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Envelope Geometry and Information Refinement of (T,I,N,F) Neutrosophic Quadruples

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Abstract. The (T,I,N,F) neutrosophic framework separates uncertainty into four components: truth T, pure indeterminacy I, neutrality N, and falsehood F. Each component may be an arbitrary subset of the unit interval, and the four components need not satisfy a normalization condition. This paper develops a structural theory of such quadruples based exclusively on their lower and upper set bounds. For a (T,I,N,F) quadruple, we introduce its lower envelope, upper envelope, envelope width, and envelope center. We establish an exact decomposition of the total envelope width into four componentwise widths and characterize crisp quadruples as precisely those having zero envelope width. An exact geometric representation of all attainable envelope-center–width pairs is derived. We further introduce a componentwise refinement relation and prove that refinement monotonically contracts the global envelope. An envelope-equivalence relation is proposed to identify different set-valued quadruples having identical boundary information. Finally, we show that marginal envelope information alone cannot determine whether the four neutrosophic components are independent, partially dependent, or totally dependent. These results provide a simple mathematical foundation for studying set-valued (T,I,N,F) information without imposing additional operators or normalization assumptions.

Keywords: neutrosophic quadruple; truth; pure indeterminacy; neutrality; falsehood; set-valued uncertainty; envelope width; information refinement; dependence.

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1. Introduction

Fuzzy sets and intuitionistic fuzzy sets provide foundational frameworks for representing graded membership, nonmembership, and hesitation [1, 2]. Interval-valued extensions subsequently replaced exact membership grades with bounded ranges, thereby preserving numerical imprecision explicitly [3, 4]. Neutrosophic sets further generalize these frameworks by representing information through truth, indeterminacy, and falsehood components [5]. Their interval-valued and single-valued realizations were developed in [6, 7].

The refined neutrosophic framework allows the original neutrosophic components to be decomposed into semantically distinct subcomponents [8]. Several four-component formulations have consequently appeared in the literature, including quadripartitioned single-valued neutrosophic sets [9], four-valued refined neutrosophic sets [10], and interval quadripartitioned neutrosophic sets [11]. These models commonly divide indeterminacy into components such as contradiction and unknownness or uncertainty.

The (T,I,N,F) representation studied in this paper employs a different semantic decomposition. It extends the classical three-component neutrosophic description by distinguishing *pure indeterminacy* from *neutrality* [12]. In this representation, an assessment is expressed as (T, I, N, F), where T denotes truth, I pure indeterminacy excluding neutrality, N neutrality, and F falsehood. Each component is taken to be a nonempty subset of the unit interval:

$$T, I, N, F \in \mathcal{P}([0,1]) \setminus \{\varnothing\}.$$

Consequently, the corresponding lower and upper values are well defined. The basic admissibility condition is

$$0 \leq \inf(T) + \inf(I) + \inf(N) + \inf(F) \leq \sup(T) + \sup(I) + \sup(N) + \sup(F) \leq 4.$$

Furthermore, the four components may be mutually independent, partially dependent, or fully dependent [12, 13].

This generality is one of the principal mathematical features of the quadruple representation. Unlike normalized fuzzy and intuitionistic fuzzy models, the total amount represented by T, I, N, F is not required to equal one. Incomplete information, complete information, and overdetermined or inconsistent information can therefore be represented within a common framework. Moreover, because each component may itself be a set rather than a single number, uncertainty occurs at two distinct levels: uncertainty arising from the four semantic components and uncertainty concerning the numerical values assigned to those components.

Representative studies of interval neutrosophic sets have developed set-theoretic operations, topological constructions, distance and similarity measures, aggregation operators, and multicriteria decision-making methods [6, 14, 15, 16]. Related interval-based constructions have also been investigated for quadripartitioned neutrosophic sets [11]. These studies demonstrate the usefulness of lower and upper component values, but they do not by themselves provide a structural analysis of the aggregate lower–upper envelope of a set-valued (T,I,N,F) quadruple.

In addition, marginal lower and upper bounds and the dependence structure among components describe different aspects of uncertainty. Dependence among neutrosophic components has been studied explicitly [13]; nevertheless, marginal endpoint information alone does not generally determine how the components depend on one another. This distinction motivates a dependence-agnostic analysis based exclusively on componentwise bounds.

The purpose of the present work is therefore to study the numerical set-valued uncertainty of the (T,I,N,F) representation directly. Instead of introducing additional logical operators, aggregation functions, distance measures, or ranking procedures, we investigate what can be deduced solely from $T, I, N, F \subseteq [0,1]$ and their lower and upper bounds.

The principal contributions of this paper are:

(1) the introduction of lower and upper envelopes for a (T,I,N,F) quadruple;

(2) an additive decomposition of the total envelope width into four componentwise widths;

(3) a characterization of single-valued quadruples through zero envelope width;

(4) an exact geometric characterization of feasible center–width pairs;

(5) the introduction of an information-refinement relation;

(6) a proof that information refinement contracts the uncertainty envelope;

(7) the introduction of envelope-equivalence classes; and

(8) a demonstration that marginal envelope information alone cannot identify the dependence structure among T, I, N, F.

Thus, the paper develops a dependence-agnostic structural theory directly from the set-valued (T,I,N,F) formulation. It complements existing operator-, similarity-, and decision-oriented neutrosophic studies by concentrating on the geometry, refinement, and information content of the componentwise uncertainty envelopes.

2. Basic (T,I,N,F) Framework

For mathematical consistency, throughout this article the four components are assumed to be nonempty subsets of $[0,1]$, so that their infima and suprema are well defined.

Definition 2.1. A *set-valued (T,I,N,F) neutrosophic quadruple* is $Q = (T, I, N, F)$, where $\varnothing \neq T, I, N, F \subseteq [0,1]$. Write

$$t^- = \inf(T), \qquad t^+ = \sup(T),$$

$$i^- = \inf(I), \qquad i^+ = \sup(I),$$

$$n^- = \inf(N), \qquad n^+ = \sup(N),$$

and

$$f^- = \inf(F), \qquad f^+ = \sup(F).$$

Since every component is contained in $[0,1]$, $0 \leq x^- \leq x^+ \leq 1$ for $x \in \{t, i, n, f\}$. Hence, the original quadruple condition becomes

$$0 \leq t^- + i^- + n^- + f^- \leq t^+ + i^+ + n^+ + f^+ \leq 4.$$

Definition 2.2. The *lower envelope* of Q is $L(Q) = t^- + i^- + n^- + f^-$, whereas its *upper envelope* is $U(Q) = t^+ + i^+ + n^+ + f^+$. Hence, the global envelope associated with Q is

$$\mathcal{E}(Q) = [L(Q), U(Q)].$$

Immediately,

$$0 \leq L(Q) \leq U(Q) \leq 4.$$

3. **Envelope Width of a Neutrosophic Quadruple**

Definition 3.1. The *envelope width* of Q is defined by $D(Q) = U(Q) - L(Q)$. For each component define

$$d_T = t^+ - t^-, \qquad d_I = i^+ - i^-, \qquad d_N = n^+ - n^-, \qquad d_F = f^+ - f^-.$$

Theorem 3.2. *For every set-valued (T,I,N,F) quadruple,* $D(Q) = d_T + d_I + d_N + d_F$.

Proof. By definition,

$$\begin{aligned} D(Q) &= U(Q) - L(Q) \\ &= (t^+ + i^+ + n^+ + f^+) - (t^- + i^- + n^- + f^-). \end{aligned}$$

Rearranging gives

$$D(Q) = (t^+ - t^-) + (i^+ - i^-) + (n^+ - n^-) + (f^+ - f^-).$$

Therefore,

$$D(Q) = d_T + d_I + d_N + d_F.$$

□

Corollary 3.3. *For every (T,I,N,F) quadruple,* $\boxed{0 \leq D(Q) \leq 4.}$

Proof. Since every component is contained in $[0, 1]$,

$$0 \leq d_T, d_I, d_N, d_F \leq 1.$$

The result follows from the Theorem 3.2. □

The maximum value $D(Q) = 4$ is attained when all four components have lower bound 0 and upper bound 1.

4. **Characterization of Single-Valued Quadruples**

Definition 4.1. A quadruple $Q = (T, I, N, F)$ is called *crisp with respect to component bounds* when every component is a singleton:

$$T = \{t\}, \qquad I = \{i\}, \qquad N = \{n\}, \qquad F = \{f\}.$$

Theorem 4.2. *A (T,I,N,F) quadruple is crisp if and only if* $D(Q) = 0$.

Proof. Suppose first that Q is crisp. Then,

$$t^- = t^+, \quad i^- = i^+, \quad n^- = n^+, \quad f^- = f^+.$$

Thus,

$$d_T = d_I = d_N = d_F = 0.$$

Hence,

$$D(Q) = 0.$$

Conversely, suppose $D(Q) = 0$. By the Theorem 3.2,

$$0 = d_T + d_I + d_N + d_F.$$

Since each component width is nonnegative,

$$d_T = d_I = d_N = d_F = 0.$$

Therefore,

$$\inf(T) = \sup(T),$$

and similarly for I, N, F. Since each component is nonempty, every component must be a singleton. □

Remark 4.3. The previous theorem gives the envelope width a direct interpretation: positive width represents genuinely set-valued boundary uncertainty, whereas zero width identifies the single-valued case.

5. Center–Width Geometry

Definition 5.1. Define the *envelope center* by $C(Q) = \frac{L(Q)+U(Q)}{2}$. Then $L(Q) = C(Q) - \frac{D(Q)}{2}$ and $U(Q) = C(Q) + \frac{D(Q)}{2}$.

Theorem 5.2 (Envelope geometry)**.** *Every (T,I,N,F) quadruple satisfies*

$$0 \le C(Q) \le 4 \text{ and } 0 \le D(Q) \le 2\min\{C(Q), 4 - C(Q)\}.$$

Proof. Since

$$L(Q) = C(Q) - \frac{D(Q)}{2} \ge 0,$$

we obtain

$$D(Q) \le 2C(Q).$$

Similarly,

$$U(Q) = C(Q) + \frac{D(Q)}{2} \le 4$$

implies

$$D(Q) \le 2(4 - C(Q)).$$

Therefore,

$$D(Q) \le 2\min\{C(Q), 4 - C(Q)\}.$$

Since $D(Q) \ge 0$, the result follows. □

Thus, the possible (C, D) pairs occupy the region

$$\mathcal{R} = \{(C, D) : 0 \le C \le 4,\ 0 \le D \le 2\min(C, 4 - C)\}.$$

Its upper boundary is

$$D = 2C, \qquad 0 \le C \le 2,$$

and

$$D = 8 - 2C, \qquad 2 \le C \le 4.$$

Theorem 5.3 (Realizability)**.** *Every pair* $(C, D) \in \mathcal{R}$ *can be realized by at least one (T,I,N,F) quadruple.*

Proof. Let

$$L = C - \frac{D}{2}, \qquad U = C + \frac{D}{2}.$$

Since $(C, D) \in \mathcal{R}$,

$$0 \leq L \leq U \leq 4.$$

Choose $l_1, l_2, l_3, l_4 \in [0, 1]$ such that

$$l_1 + l_2 + l_3 + l_4 = L.$$

This is possible because $0 \leq L \leq 4$. Since

$$U - L \leq 4 - L = \sum_{j=1}^{4}(1 - l_j),$$

there exist numbers δ_j satisfying

$$0 \leq \delta_j \leq 1 - l_j \text{ and } \sum_{j=1}^{4} \delta_j = U - L.$$

Set

$$u_j = l_j + \delta_j.$$

Now, we define

$$T = [l_1, u_1], \quad I = [l_2, u_2], \quad N = [l_3, u_3], \quad F = [l_4, u_4].$$

Then,

$$L(Q) = L, \quad U(Q) = U.$$

Therefore,

$$C(Q) = C, \quad D(Q) = D.$$

Hence, the region in the previous theorem gives the exact feasible geometry. □

6. Componentwise Information Refinement

Definition 6.1. Let $Q = (T, I, N, F)$ and $Q' = (T', I', N', F')$. We say that Q' is a *componentwise refinement* of Q, written $Q' \preceq_r Q$, if

$$T' \subseteq T, \qquad I' \subseteq I, \qquad N' \subseteq N, \qquad F' \subseteq F,$$

with all refined components nonempty.

Theorem 6.2 (Refinement contraction)**.** *If* $Q' \preceq_r Q$, *then* $(Q') \geq L(Q)$ *and* $U(Q') \leq U(Q)$*. Consequently,* $D(Q') \leq D(Q)$.

Proof. Since $T' \subseteq T$,

$$\inf(T') \geq \inf(T) \text{ and } \sup(T') \leq \sup(T).$$

The same inequalities hold for I, N, F. Summing the four lower-bound inequalities gives

$$L(Q') \geq L(Q),$$

while summing the upper-bound inequalities gives

$$U(Q') \leq U(Q).$$

Therefore,

$$\begin{aligned} D(Q') &= U(Q') - L(Q') \\ &\leq U(Q) - L(Q) \\ &= D(Q). \end{aligned}$$

Thus,

$$Q' \preceq_r Q \quad \Longrightarrow \quad \mathcal{E}(Q') \subseteq \mathcal{E}(Q).$$

□

7. Envelope Equivalence

Definition 7.1. Two quadruples $Q_1 = (T_1, I_1, N_1, F_1)$ and $Q_2 = (T_2, I_2, N_2, F_2)$ are called *envelope-equivalent*, written $Q_1 \sim_E Q_2$, whenever $\inf(X_1) = \inf(X_2)$ and $\sup(X_1) = \sup(X_2)$ for every corresponding component $X \in \{T, I, N, F\}$.

Proposition 7.2. *If $Q_1 \sim_E Q_2$, then*

$$L(Q_1) = L(Q_2), \quad U(Q_1) = U(Q_2), \; C(Q_1) = C(Q_2), \quad D(Q_1) = D(Q_2).$$

Proof. All four quantities depend only on the componentwise infima and suprema, which are equal by hypothesis. □

Remark 7.3. The converse need not hold. Equal global envelopes do not necessarily imply equality of all componentwise endpoints, since different component contributions may compensate one another.

8. Dependence and a Non-Identifiability Principle

The fundamental (T,I,N,F) framework explicitly allows the components to be totally independent, partially independent and partially dependent, or totally dependent [12]. However, the four marginal sets

$$T, \qquad I, \qquad N, \qquad F$$

do not by themselves specify how dependence is represented.

Proposition 8.1 (Dependence non-identifiability)**.** *Suppose two (T,I,N,F) descriptions have identical marginal component sets but are assigned different admissible dependence structures. Then their lower envelope, upper envelope, center, component widths, and total envelope width are identical.*

Proof. Every quantity introduced in this paper is a function only of

$$\inf(T), \sup(T), \inf(I), \sup(I), \inf(N), \sup(N), \inf(F), \sup(F).$$

None of these quantities contains an explicit dependence term. Hence changing a dependence structure while preserving the four marginal sets does not alter the envelope quantities.

Therefore, the following two notions must be distinguished:

$$\text{marginal set-valued uncertainty and dependence among } T, I, N, F.$$

□

9. Illustrative Examples

Example 9.1 (Single-valued quadruple)**.** Consider $Q_1 = (\{0.70\}, \{0.10\}, \{0.15\}, \{0.05\})$. Then, $L(Q_1) = 0.70 + 0.10 + 0.15 + 0.05 = 1$, and $U(Q_1) = 1$. Therefore, $D(Q_1) = 0$. Hence, Q_1 is crisp.

Example 9.2 (Interval-valued quadruple)**.** Let $T = [0.6, 0.8], \quad I = [0.1, 0.2], \; N = [0.2, 0.4], \quad F = [0, 0.1]$. Then,

$$L(Q_2) = 0.6 + 0.1 + 0.2 + 0 = 0.9,$$

while

$$U(Q_2) = 0.8 + 0.2 + 0.4 + 0.1 = 1.5.$$

Hence,

$$D(Q_2) = 1.5 - 0.9 = 0.6, \text{ and } C(Q_2) = \frac{0.9 + 1.5}{2} = 1.2.$$

Moreover,

$$d_T = 0.2, \quad d_I = 0.1, \quad d_N = 0.2, \quad d_F = 0.1,$$

so that

$$D(Q_2) = 0.2 + 0.1 + 0.2 + 0.1 = 0.6.$$

Example 9.3 (Refinement)**.** Consider the refinement $T' = [0.65, 0.75], \quad I' = [0.12, 0.18], \quad N' = [0.25, 0.35], \quad F' = [0.02, 0.08]$. Each new component is contained in the corresponding component of the previous example. Hence,

$$Q_3 \preceq_r Q_2.$$

We obtain

$$L(Q_3) = 0.65 + 0.12 + 0.25 + 0.02 = 1.04, \text{ and } U(Q_3) = 0.75 + 0.18 + 0.35 + 0.08 = 1.36.$$

Thus,

$$D(Q_3) = 0.32 < 0.60 = D(Q_2).$$

Example 9.4 (Different sets with the same envelope). Let $T_1 = [0.2, 0.8]$ $T_2 = \{0.2, 0.4, 0.8\}$. Clearly, $T_1 \neq T_2$, but

$$\inf(T_1) = \inf(T_2) = 0.2 \text{ and } \sup(T_1) = \sup(T_2) = 0.8.$$

If the other three components are unchanged, the resulting quadruples are envelope-equivalent even though their truth components have different internal structures.

10. Discussion

Separating indeterminacy from neutrality produces four semantic dimensions, T, I, N, F, rather than three. Because each dimension may itself be an arbitrary subset of $[0, 1]$, a (T,I,N,F) quadruple contains considerably more information than four scalar values.

The analysis developed here reveals several structural facts. First, the basic bound $0 \leq L(Q) \leq U(Q) \leq 4$ naturally generates an interval $[L(Q), U(Q)]$ describing the aggregate boundary range of the quadruple.

Second, the width of this interval is exactly additive: $D(Q) = d_T + d_I + d_N + d_F$. Thus, the spreads of truth, pure indeterminacy, neutrality, and falsehood remain individually visible while contributing to an overall boundary spread.

Third, information refinement has a precise effect:

$$Q' \preceq_r Q \quad \Longrightarrow \quad D(Q') \leq D(Q).$$

Thus, reducing the admissible values of every component cannot increase the global envelope width.

Fourth, the center–width representation produces an exact global geometry:

$$0 \leq D \leq 2\min(C, 4 - C).$$

Finally, neither this envelope nor the individual component bounds determine the dependence relationship among T, I, N, F. Dependence therefore represents additional structural information that is not recoverable solely from marginal set boundaries.

11. Conclusion

This paper developed a structural extension of the basic (T,I,N,F) neutrosophic quadruple using only the assumptions $T, I, N, F \subseteq [0, 1]$ and $0 \leq \inf(T)+\inf(I)+\inf(N)+\inf(F) \leq \sup(T)+\sup(I)+\sup(N)+\sup(F) \leq 4$. The lower and upper envelopes $L(Q)$ and $U(Q)$ were introduced, together with the global envelope width $D(Q) = U(Q) - L(Q)$. It was proved that $D(Q) = d_T + d_I + d_N + d_F$, and consequently $0 \leq D(Q) \leq 4$. Moreover, $D(Q) = 0$ was shown to characterize exactly the single-valued case. Using the envelope center $C(Q) = \frac{L(Q)+U(Q)}{2}$, an exact feasible geometry was obtained: $0 \leq D(Q) \leq 2\min\{C(Q), 4 - C(Q)\}$.

A componentwise refinement relation was also introduced and shown to satisfy

$$Q' \preceq_r Q \quad \Longrightarrow \quad [L(Q'), U(Q')] \subseteq [L(Q), U(Q)],$$

and therefore

$$D(Q') \leq D(Q).$$

Envelope equivalence was used to distinguish boundary-identical quadruples from genuinely identical set-valued quadruples. Finally, the dependence non-identifiability principle shows that whether T, I, N, F are independent, partially dependent, or totally dependent cannot be inferred solely from their marginal envelopes.

These findings show that even the elementary (T,I,N,F) formulation possesses a rich mathematical structure. The envelope, refinement, equivalence, and dependence questions developed here provide foundations for a more extensive mathematical theory of quadruple-valued neutrosophic information.

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