



Plithogenic Quasigroups with Application to Decision Making

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Abstract. In this paper, we extend the concept of neutrosophic sets by introducing *plithogenic quasigroups* by associating a quasigroup $(Q, \cdot)$ with a plithogenic structure: attributes V , dominant v_D , contradiction degrees $c_i \in [0, 1]$, and neutrosophic degrees $d(x, v_i)$. The plithogenic product $\cdot_P$ aggregates componentwise via $(1 - c_i)t_N + c_i s_N$ with $t_N = \min$, $s_N = \max$. We prove that $(P(Q), \cdot_P)$ is a quasigroup and its parastrphes; the Latin square property lifts, guaranteeing unique solvability of $a \cdot_P x = b$. Solvability holds iff each component b_i lies in the c_i -weighted interval between a_i and x_i : $c_i = 1$ forces $b_i \geq a_i$ while $c_i = 0$ forces $b_i \leq a_i$. We characterize isotopies, homomorphisms, and nuclei, showing $0 < c_i < 1$ yields trivial nuclei and contradicts associativity. We applied plithogenic quasigroup properties to portfolio selection with $V = \{\text{Risk, Return, Liquidity}\}$, setting $c_{\text{Risk}} = 0.9$ blocks buy signals whereas $c_{\text{Risk}} = 0.1$ enables solvability, where c_i functions as a parameter connecting algebraic feasibility to risk policy in decision making.

Keywords: Plithogenic sets 1; Quasigroups 2; Neutrosophic set 3 (contradiction degree, non - associative algebra, Latin square.)

1. Introduction

Quasigroups are algebraic structures that generalize groups by dropping the associativity requirement. The study of quasigroups and loops has a history spanning more than two centuries. Its origins can be traced to Eulers 1782 conjecture on the non-existence of certain pairs of mutually orthogonal Latin squares.

Algebraic structures that model uncertainty and contradiction have become central to decision science, particularly in multi-criteria problems where attributes conflict. Neutrosophic sets, introduced by Smarandache [29], capture truth, indeterminacy, and falsity degrees, while plithogenic sets [28] refine this by assigning contradiction degrees $c(v_i, v_D) \in [0, 1]$ between each attribute v_i and a dominant attribute v_D . These degrees govern how strongly an attribute opposes the dominant one during aggregation.

While plithogenic aggregation has been studied for set-theoretic and matrix operations, its extension to non-associative algebraic systems remains unexplored. Quasigroups are precisely the non-associative generalization of groups: they guarantee unique solvability of $a \cdot x = b$ and $y \cdot a = b$ but need not satisfy $(xy)z = x(yz)$. This property makes them ideal for modeling operations where order of application matters for example, sequential portfolio adjustments, layered encryption, or step-wise medical treatments.

We introduce *plithogenic quasigroups* $(P(Q), \cdot_P)$ by equipping a base quasigroup $(Q, \cdot)$ with an attribute set V , contradiction degrees c_i , and componentwise aggregation using neutrosophic t -norm $t_N = \min$ and s -norm $s_N = \max$. The plithogenic product is defined as

$$d(x \cdot_P y, v_i) = (1 - c_i) \min\{d(x, v_i), d(y, v_i)\} + c_i \max\{d(x, v_i), d(y, v_i)\},$$

so that $c_i = 0$ yields pure consensus while $c_i = 1$ yields pure dominance of the higher degree.

The theory of neutrosophic soft sets has developed significantly by many authors since it was introduced by Maji [16]. Broumi and Smarandache studied about its interval-valued [?] and intuitionistic [10] neutrosophic soft sets. Oyem et al. [23] introduced the Neutrosopic quasigroup which extends neutrosopic sets properties to non associative algebraic structures called quasigroup. Its Algebraic properties have been studied by Broumi et al. [11,12] provided its new operations and examined distributivity and associativity. Its applications to an incomplete data were considered by Saha et al. [25].

Also, fuzzy soft sets and their decision-making uses laid groundwork for algebraic treatments in soft and fuzzy settings. Recent developments include Soft Neutrosophic versions of algebraic structures [4-6,9,26] and Neutrosophic Soft versions [1,7,17,32], indicating that soft set theory offers a natural generalization of neutrosophic sets to algebra.

Oyem et al. [20], [21], [22], [24] introduced the notion of order for finite soft quasigroups and applied it to egalitarianism. They established several algebraic relationships between the order of a quasigroup and the order of its associated soft quasigroups and in other related work investigated various algebraic properties of soft quasigroups and soft paratrophes and showed that certain operations are not distributive over one another.

In conclusion, we define plithogenic quasigroups, extend classical quasigroup theory to handle multi-valued falsity, indeterminacy, and truth, and demonstrate their application to multi-attribute decision-making under uncertainty.

2. Preliminaries

. This section provides a brief review of the basic concepts and known results on quasigroups and Plithogenic sets.

A. Oyem, E. K. Osawaru, Plithogenic Quasigroups with Application to Decision Making

2.1. *Quasigroups* - [3, 13, 14, 30, 31]

Definition 2.1. (Quasigroup) Let Q be a nonempty set and let $(\cdot)$ be a binary operation on Q . The pair $(Q, \cdot)$ is called a *groupoid* or *magma*.

If for all $a, b \in Q$ the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

have unique solutions $x, y \in Q$, then $(Q, \cdot)$ is a *quasigroup*.

A quasigroup $(Q, \cdot)$ for which there exists a unique element $e \in Q$ such that

$$x \cdot e = e \cdot x = x \quad \text{for all } x \in Q$$

is called a *loop*. The element e is the identity element of the loop.

Throughout this paper we write xy for $x \cdot y$, and we adopt the convention that $\cdot$ has lower precedence than juxtaposition. For example, $x \cdot yz$ means $x(yz)$.

Definition 2.2. For a fixed element $x \in Q$, the *left translation* L_x and *right translation* R_x are the maps $Q \rightarrow Q$ defined by

$$yL_x = x \cdot y, \quad yR_x = y \cdot x.$$

A groupoid is a quasigroup if and only if L_x and R_x are permutations of Q for all $x \in Q$. Since L_x and R_x are bijective, their inverses exist. We define the left and right divisions by

$$x \backslash y = yL_x^{-1}, \quad x/y = xR_y^{-1},$$

so that

$$x \backslash y = z \iff x \cdot z = y, \quad x/y = z \iff z \cdot y = x.$$

In the language of universal algebra, a quasigroup can be presented as an algebra $(Q, \cdot, \backslash, /)$ of type $(2, 2, 2)$.

For more on quasigroups, readers can check [3, 30].

Definition 2.3. (Subquasigroup)(Pflugfelder [31])

Let $(Q, \cdot)$ be a groupoid, and let $H \subseteq Q$ with $H \neq \emptyset$. If $(H, \cdot)$ is itself a groupoid under the same operation, then H is called a *subgroupoid* of Q .

If $(Q, \cdot)$ is a quasigroup and $(H, \cdot)$ is a quasigroup, then H is called a *subquasigroup* of Q .

In either case we write $H \leq Q$.

Definition 2.4. (Neutrosophic set) A neutrosophic set A in a universe X is defined by three membership functions

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$$

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$$

Definition 2.5 (Neutrosophic Quasigroup [23]). Let $N(Q)$ be a neutrosophic set on a nonempty set Q , where each element is expressed as

$$x = \langle x, T(x), I(x), F(x) \rangle, \quad x \in Q,$$

with $T, I, F : Q \rightarrow [0, 1]$ denoting the truth, indeterminacy, and falsity membership functions.

A binary algebraic structure $(N(Q), *)$ is called a **neutrosophic quasigroup** if it satisfies:

- (1) **Closure:** $a * b \in N(Q)$ for all $a, b \in N(Q)$.
- (2) **Latin Square Property:** For all $a, b \in N(Q)$, the equations

$$a * x = b \quad \text{and} \quad y * a = b$$

have unique solutions $x, y \in N(Q)$.

Definition 2.6 (Plithogenic Set [2,15,27–29]). Let U be a universe of discourse and let $P \subseteq U$. A *plithogenic set* P is defined as the quintuple

$$P = (P, V, v_D, d, c)$$

where:

- (1) P is the set of elements;
- (2) $V = \{v_1, v_2, \dots, v_m\}$ is a set of attributes characterizing the elements of P ;
- (3) $v_D \in V$ is the dominant attribute, used as a reference for measuring contradiction;
- (4) $d : P \times V \rightarrow [0, 1]^3$ is the degree of appurtenance function, such that for each $x \in P$ and $v_i \in V$,

$$d(x, v_i) = (t_i(x), i_i(x), f_i(x))$$

denotes the neutrosophic degrees of truth, indeterminacy, and falsity of element x with respect to attribute v_i ;

- (5) $c : V \times V \rightarrow [0, 1]$ is the contradiction degree function, where $c(v_i, v_j)$ measures the degree to which attribute v_i contradicts attribute v_j . In particular, we set $c_i = c(v_i, v_D)$ as the contradiction degree of v_i relative to the dominant attribute v_D , with $c(v_D, v_D) = 0$.

For any $x, y \in P$, the *plithogenic aggregation* with respect to attribute v_i is given by

$$d_{agg}(x, y, v_i) = (1 - c_i) t_N\{d(x, v_i), d(y, v_i)\} + c_i s_N\{d(x, v_i), d(y, v_i)\},$$

where t_N and s_N are a neutrosophic t-norm and s-norm, respectively.

3. Plithogenic Quasigroups

Definition 3.1 (Plithogenic Quasigroup). Let $(Q, \cdot)$ be a quasigroup, $V = \{v_1, \dots, v_m\}$ an attribute set with dominant attribute $v_D \in V$, and $c_i = c(v_i, v_D) \in [0, 1]$ the contradiction degree of v_i relative to v_D . Choose neutrosophic t-norm $t_N = \min$ and s-norm $s_N = \max$. For each $x \in Q$, let $d(x, v_i) = (t_i, i_i, f_i) \in [0, 1]^3$ denote the neutrosophic degrees of truth, indeterminacy, and falsity for attribute v_i .

A *plithogenic quasigroup* over Q is then the structure

$$P(Q) = (Q, V, v_D, d, c, \cdot_P)$$

where the *plithogenic product* $(\cdot_P)$ is defined for $x, y \in P(Q)$ by

- (1) Base: $x \cdot_P y = x \cdot y$ in $(Q, \cdot)$;
- (2) Degrees: for each $v_i \in V$,

$$d(x \cdot_P y, v_i) = (1 - c_i) t_N\{d(x, v_i), d(y, v_i)\} + c_i s_N\{d(x, v_i), d(y, v_i)\}.$$

Theorem 3.2 (Closure of Latin Square Property). *If $(Q, \cdot)$ is a quasigroup, then $(P(Q), \cdot_P)$ is a quasigroup. That is, for any $a, b \in P(Q)$, the equations*

$$a \cdot_P x = b \quad \text{and} \quad y \cdot_P a = b$$

have unique solutions $x, y \in P(Q)$ whenever they are solvable.

Proof. Since $(Q, \cdot)$ is a quasigroup, there exists a unique $x_0 \in Q$ such that $a \cdot x_0 = b$ in the base. Construct $x \in P(Q)$ with base element x_0 .

For each $v_i \in V$, let $a_i = d(a, v_i)$, $b_i = d(b, v_i)$, and set $x_i = d(x, v_i)$. By definition of $\cdot_P$,

$$b_i = (1 - c_i) t_N\{a_i, x_i\} + c_i s_N\{a_i, x_i\}.$$

For $t_N = \min$, $s_N = \max$, this equation has a unique solution x_i provided it exists. The argument for $y \cdot_P a = b$ is symmetric. Hence $(P(Q), \cdot_P)$ satisfies the Latin square property whenever solutions exist. $\square$

Corollary 3.3 (Solvability Condition). *Let $a, b \in P(Q)$ with $a_i = d(a, v_i)$ and $b_i = d(b, v_i)$ for $v_i \in V$. Equation $a \cdot_P x = b$ is solvable in $P(Q)$ if and only if for every $v_i \in V$ and each component $* \in \{t, i, f\}$,*

$$x_i^* = \begin{cases} \frac{b_i^* - (1 - c_i)a_i^*}{c_i} \in [0, 1], & b_i^* \geq a_i^*, \\ \frac{b_i^* - c_i a_i^*}{1 - c_i} \in [0, 1], & b_i^* < a_i^*, \end{cases}$$

with the convention that $0/0$ is interpreted as any value in $[0, 1]$ when $c_i = 0$ or $c_i = 1$. If any component $x_i^ \notin [0, 1]$, the equation has no solution in $P(Q)$.*

Example 3.4. Let $Q = \{e, a, b\}$ with operation given by its Latin square:

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Take attributes $V = \{\text{Growth}, \text{Safety}\}$ with $v_D = \text{Growth}$, $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$. Define elements:

$$x_1 = [e] : G(0.90, 0.10, 0.00), S(0.80, 0.10),$$

$$x_2 = [a] : G(0.60, 0.30, 0.10), S(0.90, 0.10, 0.00),$$

$$x_3 = [b] : G(0.30, 0.40, 0.30), S(0.50, 0.30, 0.20).$$

The plithogenic product yields the following 3×3 Latin square:

TABLE 1. Plithogenic 3×3 Latin square for $(P(Q), \cdot_P)$

$\cdot_P$	x_1	x_2	x_3
x_1	$e: G(0.90, 0.10, 0.00) S(0.80, 0.10)$	$a: G(0.60, 0.30, 0.10) S(0.86, 0.10, 0.04)$	$b: G(0.30, 0.40, 0.30) S(0.62, 0.22, 0.16)$
x_2	$a: G(0.60, 0.30, 0.10) S(0.86, 0.10, 0.04)$	$b: G(0.30, 0.40, 0.30) S(0.62, 0.22, 0.16)$	$e: G(0.90, 0.10, 0.00) S(0.80, 0.10)$
x_3	$b: G(0.30, 0.40, 0.30) S(0.62, 0.22, 0.16)$	$e: G(0.90, 0.10, 0.00) S(0.80, 0.10)$	$a: G(0.60, 0.30, 0.10) S(0.86, 0.10, 0.04)$

Tables 2 and 3 verify uniqueness of solutions for right and left division:

TABLE 2. Right division test: $x_1 \cdot_P x = x_2$

Test x	Base	Growth	Safety	$= x_2$
x_1	e	(0.90, 0.10, 0.00)	(0.80, 0.10)	No
x_2	a	(0.60, 0.30, 0.10)	(0.86, 0.10, 0.04)	Yes
x_3	b	(0.30, 0.40, 0.30)	(0.62, 0.22, 0.16)	No

TABLE 3. Left division test: $y \cdot_P x_1 = x_2$

Test y	Base	Growth	Safety	$= x_2$
$y = x_1$	e	(0.90, 0.10, 0.00)	(0.80, 0.10)	No
$y = x_2$	a	(0.60, 0.30, 0.10)	(0.86, 0.10, 0.04)	Yes
$y = x_3$	b	(0.30, 0.40, 0.30)	(0.62, 0.22, 0.16)	No

Remark 3.5 (Contradiction Degree as Risk Dial). c_i determines how we can combine the attribute v_i . If

$c_i = 0$: strict AND both inputs must be high.

$c_i = 1$: strict OR the higher input prevails.

$0 < c_i < 1$: smooth interpolation between the two. A larger c_i also reduces solvability if $b_i \leq_a i$ by giving weak inputs veto power, making c_i a direct measure of risk tolerance. Corollary 3.3 shows that c_i also governs solvability: large c_i restricts solutions when $b_i < a_i$, acting as a veto. Thus c_i adjust the risk variables into the algebraic structures.

Definition 3.6 (Plithogenic Quasigroup Homomorphism). Let $P(Q_1) = (Q_1, V, v_D, d_1, c, \cdot_{P_1})$ and $P(Q_2) = (Q_2, V, v_D, d_2, c, \cdot_{P_2})$ be two plithogenic quasigroups over base quasigroups $(Q_1, \cdot_1)$ and $(Q_2, \cdot_2)$ with the same attribute set V , dominant attribute v_D , and contradiction degrees c .

A map $\phi : P(Q_1) \rightarrow P(Q_2)$ is a *plithogenic quasigroup homomorphism* if:

- (1) **Base homomorphism:** For all $x, y \in P(Q_1)$,

$$\phi(x \cdot_{P_1} y) = \phi(x) \cdot_{P_2} \phi(y)$$

on the base elements. That is, the restriction $\phi_0 : Q_1 \rightarrow Q_2$ given by $\phi_0(x) = \phi(x)$ is a quasigroup homomorphism.

- (2) **Degree preservation:** For all $x \in P(Q_1)$ and all $v_i \in V$,

$$d_2(\phi(x), v_i) = d_1(x, v_i).$$

That is, ϕ preserves the neutrosophic degrees (t, i, f) for every attribute.

If ϕ_0 is bijective, we call ϕ a *plithogenic quasigroup isomorphism*, and write $P(Q_1) \cong P(Q_2)$.

Corollary 3.7. If $\phi : P(Q_1) \rightarrow P(Q_2)$ is a plithogenic quasigroup homomorphism, then for any $a, b \in P(Q_1)$, the unique solution x to $a \cdot_{P_1} x = b$ is mapped to the unique solution to $\phi(a) \cdot_{P_2} y = \phi(b)$, namely $y = \phi(x)$.

Definition 3.8 (Kernel and Image of a Plithogenic Quasigroup Homomorphism). Let $\phi : P(Q_1) \rightarrow P(Q_2)$ be a plithogenic quasigroup homomorphism as in Definition 3.6, and let $\phi_0 : Q_1 \rightarrow Q_2$ be the induced base homomorphism.

- (1) The *kernel* of ϕ is the substructure of $P(Q_1)$ defined by

$$\ker(\phi) = \{x \in P(Q_1) : \phi(x) = e_2\},$$

where $e_2 \in P(Q_2)$ is a fixed element whose base is the identity of $(Q_2, \cdot_2)$ and $d_2(e_2, v_i) = (1, 0, 0)$ for all $v_i \in V$. If $(Q_2, \cdot_2)$ has no identity, fix any distinguished element $e_2 \in P(Q_2)$.

- (2) The *image* of ϕ is the substructure of $P(Q_2)$ given by

$$\text{Im}(\phi) = \{\phi(x) \in P(Q_2) : x \in P(Q_1)\}.$$

Then $\text{Im}(\phi)$ is a plithogenic subquasigroup of $P(Q_2)$.

First Isomorphism Theorem for Plithogenic Quasigroups. Let $\phi : P(Q_1) \rightarrow P(Q_2)$ be a surjective plithogenic quasigroup homomorphism. Then there exists a plithogenic quasigroup congruence $\sim_\phi$ on $P(Q_1)$ defined by

$$x \sim_\phi y \iff \phi(x) = \phi(y),$$

and we have a plithogenic quasigroup isomorphism

$$P(Q_1)/\sim_\phi \cong P(Q_2).$$

Moreover, $x \sim_\phi y$ if and only if $x \cdot_{P_1} y^{-1} \in \ker(\phi)$ when $(Q_1, \cdot_1)$ is a loop, and $d_1(x, v_i) = d_1(y, v_i)$ for all $v_i \in V$. $\square$

Remark 3.9. Unlike groups, $\ker(\phi)$ in a quasigroup is not automatically a subquasigroup, because quasigroups need not have identities or inverses. However, $\ker(\phi)$ is a *normal subquasigroup* in the sense of quasigroup theory if $(Q_1, \cdot_1)$ is a loop. The degree preservation condition ensures that $\ker(\phi)$ consists of elements that are neutrosophically indistinguishable from the distinguished element e_2 .

Example 3.10 (Plithogenic Quasigroup Homomorphism). Let $P(Q_1)$ be the plithogenic quasigroup from Example 3.4, with base $Q_1 = \{e, a, b\}$ and elements x_1, x_2, x_3 as defined there.

Define a second base quasigroup $(Q_2, \circ)$ by $Q_2 = \{\bar{e}, \bar{a}, \bar{b}\}$ with operation:

$\circ$	$\bar{e}$	$\bar{a}$	$\bar{b}$
$\bar{e}$	$\bar{e}$	$\bar{a}$	$\bar{b}$
$\bar{a}$	$\bar{a}$	$\bar{b}$	$\bar{e}$
$\bar{b}$	$\bar{b}$	$\bar{e}$	$\bar{a}$

which is isomorphic to $(Q_1, \cdot)$ via $\phi_0(e) = \bar{e}$, $\phi_0(a) = \bar{a}$, $\phi_0(b) = \bar{b}$.

Construct $P(Q_2)$ using the same attributes $V = \{\text{Growth}, \text{Safety}\}$ and contradiction degrees $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$. Define elements:

$$y_1 = [\bar{e}] : G(0.90, 0.10, 0.00), S(0.80, 0.10),$$

$$y_2 = [\bar{a}] : G(0.60, 0.30, 0.10), S(0.90, 0.10, 0.00),$$

$$y_3 = [\bar{b}] : G(0.30, 0.40, 0.30), S(0.50, 0.30, 0.20).$$

Define $\phi : P(Q_1) \rightarrow P(Q_2)$ by $\phi(x_i) = y_i$ for $i = 1, 2, 3$. Then ϕ is a plithogenic quasigroup homomorphism because:

- (1) **Base homomorphism:** ϕ_0 is an isomorphism of quasigroups, so $\phi(x_i \cdot_{P_1} x_j) = \phi(x_i) \cdot_{P_2} \phi(x_j)$ for all i, j . For instance,

$$\phi(x_1 \cdot_{P_1} x_2) = \phi(x_2) = y_2 = y_1 \cdot_{P_2} y_2 = \phi(x_1) \cdot_{P_2} \phi(x_2).$$

(2) **Degree preservation:** By construction, $d_1(x_i, v) = d_2(y_i, v)$ for all i and $v \in V$. For example, $d_1(x_1, \text{Growth}) = (0.90, 0.10, 0.00) = d_2(y_1, \text{Growth})$.

Hence $P(Q_1) \cong P(Q_2)$ as plithogenic quasigroups. Moreover,

$$\ker(\phi) = \{x_1\}, \quad \text{because } \phi(x_1) = y_1 = e_2,$$

and $\text{Im}(\phi) = P(Q_2)$.

Example 3.11 (Non-isomorphic Plithogenic Homomorphism). Let $P(Q_1)$ be as in Example 3.4 with x_1, x_2, x_3 . Consider a base quasigroup $(Q_2, *)$ with $Q_2 = \{\hat{e}, \hat{a}\}$ and operation:

$*$	$\hat{e}$	$\hat{a}$
$\hat{e}$	$\hat{e}$	$\hat{a}$
$\hat{a}$	$\hat{a}$	$\hat{e}$

and construct $P(Q_2)$ with the same $V = \{\text{Growth}, \text{Safety}\}$, $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$.

Suppose we try to define a base homomorphism $\phi_0 : Q_1 \rightarrow Q_2$ by

$$\phi_0(e) = \hat{e}, \quad \phi_0(a) = \hat{a}, \quad \phi_0(b) = \hat{a},$$

so that a and b collapse to the same base element. For ϕ to lift to a plithogenic quasigroup homomorphism, we must have $d_1(x, v_i) = d_2(\phi(x), v_i)$ for all x and v_i .

But in $P(Q_1)$ we have:

$$d_1(x_2, \text{Growth}) = (0.60, 0.30, 0.10),$$

$$d_1(x_3, \text{Growth}) = (0.30, 0.40, 0.30).$$

Since $\phi(x_2) = \phi(x_3) = z$ for some $z \in P(Q_2)$ with base $\hat{a}$, degree preservation forces

$$d_2(z, \text{Growth}) = (0.60, 0.30, 0.10) = (0.30, 0.40, 0.30),$$

which is impossible. Therefore no plithogenic homomorphism exists that collapses x_2 and x_3 .

If instead we try to collapse x_1 and x_3 by setting $\phi_0(e) = \phi_0(b) = \hat{e}$, we need

$$d_1(x_1, \text{Safety}) = (0.80, 0.10) = d_1(x_3, \text{Safety}) = (0.50, 0.30, 0.20),$$

again impossible.

Hence the only plithogenic homomorphisms from $P(Q_1)$ are injective on elements with distinct degrees. In this example, any ϕ must send x_1, x_2, x_3 to three distinct elements of $P(Q_2)$ with matching degrees. If $P(Q_2)$ has only two elements, no plithogenic homomorphism exists.

Remark 3.12 (Degrees Restrict Homomorphisms). In classical quasigroups, any surjective base homomorphism works. For plithogenic quasigroups, degree preservation is a strong constraint: you cannot identify two elements unless all their (t, i, f) vectors agree for every attribute. Thus c_i and d together turn the algebra into a labeled structure where homomorphisms

must respect the data. This models the real requirement that you cannot merge two portfolios with different risk scores.

Definition 3.13 (Plithogenic Quasigroup Parastrophes). Let $P(Q) = (Q, V, v_D, d, c, \cdot_P)$ be a plithogenic quasigroup over base quasigroup $(Q, \cdot)$.

A quasigroup $(Q, \cdot)$ has exactly six parastrophes determined by permuting the ternary relation $E(x, y, z) \iff x \cdot y = z$. They are:

1. $\cdot$: $x \cdot y = z$ (main operation)
2. $/$: $z/y = x$ (right division)
3. $\backslash$: $x \backslash z = y$ (left division)
4. $\circ$: $y \circ x = z$ (opposite)
5. $/$: $z/x = y$ (opposite right division)
6. $\parallel$: $y \parallel z = x$ (opposite left division)

The *plithogenic quasigroup parastrophes* of $P(Q)$ are the six operations

$$\cdot_P, \quad /_P, \quad \backslash_P, \quad \circ_P, \quad /_P, \quad \parallel_P$$

on $P(Q)$ defined by:

- (1) **Base parastrophes:** For each operation, the base elements satisfy the corresponding parastrophe of $(Q, \cdot)$. E.g., $x/_P y = z$ iff $x \cdot y = z$ in the base.
- (2) **Degree rule:** For all $v_i \in V$, the neutrosophic degrees are computed by

$$d(x *_P y, v_i) = (1 - c_i) t_N\{d(x, v_i), d(y, v_i)\} + c_i s_N\{d(x, v_i), d(y, v_i)\},$$

where $*_P \in \{\cdot_P, /_P, \backslash_P, \circ_P, /_P, \parallel_P\}$, subject to the relation $x *_P y = z$ being consistent with the base parastrophe.

Equivalently, the six operations are characterized by:

$$x \cdot_P y = z \iff z/_P y = x \iff x \backslash_P z = y \iff y \circ_P x = z \iff z/_P x = y \iff y \parallel_P z = x.$$

Theorem 3.14 (Closure of Parastrophes). If $(P(Q), \cdot_P)$ is a plithogenic quasigroup, then $(P(Q), *_P)$ is a plithogenic quasigroup for each parastrophe $*_P \in \{/_P, \backslash_P, \circ_P, /_P, \parallel_P\}$.

Proof. Each base parastrophe $(Q, *)$ is a quasigroup. The degree rule for $*_P$ is identical to $\cdot_P$, so solvability follows from Corollary 3.3 with x, y, z permuted. Uniqueness is inherited from the base. Hence $(P(Q), *_P)$ satisfies the Latin square property. $\square$

Theorem 3.15 (Isotopy Invariance of Plithogenic Structure). Let $(Q_1, \cdot_1)$ and $(Q_2, \cdot_2)$ be isotopic quasigroups via (α, β, γ) , i.e. $\alpha(x) \cdot_2 \beta(y) = \gamma(x \cdot_1 y)$ for bijections $\alpha, \beta, \gamma : Q_1 \rightarrow Q_2$.

Let $P(Q_1) = (Q_1, V, v_D, d_1, c, \cdot_{P_1})$ be a plithogenic quasigroup. Define $P(Q_2)$ with degrees $d_2(\alpha(x), v_i) = d_1(x, v_i)$ for all $x \in Q_1, v_i \in V$.

Then $P(Q_2) = (Q_2, V, v_D, d_2, c, \cdot_{P_2})$ is a plithogenic quasigroup, and the triple (α, β, γ) extends to a plithogenic isotopy $P(Q_1) \simeq P(Q_2)$.

Proof. The base is an isotopy by hypothesis. Degree preservation holds by construction. Since the aggregation rule uses only d and c , and c is fixed, $\gamma(x \cdot_{P_1} y) = \alpha(x) \cdot_{P_2} \beta(y)$. Latin square property follows from Theorem 3.2. $\square$

Theorem 3.16 (Lagrange-type Property for Finite Plithogenic Loops). *Let $P(Q)$ be a plithogenic quasigroup where $(Q, \cdot)$ is a finite loop of order n . Let $P(H)$ be a plithogenic subloop, i.e. $H \leq Q$ and $P(H)$ uses restrictions of d, c .*

Then $|H|$ divides n , and for each $v_i \in V$, the multiset of degrees $\{d(x, v_i) : x \in H\}$ is determined by $|H|$ and d .

Remark 3.17. Unlike groups, $P(H)$ need not partition $P(Q)$. The theorem only restricts $|H|$, not the degree distribution. This is why portfolio constraints must be checked via Corollary 3.3, not just by size.

Theorem 3.18 (Automorphism Group Structure). *Let $\text{Aut}(P(Q)) = \{\phi : P(Q) \rightarrow P(Q) \text{ plithogenic isomorphism}\}$. Then*

$$\text{Aut}(P(Q)) \cong \{\sigma \in \text{Aut}(Q, \cdot) : d(\sigma(x), v_i) = d(x, v_i) \forall x \in Q, v_i \in V\}.$$

That is, automorphisms are exactly base automorphisms that fix all neutrosophic degree vectors.

Proof. By Def. 3.6, any ϕ must preserve base and degrees. Conversely, any base automorphism σ preserving d extends uniquely to $\phi(x) = \sigma(x)$ with same degrees, and ϕ respects $\cdot_P$ because the aggregation rule depends only on d, c . $\square$

Theorem 3.19 (Cancellation and Idempotence Criteria). *Let $P(Q)$ be a plithogenic quasigroup. Let $a \in P(Q)$, then*

- (1) $a \cdot_P x = a \cdot_P y \Rightarrow x = y$ and $x \cdot_P a = y \cdot_P a \Rightarrow x = y$ (cancellation);
- (2) $a \cdot_P a = a$ iff $a \cdot a = a$ in $(Q, \cdot)$, since $d(a \cdot_P a, v_i) = d(a, v_i)$ by idempotence of t_N, s_N .

Hence, if $c_i = 0$ for all i , then $\cdot_P$ is idempotent iff $\cdot$ is idempotent. If $c_i = 1$ for all i , the same holds.

Proof. (1) follows from the Latin square property in Thm. 3.2.

For (2), compute degrees: $(1 - c_i)t_N(t, t) + c_i s_N(t, t) = (1 - c_i)t + c_i t = t$, similarly for i, f . Therefore, $d(a \cdot_P a) = d(a)$. So idempotence depends only on base value. $\square$

4. Application: Using Plithogenic Quasigroups as a tool for decision making

We now use Plithogenic Quasigroups properties to model a portfolio manager choosing between three stocks under conflicting criteria. The plithogenic quasigroup structure captures both the algebraic flexibility of stocks and the neutrosophic uncertainty of each attribute.

4.1. Setup: Stocks as Elements of $P(Q)$

Let $Q = \{e, a, b\}$ represent three candidate stocks:

$$e = A, \quad a = M, \quad b = T.$$

$(Q, \cdot)$ is the trivial quasigroup where $x \cdot y$ denotes replace x by y given by the Latin square in Example 3.4. This encodes the decision action: selecting a stock and then switching to another.

Let attributes $V = \{\text{Growth, Safety, Liquidity}\}$ with dominant attribute $v_D = \text{Growth}$. Contradiction degrees reflect analyst policy:

$$c_{\text{Growth}} = 0, \quad c_{\text{Safety}} = 0.4, \quad c_{\text{Liquidity}} = 0.2.$$

$c = 0$ for Growth means we require consensus: a portfolio switch only has high Growth if both stocks rate high. $c = 0.4$ for Safety means we tolerate conflict: the safer stock can dominate.

Expert evaluations give neutrosophic degrees $d(x, v_i) = (t, i, f)$:

Stock	Growth	Safety	Liquidity
$x_1 = A$	(0.90, 0.10, 0.00)	(0.80, 0.10, 0.10)	(0.95, 0.05, 0.00)
$x_2 = M$	(0.60, 0.30, 0.10)	(0.90, 0.10, 0.00)	(0.85, 0.10, 0.05)
$x_3 = T$	(0.30, 0.40, 0.30)	(0.50, 0.30, 0.20)	(0.70, 0.20, 0.10)

This defines the plithogenic quasigroup $P(Q) = (Q, V, v_D, d, c, \cdot_P)$.

4.2. Plithogenic Product as Strategy Composition for Decision Rule

The product $x_i \cdot_P x_j$ models the strategy hold x_i , then switch to x_j . By Theorem 3.2, the result is a unique portfolio with aggregated degrees.

Example: Manager holds A and considers switching to M:

$$x_1 \cdot_P x_2 = M \text{ with degrees } \begin{cases} G : \min\{(0.9, 0.1, 0.0), (0.6, 0.3, 0.1)\} = (0.6, 0.3, 0.1) \\ S : 0.6 \min + 0.4 \max = (0.86, 0.10, 0.04) \\ L : 0.8 \min + 0.2 \max = (0.87, 0.09, 0.04) \end{cases}$$

Interpretation: Growth reduces to Ms level because $c_G = 0$ enforces consensus. Safety increases because $c_S = 0.4$ lets Ms higher Safety dominate.

4.3. Solving for a Target Portfolio

Suppose the investor requires a target profile $b = \text{Target}$ with

$$d(b, G) = (0.60, 0.30, 0.10), \quad d(b, S) = (0.86, 0.10, 0.04).$$

If the current holding is $x_1 = A$, we can solve for $x_1 \cdot_P x = b$ using Corollary 3.3. Table 2 shows the uniqueness of the solution $x = x_2 = M$. So M is the unique switch that meets the target criteria.

If no parameter x_i solves the equation, then the target can not be determined under current parameter c . The manager must either change c_i or varies the target, which is exactly Remark 3.5: c_i is the risk dial.

Conclusion. In this work, we were able to combine plithogenic properties and quasigroups properties to provide a framework where the policy c_i , uncertainty (t, i, f) , and algebra $(\cdot, /, \backslash)$ are inseparable primitives for making decisions that involves uncertainty.

5. Future Work

With the extension of the theory Neutrosophic sets to plithogenic quasigroups, it opens several research directions to other algebraic structures. Future works can aim to unify plithogeny with the major non-associative algebraic theories.

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A. Oyem, E. K. Osawaru, Plithogenic Quasigroups with Application to Decision Making

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