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Implicative falling neutrosophic ideals of BCK-algebras

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Abstract: The notions of an implicative $(\in, \in)$-neutrosophic ideal and an implicative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of an implicative $(\in, \in)$-neutrosophic ideal are considered, and relations between an implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be an implicative $(\in, \in)$-neutrosophic ideal are provided, and relations between an implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and an implicative falling neutrosophic ideal are studied. Conditions for a falling neutrosophic ideal to be implicative are provided. Relations between implicative falling neutrosophic ideal, commutative falling neutrosophic ideal and positive implicative falling neutrosophic ideal are discussed.

Keywords: neutrosophic random set; neutrosophic falling shadow; (positive implicative) $(\in, \in)$-neutrosophic ideal; (positive implicative) falling neutrosophic ideal; (commutative) $(\in, \in)$-neutrosophic ideal; (commutative) falling neutrosophic ideal; (implicative) $(\in, \in)$-neutrosophic ideal; (implicative) falling neutrosophic ideal.

1 Introduction

The fuzzy set was introduced by L.A. Zadeh in 1965, where each element had a degree of membership. As a generalization of fuzzy set, the intuitionistic fuzzy set on a universe X was introduced by K. Atanassov in 1983, where besides the degree of membership $\mu_A(x) \in [0, 1]$ of each element $x \in X$ to a set A there was considered a degree of non-membership $\nu_A(x) \in [0, 1]$, but such that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. Neutrosophic set (NS) developed by Smarandache [19, 20, 21] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun et al. studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [3], [7] [16] and [18]). It is a reasonable and convenient approach for the theoretical development and the practical applications of neutrosophic sets and neutrosophic logics. Jun et al. [10] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal (see [9], [11], and [13]).Jun et al. [12] introduced the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. Bordbar et al. [2] introduced the concepts of a positive implicative $(\in, \in)$-neutrosophic ideal and a positive implicative falling neutrosophic ideal, and investigate several properties.

In this paper, we introduce the concepts of an implicative $(\in, \in)$-neutrosophic ideal and an implicative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of an implicative $(\in, \in)$-neutrosophic ideal, and discuss relations between an implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be an implicative $(\in, \in)$-neutrosophic ideal, and consider relations between an implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and an implicative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be implicative. We consider relations between implicative falling neutrosophic ideal, commutative falling neutrosophic ideal and positive implicative falling neutrosophic ideal.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [5] and [6]).

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *positive implicative* if the following assertion is valid.

$$(\forall x, y, z \in X)\left((x * z) * (y * z) = (x * y) * z\right). \tag{2.5}$$

A BCK-algebra X is said to be *implicative* if the following assertion is valid.

$$(\forall x, y \in X)\left(x = x * (y * x)\right). \tag{2.6}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.7}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I). \tag{2.8}$$

A subset I of a BCK-algebra X is called a *commutative ideal* (see [15]) of X if it satisfies (2.7) and

$$(\forall x, y \in X)(\forall z \in I)\left((x * y) * z \in I \;\Rightarrow\; x * (y * (y * x)) \in I\right). \tag{2.9}$$

Observe that every commutative ideal is an ideal, but the converse is not true (see [15]).

A subset I of a BCK-algebra X is called a *positive implicative ideal* (see [15]) of X if it satisfies (2.7) and

$$(\forall x, y, z \in X)\left(((x * y) * z \in I,\, y * z \in I \;\Rightarrow\; x * z \in I\right). \tag{2.10}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [15]).

A subset I of a BCK-algebra X is called an *implicative ideal* (see [15]) of X if it satisfies (2.7) and

$$(\forall x, y, z \in X)\left((x * (y * x)) * z \in I,\, z \in I \;\Rightarrow\; x \in I\right). \tag{2.11}$$

Observe that every implicative ideal is an ideal, but the converse is not true (see [15]).

We refer the reader to the books [4, 15] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [20]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned} T_\in(A_\sim; \alpha) &:= \{x \in X \mid A_T(x) \geq \alpha\}, \\ I_\in(A_\sim; \beta) &:= \{x \in X \mid A_I(x) \geq \beta\}, \\ F_\in(A_\sim; \gamma) &:= \{x \in X \mid A_F(x) \leq \gamma\}. \end{aligned}$$

We say $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are *neutrosophic $\in$-subsets*.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic subalgebra* of X (see [7]) if the following assertions are valid.

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x * y \in T_\in(A_\sim; \alpha_x \wedge \alpha_y), \\ x \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x * y \in I_\in(A_\sim; \beta_x \wedge \beta_y), \\ x \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x * y \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \right) \tag{2.12}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic ideal* of X (see [18]) if the following assertions are valid.

$$(\forall x \in X) \left(\begin{array}{l} x \in T_\in(A_\sim; \alpha_x) \ \Rightarrow\ 0 \in T_\in(A_\sim; \alpha_x) \\ x \in I_\in(A_\sim; \beta_x) \ \Rightarrow\ 0 \in I_\in(A_\sim; \beta_x) \\ x \in F_\in(A_\sim; \gamma_x) \ \Rightarrow\ 0 \in F_\in(A_\sim; \gamma_x) \end{array} \right) \tag{2.13}$$

and

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in T_\in(A_\sim; \alpha_x),\ y \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A_\sim; \beta_x),\ y \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ x * y \in F_\in(A_\sim; \gamma_x),\ y \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \right) \tag{2.14}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.15}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.16}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a *neutrosophic* *random set* on X (see [10]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \left(\begin{array}{l} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{array} \right). \tag{2.17}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a *neutrosophic falling shadow* (see [10]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [10]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [10]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of X (see [10]).

3 Implicative $(\in, \in)$-neutrosophic ideals

Definition 3.1. A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X is called an *implicative* $(\in, \in)$*-neutrosophic ideal* of X if it satisfies the condition (2.13) and

$$\begin{array}{l} (x * (y * x)) * z \in T_\in(A_\sim; \alpha_x),\ z \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ (x * (y * x)) * z \in I_\in(A_\sim; \beta_x),\ z \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ (x * (y * x)) * z \in F_\in(A_\sim; \gamma_x),\ z \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Example 3.2. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$(X; *, 0)$ is a BCK-algebra (see [15]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2.

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.6	0.1
1	0.7	0.6	0.1
2	0.7	0.6	0.1
3	0.5	0.2	0.6
4	0.3	0.4	0.9

Routine calculations show that $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X.

Theorem 3.3. *Every implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. It is clear by substituting x for y in (3.1) and using (2.1). □

Corollary 3.4. *Every implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X.*

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.5. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3. Then $(X; *, 0)$ is a BCK-algebra (see [15]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4. It is routine to verify that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X, but it is not an implicative $(\in, \in)$-neutrosophic ideal of X since

$$(1 * (3 * 1)) * 2 = 0 \in T_\in(A_\sim; 0.6) \text{ and } 2 \in T_\in(A_\sim; 0.65)$$

but $1 \notin T_\in(A_\sim; 0.6 \wedge 0.65) = T_\in(A_\sim; 0.6)$, and/or

$$(1 * (3 * 1)) * 2 = 0 \in F_\in(A_\sim; 0.35) \text{ and } 2 \in F_\in(A_\sim; 0.45),$$

but $1 \notin F_\in(A_\sim; 0.45) = F_\in(A_\sim; 0.35 \vee 0.45)$.

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Table 4: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.8	0.3
1	0.5	0.6	0.5
2	0.7	0.4	0.4
3	0.5	0.2	0.9
4	0.5	0.2	0.9

Theorem 3.6. *For a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\ A_T(0) \geq A_T(x),\, A_I(0) \geq A_I(x),\, A_F(0) \leq A_F(x)\ \right) \tag{3.2}$$

and

$$(\forall x, y, z \in X)\left(\begin{array}{l} A_T(x) \geq A_T((x*(y*x))*z) \wedge A_T(z) \\ A_I(x) \geq A_I((x*(y*x))*z) \wedge A_I(z) \\ A_F(x) \leq A_F((x*(y*x))*z) \vee A_F(z) \end{array}\right) \tag{3.3}$$

Proof. Assume that the non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A_\sim;A_T(a))$ and $0 \notin T_\in(A_\sim;A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly, $A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_\in(A_\sim;A_F(a))$ and $0 \notin F_\in(A_\sim;A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a) < A_T((a*(b*a))*c) \wedge A_T(c).$$

Taking $\alpha := A_T((a*(b*a))*c) \wedge A_T(c)$ implies that $(a*(b*a))*c \in T_\in(A_\sim;\alpha)$ and $c \in T_\in(A_\sim;\alpha)$ but

$a \notin T_\in(A_\sim;\alpha)$, which is a contradiction. Hence

$$A_T(x) \geq A_T((x*(y*x))*z) \wedge A_T(z)$$

for all $x,y,z \in X$. By the similar way, we can verify that

$$A_I(x) \geq A_I((x*(y*x))*z) \wedge A_I(z)$$

for all $x,y,z \in X$. Now suppose there are $x,y,z \in X$ such that

$$A_F(x) > A_F((x*(y*x))*z) \vee A_F(z) := \gamma.$$

Then $(x*(y*x))*z \in F_\in(A_\sim;\gamma)$ and $z \in F_\in(A_\sim;\gamma)$ but $x \notin F_\in(A_\sim;\gamma)$, a contradiction. Thus

$$A_F(x) \leq A_F((x*(y*x))*z) \vee A_F(z)$$

for all $x,y,z \in X$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying two conditions (3.2) and (3.3). Assume that $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty for $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. Let $x \in T_\in(A_\sim;\alpha)$, $a \in I_\in(A_\sim;\beta)$ and $u \in F_\in(A_\sim;\gamma)$ for $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_\in(A_\sim;\alpha)$, $0 \in I_\in(A_\sim;\beta)$ and $0 \in F_\in(A_\sim;\gamma)$. Let $a,b,c \in X$ be such that $(a*(b*a))*c \in T_\in(A_\sim;\alpha)$ and $c \in T_\in(A_\sim;\alpha)$ for $\alpha \in (0,1]$. Then

$$A_T(a) \geq A_T((a*(b*a))*c) \wedge A_T(c) \geq \alpha$$

by (3.3), and so $a \in T_\in(A_\sim;\alpha)$. If $(x*(y*x))*z \in I_\in(A_\sim;\beta)$ and $z \in I_\in(A_\sim;\beta)$ for all $x,y,z \in X$ and $\beta \in (0,1]$, then $A_I((x*(y*x))*z) \geq \beta$ and $A_I(z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x) \geq A_I((x*(y*x))*z) \wedge A_I(z) \geq \beta,$$

that is, $x \in I_\in(A_\sim;\beta)$. Finally, suppose that $(x*(y*x))*z \in F_\in(A_\sim;\gamma)$ and $z \in F_\in(A_\sim;\gamma)$ for all $x,y,z \in X$ and $\gamma \in (0,1]$. Then $A_F((x*(y*x))*z) \leq \gamma$ and $A_F(z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x) \leq A_F((x*(y*x))*z) \vee A_F(z) \leq \gamma.$$

Hence $x \in F_\in(A_\sim;\gamma)$. Therefore the non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. □

Theorem 3.7. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A_\sim = (A_T, A_I, A_F)$ is a implicative $(\in,\in)$-neutrosophic ideal of X if and only if the non-empty neutrosophic $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are implicative ideals of X for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be an implicative $(\in,\in)$-neutrosophic ideal of X and assume that $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty for $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$. Then there exist $x,y,z \in X$ such that $x \in T_\in(A_\sim;\alpha)$, $y \in I_\in(A_\sim;\beta)$ and $z \in F_\in(A_\sim;\gamma)$. It follows from (2.13) that $0 \in T_\in(A_\sim;\alpha)$, $0 \in I_\in(A_\sim;\beta)$ and $0 \in F_\in(A_\sim;\gamma)$. Let $x,y,z,a,b,c,u,v,w \in X$ be such that $(x*(y*x))*z \in T_\in(A_\sim;\alpha)$, $z \in T_\in(A_\sim;\alpha)$, $(a*(b*a))*c \in I_\in(A_\sim;\beta)$, $c \in I_\in(A_\sim;\beta)$, $(u*(v*u))*w \in F_\in(A_\sim;\gamma)$ and $w \in F_\in(A_\sim;\gamma)$. Then

$x \in T_\in(A_\sim; \alpha \wedge \alpha) = T_\in(A_\sim; \alpha)$, $a \in I_\in(A_\sim; \beta \wedge \beta) = I_\in(A_\sim; \beta)$, and $u \in F_\in(A_\sim; \gamma \vee \gamma) = F_\in(A_\sim; \gamma)$ by (3.1). Hence the non-empty neutrosophic $\in$-subsets $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are nonempty and are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Obviously, (2.13) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0, 1]$ be such that $(x * (y * x)) * z \in T_\in(A_\sim; \alpha_x)$ and $z \in T_\in(A_\sim; \alpha_y)$. Then $(x * (y * x)) * z \in T_\in(A_\sim; \alpha)$ and $z \in T_\in(A_\sim; \alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_\in(A_\sim; \alpha)$ is an implicative ideal of X, it follows that $x \in T_\in(A_\sim; \alpha) = T_\in(A_\sim; \alpha_x \wedge \alpha_y)$. Similarly, if $(x * (y * x)) * z \in I_\in(A_\sim; \beta_x)$ and $z \in I_\in(A_\sim; \beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0, 1]$, then $x \in I_\in(A_\sim; \beta_x \wedge \beta_y)$. Now, suppose that $(x*(y*x))*z \in F_\in(A_\sim; \gamma_x)$ and $z \in F_\in(A_\sim; \gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0, 1)$. Then $(x*(y*x))*z \in F_\in(A_\sim; \gamma)$ and $z \in F_\in(A_\sim; \gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence $x \in F_\in(A_\sim; \gamma) = F_\in(A_\sim; \gamma_x \vee \gamma_y)$ since $F_\in(A_\sim; \gamma)$ is an implicative ideal of X. Therefore $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X. □

Corollary 3.8. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X if and only if it satisfies two conditions (3.2) and (3.3).*

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be an implicative $(\in, \in)$-neutrosophic ideal.

Theorem 3.9. *If X is an implicative BCK-algebra, then every $(\in, \in)$-neutrosophic ideal is an implicative $(\in, \in)$-neutrosophic ideal.*

Proof. If X is an implicative BCK-algebra, then $x = x * (y * x)$ for all $x, y \in X$. Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X. Then

$$A_T(x) \geq A_T(x * z) \wedge A_T(z) \geq A_T((x * (y * x)) * z) \wedge A_T(z),$$

$$A_I(x) \geq A_I(x * z) \wedge A_I(z) \geq A_I((x * (y * x)) * z) \wedge A_I(z),$$

and

$$A_F(x) \leq A_F(x * z) \vee A_F(z) \leq A_F((x * (y * x)) * z) \vee A_F(z)$$

for all $x, y, z \in X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X by Corollary 3.8. □

Lemma 3.10 ([17]). *Every $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the following assertion.*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \begin{cases} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{cases} \right). \tag{3.4}$$

Lemma 3.11 ([17]). *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(A_T(0) \geq A_T(x), A_I(0) \geq A_I(x), A_F(0) \leq A_F(x) \right) \tag{3.5}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * y) \wedge A_T(y) \\ A_I(x) \geq A_I(x * y) \wedge A_I(y) \\ A_F(x) \leq A_F(x * y) \vee A_F(y) \end{array} \right) \tag{3.6}$$

Theorem 3.12. *Suppose that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Then the following assertions are equivalent. Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X satisfying the condition:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \geq A_T(x * (y * x)) \\ A_I(x) \geq A_I(x * (y * x)) \\ A_F(x) \leq A_F(x * (y * x)). \end{array} \right) \tag{3.7}$$

(3) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X satisfying the condition:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) = A_T(x * (y * x)) \\ A_I(x) = A_I(x * (y * x)) \\ A_F(x) = eA_F(x * (y * x)). \end{array} \right) \tag{3.8}$$

Proof. (1) ⇒ (2). Let $A_\sim = (A_T, A_I, A_F)$ be an implicative $(\in, \in)$-neutrosophic ideal of X. Then $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.3. Using (3.2) and (3.3) implies that

$$A_T(x) \geq A_T((x * (y * x)) * 0) \wedge A_T(0) = A_T(x * (y * x)) \wedge A_T(0) = A_T(x * (y * x)),$$

$$A_I(x) \geq A_I((x * (y * x)) * 0) \wedge A_I(0) = A_I(x * (y * x)) \wedge A_I(0) = A_I(x * (y * x))$$

and

$$A_F(x) \leq A_F((x * (y * x)) * 0) \vee A_F(0) = A_F(x * (y * x)) \vee A_F(0) = A_F(x * (y * x))$$

for all $x, y \in X$.

(2) ⇒ (3). Observe that $x * (y * x) \leq x$ for all $x, y \in X$. Using Lemma 3.10, we have $A_T(x) \leq A_T(x * (y * x))$, $A_I(x) \leq A_I(x * (y * x))$ and $A_F(x) \geq A_F(x * (y * x))$. It follows from (3.7) that $A_T(x) = A_T(x * (y * x))$, $A_I(x) = A_I(x * (y * x))$ and $A_F(x) = A_F(x * (y * x))$ for all $x, y \in X$.

(3) ⇒ (1). Let $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X satisfying the condition (3.8).

Then

$$\begin{aligned} A_T(x) &= A_T(x*(y*x)) \geq A_T((x*(y*x))*z) \wedge A_T(z),\\ A_I(x) &= A_I(x*(y*x)) \geq A_I((x*(y*x))*z) \wedge A_I(z),\\ A_F(x) &= A_F(x*(y*x)) \leq A_F((x*(y*x))*z) \vee A_F(z) \end{aligned}$$

for all $x, y, z \in X$ by (3.8) and (3.6). Therefore $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X. □

Lemma 3.13 ([14]). *Let I and A be ideals of a BCK-algebra X such that $I \subseteq A$. If I is an implicative ideal of X, then so is A.*

Theorem 3.14. *Let $A_\sim = (A_T, A_I, A_F)$ and $B_\sim = (B_T, B_I, B_F)$ be $(\in, \in)$-neutrosophic ideals of X such that $A_\sim \sqsubseteq B_\sim$, that is, $A_T(x) \leq B_T(x)$, $A_I(x) \leq B_I(x)$ and $A_F(x) \geq B_F(x)$ for all $x \in X$. If $A_\sim = (A_T, A_I, A_F)$ is implicative, then so is $B_\sim = (B_T, B_I, B_F)$.*

Proof. It is sufficient to show that the non-empty neutrosophic $\in$-subsets $T_\in(B_\sim; \alpha)$, $I_\in(B_\sim; \beta)$ and $F_\in(B_\sim; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. If $x \in T_\in(A_\sim; \alpha)$, then $B_T(x) \geq A_T(x) \geq \alpha$ and so $T_\in(A_\sim; \alpha) \subseteq T_\in(B_\sim; \alpha)$. Similarly, $I_\in(A_\sim; \beta) \subseteq I_\in(B_\sim; \beta)$. If $x \in F_\in(A_\sim; \gamma)$, then $B_F(x) \leq A_F(x) \leq \gamma$ and thus $F_\in(A_\sim; \gamma) \subseteq F_\in(B_\sim; \gamma)$. Since $A_\sim = (A_T, A_I, A_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X, it follows from Theorem 3.7 that $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are implicative ideals of X. Therefore $T_\in(B_\sim; \alpha)$, $I_\in(B_\sim; \beta)$ and $F_\in(B_\sim; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, and hence $B_\sim = (B_T, B_I, B_F)$ is an implicative $(\in, \in)$-neutrosophic ideal of X. □

4 Implicative falling neutrosophic ideals

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, that is,

$$\begin{aligned} \tilde{H}_T(x_T) &= P(\omega_T \mid x_T \in \xi_T(\omega_T)),\\ \tilde{H}_I(x_I) &= P(\omega_I \mid x_I \in \xi_I(\omega_I)),\\ \tilde{H}_F(x_F) &= 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)) \end{aligned} \tag{4.1}$$

is called an *implicative falling neutrosophic ideal* of X.

Example 4.2. Consider a set $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 5. Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0, 1] \to \mathcal{P}(X),\ \ x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.25),\\ \{0, 1\} & \text{if } t \in [0.25, 0.55),\\ \{0, 1, 3\} & \text{if } t \in [0.55, 0.95),\\ X & \text{if } t \in [0.95, 1], \end{cases}$$

Table 5: Cayley table for the binary operation “$*$”

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.45), \\ \{0,2\} & \text{if } t \in [0.45, 0.65), \\ \{0,2,3\} & \text{if } t \in [0.65, 0.95), \\ X & \text{if } t \in [0.95, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,1,2\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are implicative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is an implicative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.75 & \text{if } x = 1, \\ 0.05 & \text{if } x = 2, \\ 0.35 & \text{if } x = 3, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.05 & \text{if } x = 1, \\ 0.55 & \text{if } x = 2, \\ 0.35 & \text{if } x = 3, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\Omega(x;\xi_T) := \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\},$$
$$\Omega(x;\xi_I) := \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\},$$
$$\Omega(x;\xi_F) := \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}.$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [10]).

Proposition 4.3. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is an implicative falling neutrosophic ideal of X, then*

$$(\forall x, y, z \in X) \begin{pmatrix} \Omega((x*(y*x))*z;\xi_T) \cap \Omega(z;\xi_T) \subseteq \Omega(x;\xi_T) \\ \Omega((x*(y*x))*z;\xi_I) \cap \Omega(z;\xi_I) \subseteq \Omega(x;\xi_I) \\ \Omega((x*(y*x))*z;\xi_F) \cap \Omega(z;\xi_F) \subseteq \Omega(x;\xi_F) \end{pmatrix}, \tag{4.2}$$

$$(\forall x, y, z \in X) \begin{pmatrix} \Omega(x;\xi_T) \subseteq \Omega((x*(y*x))*z;\xi_T) \\ \Omega(x;\xi_I) \subseteq \Omega((x*(y*x))*z;\xi_I) \\ \Omega(x;\xi_F) \subseteq \Omega((x*(y*x))*z;\xi_F) \end{pmatrix}. \tag{4.3}$$

Proof. Let $\omega_T \in \Omega((x*(y*x))*z;\xi_T) \cap \Omega(z;\xi_T)$, $\omega_I \in \Omega((x*(y*x))*z;\xi_I) \cap \Omega(z;\xi_I)$ and $\omega_F \in \Omega((x*(y*x))*z;\xi_F) \cap \Omega(z;\xi_F)$ for all $x, y, z \in X$. Then
$(x*(y*x))*z \in \xi_T(\omega_T)$ and $z \in \xi_T(\omega_T)$,
$(x*(y*x))*z \in \xi_I(\omega_I)$ and $z \in \xi_I(\omega_I)$,
$(x*(y*x))*z \in \xi_F(\omega_F)$ and $z \in \xi_F(\omega_F)$.
Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals of X, it follows from (2.11) that $x \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$ and so that $\omega_T \in \Omega(x;\xi_T)$, $\omega_I \in \Omega(x;\xi_I)$ and $\omega_F \in \Omega(x;\xi_F)$. Hence (4.2) is valid. Now let $x, y, z \in X$ be such that $\omega_T \in \Omega(x;\xi_T)$, $\omega_I \in \Omega(x;\xi_I)$, and $\omega_F \in \Omega(x;\xi_F)$. Then $x \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Note that

$$\begin{aligned} &((x*(y*x))*z)*x = ((x*(y*x))*x)*z \\ &= ((x*x)*(y*x))*z = (0*(y*x))*z = 0*z = 0, \end{aligned}$$

and thus

$$((x*(y*x))*z)*x = 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals and hence ideals of X, it follows that $(x*(y*x))*z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Hence $\omega_T \in \Omega((x*(y*x))*z;\xi_T)$, $\omega_I \in \Omega((x*(y*x))*z;\xi_I)$, and $\omega_F \in \Omega((x*(y*x))*z;\xi_F)$. Therefore (4.3) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.4}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\, ((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.5}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [8]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset A of X and $g_T, g_I, g_F \in \mathcal{F}(X)$, consider the followings:

$$A_T^g := \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\},$$
$$A_I^g := \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\},$$
$$A_F^g := \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}$$

and

$$\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\},$$
$$\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\},$$
$$\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.$$

Then A_T^g, A_I^g, $A_F^g \in \mathcal{A}$ (see [10]).

Theorem 4.4. *If K is an implicative ideal of a BCK-algebra X, then*

$$\xi_T(\omega_T) = \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\},$$
$$\xi_I(\omega_I) = \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\},$$
$$\xi_F(\omega_F) = \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\}$$

are implicative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is an implicative ideal of a BCK-algebra X. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have

$$\theta \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that $(f_T \circledast (g_T \circledast f_T)) \circledast h_T \in \xi_T(\omega_T)$ and $h_T \in \xi_T(\omega_T)$. Then

$$(f_T(\omega_T) * (g_T(\omega_T) * f_T(\omega_T))) * h_T(\omega_T) = ((f_T \circledast (g_T \circledast f_T)) \circledast h_T)(\omega_T) \in K$$

and $h_T(\omega_T) \in K$. Since K is an implicative ideal of X, it follows from (2.11) that $f_T(\omega_T) \in K$, that is, $f_T \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is an implicative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is an implicative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast (g_F \circledast f_F)) \circledast h_F \in \xi_F(\omega_F)$ and $h_F \in \xi_F(\omega_F)$. Then

$$(f_F(\omega_F) * (g_F(\omega_F) * f_F(\omega_F))) * h_F(\omega_F) = ((f_F \circledast (g_F \circledast f_F)) \circledast h_F)(\omega_F) \in K$$

and $h_F(\omega_F) \in K$. Hence $f_F(\omega_F) \in K$, i.e., $f_F \in \xi_F(\omega_F)$. Therefore $\xi_F(\omega_F)$ is an implicative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 4.5. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra is an implicative falling neutrosophic ideal.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be an implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Then $T_\in(\tilde{H}; \alpha)$, $I_\in(\tilde{H}; \beta)$ and $F_\in(\tilde{H}; \gamma)$ are implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ by Theorem 3.7. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H}; \alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H}; \beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H}; \gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$, and so $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is an implicative falling neutrosophic ideal of X. □

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 6.

Table 6: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.25), \\ \{0,2\} & \text{if } t \in [0.25, 0.55), \\ \{0,2,4\} & \text{if } t \in [0.55, 0.7), \\ \{0,1,2,3\} & \text{if } t \in [0.7, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,2\} & \text{if } t \in [0, 0.28), \\ \{0,4\} & \text{if } t \in [0.28, 0.68), \\ \{0,1,2,3\} & \text{if } t \in [0.68, 1] \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.75, 1], \\ \{0,4\} & \text{if } t \in (0.63, 0.75], \\ \{0,2,4\} & \text{if } t \in (0.44, 0.63], \\ \{0,1,4\} & \text{if } t \in (0.23, 0.44], \\ \{0,1,2,3\} & \text{if } t \in [0, 0.23]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are implicative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is an implicative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.55 & \text{if } x = 1, \\ 0.75 & \text{if } x = 2, \\ 0.3 & \text{if } x = 3, \\ 0.15 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.32 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.32 & \text{if } x = 3, \\ 0.4 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.56 & \text{if } x = 1, \\ 0.58 & \text{if } x = 2, \\ 0.77 & \text{if } x = 3, \\ 0.48 & \text{if } x = 4. \end{cases}$$

If $\alpha \in [0, 0.55)$, then $T_\in(\tilde{H}_T; \alpha) = \{0, 1, 2\}$ is not an implicative ideal of X since

$$(3 * (2 * 3)) * 1 = (3 * 0) * 1 = 3 * 1 = 2 \in T_\in(\tilde{H}_T; \alpha)$$

and $1 \in T_\in(\tilde{H}_T; \alpha)$, but $3 \notin T_\in(\tilde{H}_T; \alpha)$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an implicative $(\in, \in)$-neutrosophic ideal of X by Theorem 3.7.

We provide relations between a falling neutrosophic ideal and an implicative falling neutrosophic ideal .

Theorem 4.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is an implicative falling neutrosophic ideal of X, then it is a falling neutrosophic ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be an implicative falling neutrosophic ideal of a BCK-algebra X. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are implicative ideals of X, and so $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

The following example shows that the converse of Theorem 4.7 is not true in general.

Example 4.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7. Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a

Table 7: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.37), \\ \{0,1,2,3\} & \text{if } t \in [0.37, 0.67), \\ \{0,1,2\} & \text{if } t \in [0.67, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,4\} & \text{if } t \in [0.45, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.74, 1], \\ \{0,3\} & \text{if } t \in (0.66, 0.74], \\ \{0,1,2\} & \text{if } t \in (0.48, 0.66], \\ \{0,1,2,3\} & \text{if } t \in [0, 0.48]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic ideal of X. But it is not an implicative falling neutrosophic ideal of X because if $\alpha \in [0.67, 1]$, $\beta \in [0, 0.45)$ and $\gamma \in (0.66, 0.74]$, then $\xi_T(\alpha) = \{0,1,2\}$, $\xi_I(\beta) = \{0,1,2\}$ and $\xi_F(\gamma) = \{0,3\}$ are not implicative ideals of X respectively.

Since every ideal is implicative in an implicative BCK-algebra (see [15]), we have the following theorem.

Theorem 4.9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on an implicative BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is an implicative falling neutrosophic ideal of X.*

Corollary 4.10. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. For any BCK-algebra X which satisfies one of the following assertions*

$$(\forall x, y \in X)(y * (y * x) = (x * (x * y)) * (x * y)),$$
$$(\forall x, y \in X)((x * (x * y)) * (y * x) = y * (y * x)),$$
$$(\forall x, y \in X)((x * (x * y)) * (x * y) = (y * (y * x)) * (y * x)),$$

let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X.

If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is an implicative falling neutrosophic ideal of X.

Definition 4.11 ([12]). Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *commutative falling neutrosophic ideal* of X if $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$.

Definition 4.12 ([2]). Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of X for all ω_T, ω_I, $\omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, that is,

$$\begin{aligned} &\tilde{H}_T(x_T) = P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I(x_I) = P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F(x_F) = 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)) \end{aligned} \tag{4.6}$$

is called a *positive implicative falling neutrosophic ideal* of X.

Since every implicative ideal is both a commutative ideal and a positive implicative ideal in BCK-algebras (see [15]), the following theorem is straightforward.

Theorem 4.13. *Every implicative falling neutrosophic ideal is both a commutative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal.*

The following example shows that there exist a commutative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal which is not an implicative falling neutrosophic ideal.

Example 4.14. (1) Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.2. Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.25), \\ \{0,4\} & \text{if } t \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } t \in [0.55, 0.85), \\ \{0,3,4\} & \text{if } t \in [0.85, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,3\} & \text{if } t \in [0.45, 0.75), \\ \{0,1,2,4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,4\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X (see [12]). If $t \in [0.85, 1]$, then $\xi_T(t) = \{0, 3, 4\}$ is not an implicative ideal of X. Also, if $t \in (0.5, 0.7]$, then $\xi_F(t) = \{0, 4\}$ is not an implicative ideal of X. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an implicative falling neutrosophic ideal of X.

(2) Let $X = \{0, 1, 2, 3\}$ be a set with the binary operation $*$ which is given in Table 8.

Table 8: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [15]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.35), \\ \{0, 2\} & \text{if } t \in [0.35, 0.55), \\ \{0, 1, 2\} & \text{if } t \in [0.55, 0.95), \\ X & \text{if } t \in [0.95, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 1\} & \text{if } t \in [0, 0.2), \\ \{0, 2\} & \text{if } t \in [0.2, 0.5), \\ \{0, 1, 2\} & \text{if } t \in [0.5, 0.9), \\ X & \text{if } t \in [0.9, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.95, 1], \\ \{0, 1\} & \text{if } t \in (0.6, 0.95], \\ \{0, 2\} & \text{if } t \in (0.4, 0.6], \\ \{0, 1, 2\} & \text{if } t \in (0.1, 0.4], \\ X & \text{if } t \in [0, 0.1]. \end{cases}$$

Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a positive implicative falling neutrosophic ideal of X. If $t \in [0.35, 0.55)$, then $\xi_T(t) = \{0, 2\}$ is not an implicative ideal of X. If $t \in [0.2, 0.5)$, then $\xi_I(t) = \{0, 2\}$ is not an implicative ideal of X. Also, if $t \in (0.6, 0.95]$, then $\xi_F(t) = \{0, 1\}$ is not an implicative ideal of X. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an implicative falling neutrosophic ideal of X.

The notions of a commutative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal are independent, that is, a commutative falling neutrosophic ideal need not be a positive implicative falling neutrosophic ideal, and vice versa. In fact, the commutative falling neutrosophic ideal $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ in

Example 4.14(1) is not a positive implicative falling neutrosophic ideal. Also the positive implicative falling neutrosophic ideal $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ in Example 4.14(2) is not a commutative implicative falling neutrosophic ideal.

Theorem 4.15. *If the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is both a commutative implicative falling neutrosophic ideal and a positive implicative falling neutrosophic ideal, then it is an implicative falling neutrosophic ideal.*

Proof. It is straightforward because if any ideal is both commutative and position implicative, then it is implicative. □

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