four strong intrinsic definitions agree term by term with NM, NR, NH, and NS, respectively.

For the weak properties, Proposition 5.2 identifies the admissible input sets precisely through their positive supports $D_n = A^+_{n,\Phi}$, which are ordinary dense subsets of X. If a classical M-, R-, H-, or S-selection is made from the D_n, then whenever a selected point lies in a crisp open set U its overlap with $\widehat{U}$ is strictly positive. The same selected sets or points therefore satisfy the corresponding weak intrinsic condition. Conversely, apply a weak intrinsic property to the crisp embeddings $\widehat{D_n}$ of an arbitrary sequence of ordinary dense sets. All non-zero overlaps are then equal to 1, so the weak intrinsic condition reduces exactly to the classical M-, R-, H-, or S-selection requirement. This proves all the asserted equivalences. □

The intrinsic theory is not determined by the ordinary support of an SVNS. This is where the new topological layer differs essentially from the transfer theorem.

Example 5.7 (Same support, different strong intrinsic density)**.** Take $X = \{a, b\}$, $\Phi = \Phi_{\min}$ and the NTS $\mathcal{T} = \{\mathbf{0}_N, U, \mathbf{1}_N\}$, where $U(a) = (0.8, 0, 0)$ and $U(b) = (0.2, 0, 0)$. This is a neutrosophic topology because it contains the null and unit sets and is closed under the required operations. Let A have confidences 0.8 at a and 1 at b, and let B have confidences 0.4 at a and 1 at b; for instance take triples $(r, 0, 0)$ realizing these values. Both A and B have the same positive support X. For the unit open set both reach confidence one. However, $h_\Phi(U) = 0.8$, while $\mathrm{ov}_\Phi(A, U) = 0.8$ and $\mathrm{ov}_\Phi(B, U) = 0.4$. Hence A is strongly intrinsically

N-dense and B is not, although both are weakly intrinsically N-dense. The distinction has no support-topological explanation.

Remark 5.8 (The product aggregator can change local ordering)**.** In Example 5.7 all nontrivial triples have the form $(r,0,0)$, so $\Phi_{\min}(r,1,1) = r$ and the product aggregator $\Phi_{\Pi}(a,b,c) = abc$ gives the same confidence. Thus that particular example is unchanged by replacing the minimum with the product. In general, however, the ordering can reverse. The triple $C = (0.6, 0.4, 0)$ has evidence coordinates $(0.6, 0.6, 1)$ and therefore $\rho_C^{\Phi_{\min}} = 0.6$ but $\rho_C^{\Phi_{\Pi}} = 0.36$, whereas $D = (0.5, 0, 0)$ has confidence 0.5 for both aggregators. Hence the minimum ranks C above D, while the product ranks D above C. Intrinsic overlap and the identity of optimal local witnesses may therefore depend on Φ, even though the conservative transfer theorem does not.

The preceding remark establishes dependence at the level of local confidence ordering and admissible witnesses, but it does not by itself show that the *class of intrinsically separable NTSs* depends on the aggregator. To make this distinction explicit, whenever the choice of aggregator is under discussion we write $\mathrm{iN}M_{\Phi}$, $\mathrm{iN}H_{\Phi}$, $\mathrm{iN}R_{\Phi}$ and $\mathrm{iN}S_{\Phi}$ for the corresponding intrinsic properties.

Question 5.9 (Aggregator invariance)**.** *Does there exist a non-crisp NTS $(X, \mathcal{T})$ and two admissible aggregators Φ and Ψ—already $\Phi_{\min}$ and Φ_{Π} would be of particular interest—such that $(X, \mathcal{T})$ is $\mathrm{iN}M_{\Phi}$-separable but not $\mathrm{iN}M_{\Psi}$-separable? More generally, which conditions on Φ and Ψ guarantee that the strong or weak intrinsic selective classes are invariant under the change of aggregator?*

Example 5.7 also shows why Theorem 3.4 should not be expected in a general NTS: the topology itself carries graded information and therefore changes which uncertain sets are strongly dense.

6. Neutrosophic networks and network selection

We now incorporate network-based selection principles. A classical network on a topological space X is a family $\mathcal{N}$ of subsets such that, whenever $x \in U$ and U is open, some $N \in \mathcal{N}$ satisfies $x \in N \subseteq U$. The classical network-selective framework motivating this section was initiated in [3]. Our aim is to preserve the local graded information without making countability vacuous.

A literal requirement involving every neutrosophic point $x_{t,i,f}$ may force a network to resolve continuum many point profiles. For countable network selection it is more natural to fix a countable dense *calibration set*

$$\mathcal{C} \subseteq [0,1]^3 \setminus \{(0,1,1)\}, \qquad (1,0,0) \in \mathcal{C}.$$

Our canonical choice is $\mathcal{C}_{\mathbb{Q}} = ([0,1] \cap \mathbb{Q})^3 \setminus \{(0,1,1)\}$.

Definition 6.1. Let $(X, \mathcal{T})$ be an NTS.

(a) A family $\mathscr{N} \subseteq \mathcal{SVN}(X)$ is a *full neutrosophic network* if for every neutrosophic point p and every $U \in \mathcal{T}$ with $p \in_N U$, there exists $N \in \mathscr{N}$ such that $p \in_N N \preceq_N U$.

(b) Given a calibration set $\mathcal{C}$, the family $\mathscr{N}$ is a *$\mathcal{C}$-calibrated neutrosophic network* if the same condition is required only for points $p = x_{t,i,f}$ with $(t,i,f) \in \mathcal{C}$.

(c) The *$\mathcal{C}$-calibrated neutrosophic network weight* $\mathrm{nnw}_{\mathcal{C}}(X, \mathcal{T})$ is the least cardinality of a $\mathcal{C}$-calibrated neutrosophic network.

Every full network is $\mathcal{C}$-calibrated. The converse need not hold. The countable rational calibration separates two issues that should not be conflated: countability of the network family and continuum many possible numerical descriptions of the same underlying point. More conceptually, this is not merely a technical device but a separation between *spatial resolution* and *data precision*. The underlying set and neutrosophic topology determine which locations and neighbourhood relations must be resolved, whereas $\mathcal{C}$ specifies the precision with which the truth–indeterminacy–falsity profile of a point is tested. Choosing the dense countable set $\mathcal{C}_{\mathbb{Q}}$ keeps these two scales independent: it provides profile tests at arbitrarily fine rational levels while the network is not forced to encode continuum many numerical degrees as separate cardinal requirements. This density is meant as a calibration of numerical precision; without additional regularity assumptions, a $\mathcal{C}_{\mathbb{Q}}$-calibrated network need not satisfy the full-network condition for every irrational profile. Full networks remain available when the cardinality of the complete point-profile geometry, rather than countable selection, is the object of study.

In the rest of this section $\mathcal{C} = \mathcal{C}_{\mathbb{Q}}$ and we write simply *calibrated network* and $\mathrm{nnw}_{\mathbb{Q}}$.

Definition 6.2. Assume $\mathrm{nnw}_{\mathbb{Q}}(X, \mathcal{T}) \leq \omega$. The NTS $(X, \mathcal{T})$ is

(a) *M-Nnw-selective* if for every sequence $(\mathscr{N}_n)$ of countable calibrated neutrosophic networks there are finite $\mathscr{F}_n \subseteq \mathscr{N}_n$ such that $\bigcup_n \mathscr{F}_n$ is a calibrated neutrosophic network;

(b) *R-Nnw-selective* if for every such sequence one can choose $N_n \in \mathscr{N}_n$ so that $\{N_n : n \in \omega\}$ is a calibrated neutrosophic network;

(c) *H-Nnw-selective* if for every such sequence there are finite $\mathscr{F}_n \subseteq \mathscr{N}_n$ such that, whenever $p = x_{t,i,f} \in_N U$ with $(t,i,f) \in \mathcal{C}_{\mathbb{Q}}$ and $U \in \mathcal{T}$, there is k such that for every $n \geq k$ there exists $N \in \mathscr{F}_n$ satisfying $p \in_N N \preceq_N U$.

Proposition 6.3 (Crisp recovery of calibrated networks)**.** *Let (X, τ) be an ordinary topological space. If $\mathcal{N}$ is a classical network, then $\widehat{\mathcal{N}} = \{\widehat{N} : N \in \mathcal{N}\}$ is a calibrated neutrosophic network for $(X, \widehat{\tau})$. Conversely, every calibrated neutrosophic network consisting only of crisp embedded sets corresponds to a classical network. Consequently, when Definition 6.2 is restricted to crisp networks on $(X, \widehat{\tau})$, it is exactly the classical M-, R-, or H-nw-selective property, respectively.*

Proof. Suppose first that $\mathcal{N}$ is a classical network and let $p = x_{t,i,f} \in_N \widehat{U}$ have a calibrated profile. Since the null profile is excluded, $p \in_N \widehat{U}$ implies $x \in U$. Choose $N \in \mathcal{N}$ with $x \in N \subseteq U$. Because $\widehat{N}(x) = \mathbf{1}_N$, we have $p \in_N \widehat{N}$, and $N \subseteq U$ gives $\widehat{N} \preceq_N \widehat{U}$. Thus $\widehat{\mathcal{N}}$ is calibrated.

Conversely, let $\mathscr{N}$ be a calibrated network all of whose members are crisp embeddings, say $\mathscr{N} = \{\widehat{N} : N \in \mathcal{N}\}$. If $x \in U \in \tau$, then the full profile $x_{1,0,0}$ belongs to $\widehat{U}$ and $(1,0,0) \in \mathcal{C}_{\mathbb{Q}}$. Calibration therefore yields some $\widehat{N} \in \mathscr{N}$ with $x_{1,0,0} \in_N \widehat{N} \preceq_N \widehat{U}$. The first relation gives $x \in N$, and the second gives $N \subseteq U$. Hence $\mathcal{N}$ is a classical network.

This equivalence also applies after making selections from a sequence of crisp networks. In the M and R cases it identifies the selected union or selected family with a classical network; in the H case the eventual calibrated-point condition, tested in particular at $x_{1,0,0}$, is exactly the classical eventual network condition. Therefore the three selective properties reduce as claimed. □

The unrestricted full-network versions can still be studied, but their countable instances may be exceptional. The calibrated version is therefore the default object for selective questions, while the full version records the stronger point-profile geometry.

Question 6.4. *Which classical cardinal bounds for M-, R-, and H-nw-selectivity survive for calibrated M-, R-, and H-Nnw-selectivity? Can genuinely neutrosophic network degrees produce ZFC examples that do not arise from crisp embeddings?*

Question 6.5. *Under which hypotheses is H-Nnw-selectivity preserved by finite neutrosophic products, and can failure or preservation phenomena from the classical network-selective program be recovered by crisp calibration? The question is posed relative to the foundational network framework cited above; no classical product-preservation theorem is assumed here.*

7. Filter-parametrized selective separability

Aurichi, Maesano and Zdomskyy [1] introduce selection properties parametrized by free filters on ω. Recall that, for a free filter $\mathcal{G}$, a space is classically $S_{\mathcal{G}}$-separable if for every sequence (D_n) of dense subsets there are finite $F_n \subseteq D_n$ such that $\{n : U \cap F_n \neq \varnothing\} \in \mathcal{G}$ for every non-empty open U. We first retain this index-filter viewpoint and then pass to genuinely single-valued neutrosophic filters on the index set.

Definition 7.1. Let $\mathcal{G}$ be a free filter on ω. A topological space (X,τ) is N$S_{\mathcal{G}}$*-separable* if for every sequence (A_n) of N-dense SVNSs there are finite $F_n \subseteq A^+_{n,\Phi}$ such that, for every non-empty open U and every $\varepsilon > 0$,

$$\{n \in \omega : c_\Phi(A_n, F_n; U) > 1 - \varepsilon\} \in \mathcal{G}.$$

Theorem 7.2 (Filter transfer). *Let $\mathcal{G}$ be any free filter on ω and let Φ be any admissible confidence aggregator. Then X is $\mathrm{N}S_{\mathcal{G}}$-separable if and only if it is classically $S_{\mathcal{G}}$-separable.*

Proof. Suppose first that X is classically $S_{\mathcal{G}}$-separable, and let (A_n) be a sequence of N-dense SVNSs. Choose $\alpha_n \nearrow 1$ and set $D_n = A_{n,\Phi}^{>\alpha_n}$. By Lemma 3.2, each D_n is dense. Choose finite $F_n \subseteq D_n$ so that, for every non-empty open U,

$$H_U = \{n : F_n \cap U \neq \varnothing\} \in \mathcal{G}.$$

Fix U and $\varepsilon > 0$. Since $\alpha_n \to 1$, the set $C_\varepsilon = \{n : \alpha_n > 1 - \varepsilon\}$ is cofinite and hence belongs to the free filter $\mathcal{G}$. Therefore $H_U \cap C_\varepsilon \in \mathcal{G}$. If $n \in H_U \cap C_\varepsilon$, choose $x \in F_n \cap U$. Then $x \in D_n$ and hence $c_\Phi(A_n, F_n; U) \geq \rho_{A_n}^{\Phi}(x) > \alpha_n > 1 - \varepsilon$. Thus

$$H_U \cap C_\varepsilon \subseteq \{n : c_\Phi(A_n, F_n; U) > 1 - \varepsilon\}.$$

By upward closure of $\mathcal{G}$, the set on the right belongs to $\mathcal{G}$, proving $\mathrm{N}S_{\mathcal{G}}$-separability.

Conversely, suppose that X is $\mathrm{N}S_{\mathcal{G}}$-separable and let (D_n) be a sequence of ordinary dense sets. Apply the definition to $A_n = \widehat{D_n}$. Then $A_{n,\Phi}^{+} = D_n$. For the resulting finite $F_n \subseteq D_n$ and every non-empty open U, take $\varepsilon = 1/2$. Since crisp confidences are 0 or 1,

$$\{n : c_\Phi(\widehat{D_n}, F_n; U) > 1/2\} = \{n : F_n \cap U \neq \varnothing\}.$$

The left-hand set belongs to $\mathcal{G}$, so (F_n) witnesses classical $S_{\mathcal{G}}$-separability. □

Consequently, classical results obtained through free filters can be transported to the crisp-topological confidence layer by the same cut argument. In particular, Aurichi, Maesano and Zdomskyy prove under Near Coherence of Filters that, for countable spaces, S-separability, its monotone version, and M-separability are equivalent [1, Theorem 1.1]. Together with Theorem 3.4, the filter transfer theorem shows that the corresponding confidence formulations inherit these classical equivalences whenever the analogous monotone formulation is used. The filter in Theorem 7.2 is still, however, a classical filter of index sets. Earlier notions of neutrosophic filters were investigated by Salama and Alagamy [11]; the single-valued neutrosophic filter framework adopted here is the one developed in [10]. We next make explicit how this framework can enter the selection mechanism itself.

Definition 7.3 (SVN-filter and crisp trace). In the evidence convention of Definition 2.3, a *single-valued neutrosophic filter* on ω is a non-empty family $\mathfrak{F} \subseteq \mathcal{SVN}(\omega)$ which does not contain $\mathbf{0}_N$, is upward closed under $\preceq_N$, and is closed under finite neutrosophic intersections. This is the order-isomorphic reparametrization of the single-valued neutrosophic filter convention used in [10]. Its *crisp trace* is

$$\mathrm{Tr}(\mathfrak{F}) = \{H \subseteq \omega : \widehat{H} \in \mathfrak{F}\}.$$

Proposition 7.4. *The crisp trace of every SVN-filter on ω is a classical filter on ω.*

Proof. Since $\mathfrak{F}$ is non-empty and upward closed, it contains $\mathbf{1}_N$, so $\omega \in \mathrm{Tr}(\mathfrak{F})$. It does not contain $\varnothing$ because $\widehat{\varnothing} = \mathbf{0}_N$. If H, K are in the trace, then $\widehat{H \cap K} = \widehat{H} \sqcap_N \widehat{K} \in \mathfrak{F}$. Finally, $H \subseteq K$ implies $\widehat{H} \preceq_N \widehat{K}$, and upward closure gives K in the trace. □

Let $(X, \mathcal{T})$ be an NTS, let (A_n) be strongly intrinsically N-dense, and let (F_n) be finite selections. For every effective open U put $q_U(n) = \gamma_\Phi(A_n, F_n; U)$ and define the *graded response SVNS* R_U on ω by

$$R_U(n) = \big(q_U(n), 1 - q_U(n), 1 - q_U(n)\big).$$

Larger local selection quality therefore gives a larger response in the evidence order.

Definition 7.5 (Intrinsic selection controlled by an SVN-filter)**.** Let $\mathfrak{F}$ be an SVN-filter on ω. The NTS $(X, \mathcal{T})$ is

(a) *trace*-iN$S_{\mathfrak{F}}$*-separable* if for every sequence (A_n) of strongly intrinsically N-dense SVNSs there are finite $F_n \subseteq A^+_{n,\Phi}$ such that, for every effective open U and every $\varepsilon > 0$,

$$\{n : q_U\widehat{(n) >} 1 - \varepsilon\} \in \mathfrak{F};$$

(b) *graded*-iN$S_{\mathfrak{F}}$*-separable* if the selections can be chosen so that $R_U \in \mathfrak{F}$ for every effective open U.

The trace version is conservative with respect to the index filter encoded by $\mathfrak{F}$, whereas the graded version lets the filter act directly on the entire sequence of confidence scores.

Proposition 7.6. *Trace*-iN$S_{\mathfrak{F}}$*-separability is exactly the intrinsic filter-selection condition obtained by using the classical filter* $\mathrm{Tr}(\mathfrak{F})$ *on the threshold sets* $\{n : q_U(n) > 1 - \varepsilon\}$.

Proof. This is immediate from Definition 7.3: a threshold set belongs to $\mathrm{Tr}(\mathfrak{F})$ precisely when its crisp embedding belongs to $\mathfrak{F}$. □

The graded version retains information that the crisp trace deliberately discards: membership is tested on the full response SVNS rather than only on its threshold sets. It is therefore a natural candidate for a fully neutrosophic analogue of filter-controlled selective separability.

Question 7.7. *Which structural assumptions on an SVN-filter $\mathfrak{F}$ make the graded and trace versions of* iN$S_{\mathfrak{F}}$ *equivalent? In the classical setting, Near Coherence of Filters is used by Aurichi, Maesano and Zdomskyy [1] to relate filter-parametrized selection to S- and M-separability. The graded* iN$S_{\mathfrak{F}}$ *construction suggests a genuinely graded neutrosophic counterpart of NCF: rather than testing coherence only after passing to crisp index sets, one could require suitable confidence-preserving images of SVN-filters to become coherent at the level of* *graded responses. Can such a principle be formulated so that it reduces to classical NCF on crisp traces and yields intrinsic NTS analogues of the classical NCF equivalences?*

8. Consequences for Fréchet–Urysohn and function spaces

The conservative layer immediately transfers several recent classical results while the intrinsic layer identifies where new work is required.

Corollary 8.1. *For every admissible Φ, every countable Hausdorff Fréchet–Urysohn space is* $\mathrm{N}S$*-separable and therefore* $\mathrm{Np}M$*-separable.*

Proof. Aurichi, Maesano and Zdomskyy prove that every countable Hausdorff Fréchet–Urysohn space is S-separable [1, Theorem 1.8]. Apply Theorems 3.4 and 4.3. □

Corollary 8.2. *For every admissible Φ, there exists in ZFC a countable Hausdorff Fréchet–Urysohn space without isolated points which is* $\mathrm{N}R$*-separable but not* $\mathrm{N}H$*-separable.*

Proof. Bardyla, Maesano and Zdomskyy construct in ZFC a countable Hausdorff Fréchet–Urysohn space without isolated points which is not H-separable [2, Theorem 2.1]. Their paper also recalls that separable Fréchet–Urysohn spaces are R-separable. Apply Theorem 3.4. □

For a Tychonoff space T, let $C_p(T)$ denote the space of real-valued continuous functions on T with the topology of pointwise convergence.

Corollary 8.3. *For every admissible Φ and every Tychonoff space T, the space $C_p(T)$ is* $\mathrm{N}S$*-separable if and only if it is* $\mathrm{N}M$*-separable.*

Proof. Aurichi, Maesano and Zdomskyy prove that, for $C_p(T)$, classical S-separability and M-separability are equivalent [1, Theorem 1.6]. Apply Theorem 3.4. □

The corresponding statements for $\mathrm{iN}S$, $\mathrm{iN}M$ or the network properties are no longer automatic because a genuine neutrosophic topology on the function space must first be specified. The computational constructions in [8, 9] provide one practical route for finite or discretized examples.

9. Concluding remarks

The paper separates a conservative confidence lift from an intrinsic neutrosophic-topological extension. The conservative layer is useful because its exact transfer theorems prove that no classical space class is changed artificially. Example 3.6 nevertheless shows that the witnesses are strictly filtered by confidence, which is the operational content of the lift. The invariance of Theorem 3.4 under every admissible aggregator also shows that this conclusion is not an artefact of the minimum operator.

The intrinsic layer now has two density scales. Strong intrinsic density requires full use of the confidence height of every effective open set, while weak intrinsic density only requires positive overlap. These notions are distinct even on a crisp-induced topology for individual SVNSs, as Example 5.3 shows. Nevertheless, their associated selection principles recover the expected classical space classes in the crisp case. A resolution parameter $\delta > 0$ makes normalized scores numerically robust by excluding open sets whose confidence height is below the prescribed tolerance.

The comparison in Remark 5.8 shows that local confidence ordering and the identity of preferred witnesses may depend on the chosen admissible aggregator: the minimum and product rules can reverse the ordering of neutrosophic evidence. Whether this local dependence can change the *intrinsic separability class* of a non-crisp NTS is a separate question, isolated in Question 5.9. This distinction is invisible to the conservative transfer theorem and is a natural source of genuinely neutrosophic problems.

For networks, countable calibration should be read as a separation between spatial resolution and numerical profile precision, not only as a device for avoiding continuum-sized families. The calibrated theory retains exact crisp recovery while leaving full neutrosophic networks available for cardinal investigations. Product behaviour, cardinal bounds and genuinely non-crisp examples remain open. The filter theory now has two levels as well: a trace-based form that reduces to an ordinary index filter, and a graded form in which a single-valued neutrosophic filter acts directly on the sequence of local confidence scores.

Several directions remain open. A systematic study should identify aggregator conditions weaker than meet-compatibility that preserve natural neutrosophic products, compare strong and weak density on non-crisp NTSs, and determine when calibrated and full network notions agree. On the filter side, the main problem is to characterize SVN-filters for which graded and trace selection coincide and to formulate a graded neutrosophic analogue of Near Coherence of Filters that reduces to the classical principle on crisp traces. These questions go beyond a formal relabeling of classical properties and appear to be the natural next step for the theory.

Acknowledgments: This research was supported by Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (G.N.S.A.G.A.) of Istituto Nazionale di Alta Matematica (INdAM) "F. Severi", Italy.

References

1. L. F. Aurichi, F. Maesano and L. Zdomskyy, *A density counterpart of the Scheepers covering property*, Topology Appl. **391** (2026), 109913, doi:10.1016/j.topol.2026.109913.
2. S. Bardyla, F. Maesano and L. Zdomskyy, *Selective separability properties of Fréchet–Urysohn spaces and their products*, Fund. Math. **263** (2023), no. 3, 271–299, doi:10.4064/fm230522-13-10.
3. M. Bonanzinga and D. Giacopello, *A generalization of M-separability by networks*, Atti Accad. Pelorit. Pericol. Cl. Sci. Fis. Mat. Nat. **101** (2023), no. 2, A11, doi:10.1478/AAPP.1012A11.
4. K. Chandran, S. S. Sundaramoorthy, F. Smarandache and S. Jafari, *On product of smooth neutrosophic topological spaces*, Symmetry **12** (2020), 1557, doi:10.3390/sym12091557.
5. F. Gallego Lupiáñez, *On neutrosophic topology*, Kybernetes **37** (2008), no. 6, 797–800, doi:10.1108/03684920810876990.
6. G. Gruenhage and M. Sakai, *Selective separability and its variations*, Topology Appl. **158** (2011), 1352–1359, doi:10.1016/j.topol.2011.05.009.
7. S. Jafari, G. Nordo and S. S. Thakur, *More on neutrosophic topology*, Neutrosophic Sets Syst. **73** (2024), 584–591, doi:10.5281/zenodo.13989337.
8. G. Nordo, S. Jafari, A. Mehmood and B. Basumatary, *A Python framework for neutrosophic sets and mappings*, Neutrosophic Sets Syst. **65** (2024), 199–236, doi:10.5281/zenodo.10858945.
9. G. Nordo, S. Jafari and M. Y. Leyva Vázquez, *A Python framework enhancement for neutrosophic topologies*, Neutrosophic Sets Syst. **78** (2025), 252–273, doi:10.5281/zenodo.14271018.
10. G. Nordo, A. Mehmood and S. Broumi, *Single valued neutrosophic filters*, Int. J. Neutrosophic Sci. **6** (2020), no. 1, 8–21, doi:10.54216/IJNS.060101.
11. A. A. Salama and H. Alagamy, *Neutrosophic filters*, Int. J. Comput. Sci. Eng. Inf. Technol. Res. **3** (2013), no. 1, 307–312.
12. A. A. Salama and S. A. Alblowi, *Neutrosophic set and neutrosophic topological spaces*, IOSR J. Math. **3** (2012), no. 4, 31–35, doi:10.9790/5728-0343135.
13. D. Repovš and L. Zdomskyy, *On M-separability of countable spaces and function spaces*, Topology Appl. **157** (2010), 2538–2541, doi:10.1016/j.topol.2010.07.036.
14. D. Repovš and L. Zdomskyy, *Products of H-separable spaces in the Laver model*, Topology Appl. **239** (2018), 115–119, doi:10.1016/j.topol.2018.02.021.
15. D. Repovš and L. Zdomskyy, *M-separable spaces of functions are productive in the Miller model*, Ann. Pure Appl. Logic **171** (2020), no. 7, 102806, doi:10.1016/j.apal.2020.102806.
16. M. Scheepers, *Combinatorics of open covers (VI): selectors for sequences of dense sets*, Quaest. Math. **22** (1999), 109–130, doi:10.1080/16073606.1999.9632063.
17. F. Smarandache, *A Unifying Field in Logics: Neutrosophy, Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, 1999.
18. H. Wang, F. Smarandache, Y.-Q. Zhang and R. Sunderraman, *Single valued neutrosophic sets*, Multispace Multistruct. **4** (2010), 410–413.

Received: Jan 25, 2026. Accepted: Aug 14, 2026