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Algebraic Properties of Quasigroup Under $Q-$neutrosophic Soft Set

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Abstract. The novel concept called neutrosophic set was launched to take care of indeterminate factors in real-life data. The hybrid model of neutrosophic set and soft set has been widely studied in different areas of algebra, especially in associative structures such as fields, groups, rings, and modules. In this current paper, the novel concept is further introduce to a non-associative structure termed $Q-$neutrosophic soft quasigroup ($Q-$NS$\hat{G}$) and investigate its different algebraic properties of the quasigroups. We shown the conditions for the sets of $\alpha-$level cut of $Q-$NS$\hat{G}$ to be subquasigroups, the condition for each set of subquasigroups of a quasigroup to be $Q-$level cut neutrosophic soft subquasigroup were established. It was shown that $Q-$NS$\hat{G}$ obeys alternative property and flexible law. In addition, We defined $Q-$neutrosophic soft loop and investigate some of its characteristics. In particular, it was shown that $Q-$neutrosophic soft loop obeys inverse, weak inverse and cross inverse properties. We established the condition for a $Q-$neutrosophic soft loop to obey anti-automorphic inverse, semi-automorphic inverse and super anti-automorphic inverse properties. The necessary and sufficient condition for $Q-$neutrosophic soft set under a loop $(\hat{G}, \circ, /, \backslash)$ to be a $Q-$neutrosophic soft loop was also established.

Keywords: Q- set; Soft set; Neutrosophic set; Quasigroup; Loop.

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1. Introduction

Let $\hat{G}$ be a non-empty set and $(\circ)$ be an operation on $\hat{G}$. If $w \circ t \in \hat{G}$ for all $w, t \in G$, then $(\hat{G}, \circ)$ is called a *groupoid*. A groupoid $(\hat{G}, \circ)$ is called quasigroup, if there exist $a, b \in \hat{G}$ such that each of the equations:

$$a \circ w = b \qquad \text{and} \qquad t \circ a = b$$

have unique solution w, t respectively. Furthermore, the quasigroup is called a loop if there is a unique element $e \in \hat{G}$ called the *identity element* such that $\forall\ w \in G$,

$$w \circ e = e \circ w = w$$

In what follows, wt is written instead of $w \circ t$, which stipulates that $\circ$ has lower priority than juxtaposition amongst factors to be multiplied. For example we write, $p \circ qr$ stands for $p(qr)$.

Suppose that w is a fixed element in the groupoid $(\hat{G}, \circ)$, a translation map of $w \in \hat{G}$, called the left(right) translation maps written as L_w and R_w respectively are defined as

$$tL_w = w \circ t \qquad \text{and} \qquad tR_w = t \circ w.$$

Obviously, it implies that if the left and right translations maps are permutations, then a groupoid $(\hat{G}, \circ)$ is a quasigroup. And if the left and right translation maps of a quasigroup are bijections, it means that the inverse mappings L_w^{-1} and R_w^{-1} exist. Let

$$w \backslash t = tL_w^{-1} \qquad \text{and} \qquad w/t = wR_t^{-1}$$

and note that

$$w \backslash t = z \Leftrightarrow w \circ z = t \qquad \text{and} \qquad w/t = z \Leftrightarrow z \circ t = w.$$

Consequently, $(\hat{G}, \backslash)$ and $(\hat{G}, /)$ are also quasigroups.

A consideration of Fuzzy set was first initiated by Zadeh in [2], and the notion was designed to handle the challenges of uncertainty in real life data while the generalization of fuzzy set was considered by Atanassov in [4, 6] which is called intuitionistic fuzzy set. In 1971, Rosenfeld [5] for the time considered the concept of fuzzy set under the theoretical study of a group structure and established different properties and conditions for a subset of fuzzy set defined under a groups to be fuzzy subgroup. Since them, the concept has been extended to different field in mathematics. As away of generalizing the work in [5], the fuzzification of quasigroup was initiated by Dudek in 1998 [23] while 1999, Dudek and Jun [24] introduced fuzzy subquasigroup under norms to further the results in [23]. In 2000, the consideration of intuitionistic fuzzy set in a quasigroup was studied by Kyung et al. [27] as an extended method of fuzzy subquasigroup. In [23], research on intuitionistic fuzzy subquasigroup was furthered studied by Dudek [28] in 2005. It was revealed in [3] that each of these notions and their hybrid methods has their respective limitations and difficulties, and to address some of those difficulties, Molodtsov [3] launched the notion of soft set. It was reported that the notion of soft set theory is a better method for handling problems involving uncertainty, incompatible and incomplete data. Although, the study of soft set theory is not suitable for characterizing the degree of membership values as in the case of intuitionistic fuzzy set. Also, the notion is not capable for handling problems involving indeterminate data and as a result of that, a generalized concept called neutrosophic set was called out by Smarandache in 1998 [14,15], as a mathematical notion for dealing with indeterminacy occurrence. Neutrosophic set is more complex and the only generalized concept of the classical set theory found in the literature for dealing with problems involving indeterminate. The character of the degree values of a neutrosophic set are represented by the true membership T, indeterminate membership I and falsity membership F.

The different methods of determining the indeterminate factors of neutrosophic set in real-life data have been widely applied in different area in mathematics and its related field. For example, the work of Jidid et al. in [8] applied neutrosophy concept to handle the product quality control on inspection assignment form while Dey and Ray in 2023 [9] used the concept to characterized the separation axioms of neutrosophic topological spaces. The concept were used in the area of operation research in management in [10].

The hybrid model of neutrosophic sets, especially the neutrosophic consideration of soft set structure has been widely and sporadically flagged by algebraist in the recent past, (see the following articles [7,11,13,33]). However, it is important to mention the efforts of Muhammad et al. [20] and Mumtaz et al. [19], where set components of neutrosophy study, was replicated using groupoids, groups and bigroups. Furthermore, in 2020 Oyem et al. [29] conducted algebraic characterization of soft quasigroup while the generalization of his study was considered in [30]. Most recently, a study pattern of $Q-$fuzzy groups and their hybrid methods was called out by Solairaju et al. [16] and Thirunemi and Solairaju [17]. Then, was later escalated to $Q-$neutrosophic soft group in 2020 [18] to handle indeterminate data. The extension of $Q-$NS group to $Q-$NS quasigroup was recently announced by Oyebo et al. [25], which by tradition a generalization of the former.

In this present research, results of fuzzy quasigroup and its generalizations studied in the following articles [23, 24, 27, 28] are extended to neutrosophic soft quasigroup of two universal sets. Since the definition of $Q-$neutrosophic soft quasigroup was flagged up by Oyebo et. al [25], the question whether the concept obeys the following algebraic properties of quasigroup such as left(right) alternative property LAP(RAP), and flexible law are not yet known for the best of our searching. The result on characterization of supremum and infimum of fuzzy quasigroup studied by Dudek were extended to $Q-$neutrosophic soft quasigroup by capturing the behavior of an indeterminate factor of two universal sets that was lacking in structure of fuzzy quasigroup and intuitionistic fuzzy quasigroup. In addition, this paper is for the time introduced the concept of $Q-$neutrosophic soft loop which is a $Q-$neutrosophic soft quasigroup with an identity element without associative property. Also, the work of Dudek [23,24] and the generalized version in [25] did not shown results on the algebraic characteristics of the following class of quasigroup called left inverse property (LIP), right inverse property

(RIP), cross inverse property (CIP), weak inverse property (WIP), automorphic inverse prperty (AIP), anti-automorphic inverse property (AAIP), semi-automorphic inverse property (SAIP) and super anti- automorphic inverse property loop (SAAIP). In order to close up the gap, we investigate whether $Q-$neutrosophic soft quasigroup obey the properties of quasigroup mentioned above. In addition, we also pay attention to the necessary and sufficient condition for $Q-$neutrosophic soft set under a loop $(G, \circ)$ to be $Q-$neutrosophic soft loop.

The table below shown some set structures studied in the literature with their respective characterizations and generalizations.

Table 1. Properties of some set theories

Set structures	Membership function	uncertainty	inconsistency	indeterminacy	sum of membership ≤	independence (i)/ dependence(d)
Fuzzy	✓	✓	✓	×	1	d
intuitionistic fuzzy	✓	✓	✓	×	1	d
Soft	×	✓	✓	×	not applicable	not applicable
Rough	not applicable	✓	✓	×	not applicable	not applicable
interval fuzzy	✓	✓		×	1	d
vaque set	✓	✓	✓	×	1	d
Pythagorean fuzzy	✓	✓	✓	×	1	d
Neutrosophy	✓	✓	✓	✓	3	i
Spherical fuzzy set	✓	✓	✓	×	1	d

2. Preliminaries

Definition 2.1. A quasigroup(loop) $(\hat{G}, \circ)$ is said to have

(1) LIP if $\exists$ a map $J_\lambda : u \mapsto u^\lambda$ such that $u^\lambda \circ uv = v$ for all $u, v \in \hat{G}$
(2) RIP if $\exists$ a map $J_\rho : u \mapsto u^\rho$ such that $uv \circ u^\rho = v$ for all $u, v \in \hat{G}$,
(3) RAP if $t \circ ww = tw \circ w$ for all $w, t \in \hat{G}$,
(4) LAP if $ww \circ t = w \circ wt$ for all $w, t \in \hat{G}$,
(5) flexible if $uv \circ u = u \circ vu$ for all $u, v \in \hat{G}$,
(6) IPL if it satisfies $wt \circ w^{-1} = t$ or $w^{-1} \circ tw = t$ for all $w, t \in \hat{G}$ and
(7) WIPL if it satisfies the identity $t \circ (wt)^{-1} = w^{-1}$ for all $w, t \in \hat{G}$

Definition 2.2. The following identities hold in a loop $(\hat{G}, \circ)$ it is called:

(1) AIPL if $(wt)^{-1} = w^{-1}t^{-1} \;\; \forall w, t \in \hat{G}$,
(2) an AAIPL if $(wt)^{-1} = t^{-1}w^{-1}$ for all $w, t \in \hat{G}, \;\; \forall w, t \in \hat{G}$
(3) a SAAIPL if $(w \circ tz)^{-1} = z^{-1} \circ \left(t^{-1}w^{-1}\right)$, for all $w, t, z \in \hat{G}$ [see [26]]
(4) a SAIPL if $(wt \circ w)^{-1} = w^{-1}t^{-1} \circ w^{-1}$, for all $w, t \in \hat{G}$

Theorem 2.3. *[32] Let $(\hat{G}, \circ)$ be a quasigroup and $\hat{G}$ be a non empty subset of $\hat{G}$. Then, $\hat{G}$ is a subquasigroup of $(\hat{G}, \circ)$ if and only if $(\hat{G}, \circ)$, $(\hat{G}, /)$ and $(\hat{G}, \backslash)$ are groupoids*

Definition 2.4. [32] Let $(\hat{G}, \circ)$ be a quasigroup and $\emptyset \neq H \subseteq \hat{G}$. Then, H is called subquasigroup of $\hat{G}$ if $(H, \circ)$ is a quasigroup. Also, suppose that D and E are non empty subsets of $\hat{G}$, then $D \circ E = \{d \circ e \mid d \in D, e \in E\}, D/E = \{d/e \mid d \in D, e \in E\}$ and $E\backslash D = \{e\backslash d \mid d \in D, e \in E\}$

Definition 2.5. Let $M = [0, 1]$ and S be a subset of M. Then: the supremum of S denoted by $\sup S$ is a number $\beta_0 \in [0, 1]$ satisfying the conditions

(1) β_0 is an upper bound for S;
(2) for all $\epsilon > 0$, the number $\beta_0 - \epsilon$ is not an upper bound for S

the infimum of S denoted by $\inf S$ is a number $\alpha_0 \in [0, 1]$ satisfying the conditions

(1) α_0 is an upper bound for S;
(2) for all $\epsilon > 0$, the number $\alpha_0 + \epsilon$ is not a lower bound for S

Definition 2.6. [3] Given a set M and a parameter set $\mathfrak{A}$ of M. If $F : \mathfrak{A} \to P(M)$, where $P(M)$ is power set of M then the pair $(F, \mathfrak{A})$ is called a soft set .

Definition 2.7. [15] Given a set M. A neutrosophic set Φ (NS) on M is an object of the form
$\Phi = \{\langle m, (T_\Phi(m), I_\Phi(m), F_\Phi(m))\rangle : m \in M\}$ and the membership degree is described by $T_\Phi, I_\Phi, F_\Phi : W \to]^{-}0, 1^{+}[$.

Definition 2.8. [7] Given a set M and $\mathfrak{A}$ parameter sets. A neutrosophic soft set $(\Phi, \mathfrak{A})$ is described as $(\Phi, \mathfrak{A}) = \{\langle w, (T_\Phi(m), I_\Phi(m), F_\Phi(m))\rangle : m \in M\}$

Definition 2.9. [1] Let W be a universe of discourse and Q be a non-empty set and $\mathfrak{A} \subset E$ be a set of parameters. Let $\rho^l QNS(W)$ be the set of all multi-Q-NSs on W with dimension $l = 1$. A pair $(\Phi^Q, \mathfrak{A})$ is called a $Q-$neutrosophic soft set $(Q - NSS)$ denoted by $(\Phi^Q, A) = \{(a, \Phi^Q(a)) : a \in \mathfrak{A}, \Phi^Q(a) \in \rho^l QNS(W)\}$ over W, where $\Phi^Q : A \to \rho^l QNS(W)$ is a map such that $\Phi^Q(a) = \emptyset$ if $a \notin A$.

3. Results

Definition 3.1. Suppose that $(\hat{G}, \circ, \backslash, /)$ is a quasigroup and $(\Phi^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft set over $(\hat{G}, \circ, \backslash, /)$. Then, $(\Phi^Q, \mathfrak{A})$ is called a $Q - NS\hat{G}$ of $\hat{G}$ if for all $a \in \mathfrak{A}, w_1, w_2 \in \hat{G}, v \in Q$ satisfies the following conditions

(1) $T_{\Phi^Q(a)}(w_1 * w_2, v) \geq \min\{T_{\Phi^Q(a)}(w_1, v), T_{\Phi^Q(a)}(w_2, v)\}$
(2) $I_{\Phi^Q(a)}(w_1 * w_2, v) \leq \max\{I_{\Phi^Q(a)}(w_1, v), I_{\Phi^Q(a)}(w_1, v)\}$

(3) $F_{\Phi^Q(a)}(w_1 * w_2, v) \leq \max\{F_{\Phi^Q(a)}(w_1, v), F_{\Phi^Q(a)}(w_2, v)\}$

where $* \in \{\circ, /, \backslash\}$

Definition 3.2. Let $(\Lambda^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over $\hat{G}$ such that there exist $\alpha, \beta, \gamma \in [0,1]$ with the restriction that $\alpha_Q + \beta_Q + \gamma_Q \leq 3$. Then, $(\Lambda^Q, \mathfrak{A})_{(\alpha,\beta,\gamma)}$ is $Q-$level soft set defined as

$$(\Lambda_Q, \mathfrak{A})_{(\alpha,\beta,\gamma)} = \{f_1 \in \hat{G}, v \in Q : T_{\Lambda^Q(a)}(f_1, v) \geq \alpha, I_{\Lambda^Q(a)}(f_1, v) \leq \beta, F_{\Lambda^Q(a)}(f_1, v) \leq \gamma\}$$

for all $a \in \mathfrak{A}$

Suppose that $\alpha = \beta = \gamma$ for any $\alpha \in [0,1]$ with $\alpha + \alpha + \alpha \leq 3$ such that $(\Lambda^Q, \mathfrak{A})_\alpha = \{f_1 \in \hat{G}, v \in Q : T_{\Lambda^Q(a)}(f_1, v) \geq \alpha, I_{\Lambda^Q(a)}(f_1, v) \leq \alpha, F_{\Lambda^Q(a)}(f_1, v) \leq \alpha\}$, then $(\Lambda^Q, \mathfrak{A})_\alpha$ is called α -level set of Λ.

In neutrosophic soft set, the set

$$T(\Lambda^Q, \alpha) = \{f_1 \in \hat{G}, v \in Q : \Lambda^Q(a)(f_1, v) \geq \alpha\},$$
$$F(\Lambda^Q, \alpha) = \{f_1 \in \hat{G}, v \in Q : \Lambda^Q(a)(f_1, v) \leq \alpha\} \text{ and}$$
$$I(\Lambda^Q, \alpha) = \{f_1 \in \hat{G}, v \in Q : \Lambda^Q(a)(f_1, v) \leq \alpha\}$$

are respectively called the truth, falsity and indeterminacy α-levels cut of Λ

Theorem 3.3. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over $\hat{G}$. Then, the sets $U(\Lambda^Q, \alpha), I(\Lambda^Q, \alpha)$ and $L(\Lambda^Q, \alpha)$ are subquasigroups for all $\alpha \in Im(T_{\Lambda^Q(a)}(f_1, v)) \cap Im(I_{\Lambda^Q(a)}(f_1, v)) \cap Im(F_{\Lambda^Q(a)}(f_1, v))$, where Im donate the image under the map of membership degree.*

Proof: Let $\alpha \in Im(T_{\Lambda^Q(a)}(f_1, v)) \cap Im(I_{\Lambda^Q(a)}(f_1, v)) \cap Im(F_{\Lambda^Q(a)}(f_1, v)) \subseteq [0,1]$. Obviously, the sets $U(\Lambda^Q, \alpha), I(\Lambda^Q, \alpha)$ and $L(\Lambda^Q, \alpha)$ are non-empty and let $q \in Q$ and $f_1, h_1 \in U(\Lambda^Q, \alpha)$. Then, $T_{\Lambda^Q(a)}(f_1, v) \geq \alpha$ and $T_{\Lambda^Q(a)}(h_1, v) \geq \alpha$ for all $a \in \mathfrak{A}$. Using Definition 3.1, we have

$$T_{\Lambda^Q(a)}(f_1 h_1, v) \geq \min\{T_{\Lambda^Q(a)}(f_1, v), T_{\Lambda^Q(a)}(h_1, v)\} \geq \alpha \textit{ so that } f_1 \circ h_1 \in U(\Lambda^Q, \alpha)$$

Suppose that $f_1, h_1 \in I(\Lambda^Q, \alpha)$, then $I_{\Lambda^Q(a)}(f_1, v) \leq \alpha$ and $I_{\Lambda^Q(a)}(h_1, v) \leq \alpha$, by definition, we have

$$I_{\Lambda^Q(a)}(f_1 h_1, v) \leq \max\{I_{\Lambda^Q(a)}(f_1, v), I_{\Lambda^Q(a)}(h_1, v)\} \leq \alpha$$

Hence $f_1 \circ h_1 \in I(\Lambda^Q, \alpha)$.

Let $f_1, h_1 \in F(\Lambda^Q, \alpha)$, then $F_{\Lambda^Q(a)}(f_1, v) \leq \alpha$ and $F_{\Lambda^Q(a)}(h_1, v) \leq \alpha$. From definition, it follows that

$$F_{\Lambda^Q(a)}(f_1 h_1, v) \leq \max\{F_{\Lambda^Q(a)}(f_1, v), F_{\Lambda^Q(a)}(h_1, v)\} \leq \alpha$$

Hence, $f_1 \circ h_1 \in F(\Lambda^Q, \alpha)$. Thus, $U(\Lambda^Q, \alpha), I(\Lambda^Q, \alpha)$ and $L(\Lambda^Q, \alpha)$ are subquasigroups of $\hat{G}$

Theorem 3.4. *Let* $(\Lambda^Q, \mathfrak{A})$ *be a* $Q-$*NSS over* $\hat{G}$ *such that a nonempty set* (Λ^Q, α^K) *is a subquasigroup of* $\hat{G}$ *for all* $\alpha \in [0,1]$ *. Then,* $(\Lambda^Q, \mathfrak{A})$ *is a* $Q-$*neutrosophic soft subquasigroup of* $\hat{G}$ *for all* $a \in \mathfrak{A}$

Proof: We assume that the nonempty set (Λ^Q, α^K) is a subquasigroup of $\hat{G}$ for all $\alpha \in [0,1]$. We want to show that $(\Lambda^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft subquasigroup of $\hat{G}$ for all $f_1, h_1' \in \hat{G}, v \in Q$ and $a \in \mathfrak{A}$. On the contrary, suppose that Definition 3.1 does not hold and there exist $f_1, h_1' \in \hat{G}, v \in Q$, and $a \in \mathfrak{A}$ such that

$$\begin{cases} T_{\Psi^Q(a)}(f_1 \circ h_1', v) < \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1', v)\} \\ I_{\Psi^Q(a)}(f_1 \circ h_1', v) > \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1', v)\} \\ F_{\Psi^Q(a)}(f_1 \circ h_1', v) > \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1', v)\} \end{cases} \tag{1}$$

Let

$$T_{\Psi^Q(a)}(f_1, v) = \alpha_1, T_{\Psi^Q(a)}(h', v) = \beta_1 \text{ and } T_{\Psi^Q(a)}(f_1 \circ h_1\prime, v) = \gamma_1$$
$$I_{\Psi^Q(a)}(f_1, v) = \alpha_2, I_{\Psi^Q(a)}(h_1', v) = \beta_2 \text{ and } I_{\Psi^Q(a)}(f_1 \circ h_1\prime, v) = \gamma_2$$
$$F_{\Psi^Q(a)}(f_1, v) = \alpha_3, F_{\Psi^Q(a)}(h_1', v) = \beta_3 \text{ and } F_{\Psi^Q(a)}(f_1 \circ h_1\prime,) = \gamma_3$$

Then, its follows from equation 1

$$\gamma_1 < \min\{\alpha_1, \beta_1\}, \gamma_2 > \max\{\alpha_2, \beta_2\} \text{ and } \gamma_3 > \max\{\alpha_3, \beta_3\} \tag{2}$$

Put

$$\begin{cases} \gamma_1^* = \frac{1}{2}\Big[T_{\Psi^Q(a)}(f_1 \circ h', v) + \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1', v)\}\Big] \\ \gamma_2^* = \frac{1}{2}\Big[I_{\Psi^Q(a)}(f_1 \circ h_1', v) + \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1', v)\}\Big] \\ \gamma_3^* = \frac{1}{2}\Big[F_{\Psi^Q(a)}(f_1 \circ h_1', v) + \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1', v)\}\Big] \end{cases} \tag{3}$$

Therefore,

$\gamma_1^* = \frac{1}{2}\Big[(\gamma_1, v) + \min\{\alpha_1, v), (\beta_1, v)\}\Big]$

$\gamma_2^* = \frac{1}{2}\Big[(\gamma_2, v) + \max\{\alpha_2, v), (\beta_2, v)\}\Big]$

$\gamma_3^* = \frac{1}{2}\Big[(\gamma_3, v) + \max\{\alpha_3, v), (\beta_3, v)\}\Big]$

Then,

$\alpha_1 > \gamma_1^* = \frac{1}{2}\Big[(\gamma_1, v) + \min\{\alpha_1, v), (\beta_1, v)\}\Big] > \gamma_1$

$\alpha_2 < \gamma_2^* = \frac{1}{2}\Bigg[(\gamma_2, v) + \min\{\alpha_2, v), (\beta_2, v)\}\Bigg] < \gamma_3$

$\alpha_3 < \gamma_3^* = \frac{1}{2}\Bigg[(\gamma_3, v) + \min\{\alpha_3, v), (\beta_3, v)\}\Bigg] < \gamma_3$

Thus,

$$T_{\Psi^Q(a)}(f_1 \circ h_1', v) < \gamma_1^* < \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1', v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1', v) > \gamma_1^* > \min\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1', v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1', v) > \gamma_1^* > \min\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1', v)\}$$

It follows that $f_1, h_1' \in (\Lambda^Q, \alpha^K)$, but $f_1 \circ h_1' \notin (\Lambda^Q, \alpha^K)$ a contradiction base on the fact that

$$T_{\Psi^Q(a)}(f_1, v) = \alpha_1 \geq \min\{(\alpha_1, v), (\beta_1, v)\} > \gamma_1^*$$
$$I_{\Psi^Q(a)}(f_1, v) = \alpha_2 \leq \max\{(\alpha_2, v), (\beta_2, v)\} < \gamma_2^*$$
$$F_{\Psi^Q(a)}(f_1, v) = \alpha_3 \leq \max\{(\alpha_3, v), (\beta_3, v)\} < \gamma_3^*$$

this implies that $f_1, h_1' \in (\Lambda^Q, \alpha^K)$. Thus, condition 3.1 hold. The prof is complete

Theorem 3.5. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q-$NSS over $\hat{G}$. Then, each subquasigroup H of $\hat{G}$ is a $Q-$level neutrosophic soft subquasigroup for all $\alpha, \beta, \gamma \in [0, 1]$ and $a \in \mathfrak{A}$*

Proof: Let $(\Lambda^Q, \mathfrak{A})$ be defined by

$$T_{\Phi_{Q(a)}}(f_1, v) = \begin{cases} \alpha, & \text{if } f_1 \in H \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\Phi_{Q(a)}}(f_1, v) = \begin{cases} \beta, & \text{if } f_1 \in H \\ 0, & \text{otherwise.} \end{cases}$$

$$F_{\Phi_{Q(a)}}(f_1, v) = \begin{cases} \gamma, & \text{if } f_1 \in H \\ 0, & \text{otherwise.} \end{cases}$$

where $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma \leq 3$, for all $f_1 \in \hat{G}, v \in Q$ and $a \in \mathfrak{A}$

We consider the following cases to show that $(\Lambda^Q, \mathfrak{A})$ is a Q- neutrosophic soft quasigroup over $\hat{G}$.

Case 1: Suppose that $f_1, h_1 \in H$, then $f_1 \circ h_1 \in H$. So,

$$T_{\Psi^Q(a)}(f_1 \circ h_1, v) = \alpha = \min\{\alpha, \alpha\} = \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1, v) = \beta = \min\{\beta, \beta\} = \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1, v) = \beta = \min\{\beta, \beta\} = \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$$

Case 2: If $f, h \notin H$, then
$T_{\Psi^Q(a)}(f_1, v) = 0 = T_{\Psi^Q(a)}(h, v), I_{\Psi^Q(a)}(f_1, v) = 0 = I_{\Psi^Q(a)}(h, v)$ and $F_{\Psi^Q(a)}(f_1, v) = 0 = F_{\Psi^Q(a)}(h, v)$. Therefore,

$$T_{\Psi^Q(a)}(f_1 \circ h_1, v) \geq 0 = \min\{0, 0\} = \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{0, 0\} = \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{0, 0\} = \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$$

Case 3: If $f_1 \in H$ and $h_1 \notin H$, then $T_{\Psi^Q(a)}(f_1, v) = \alpha, I_{\Psi^Q(a)}(f_1, v) = \beta$ and $F_{\Psi^Q(a)}(f_1, v) = \gamma, F_{\Psi^Q(a)}(h_1, v) = 0 = T_{\Psi^Q(a)}(h_1, v) = I_{\Psi^Q(a)}(h_1, v)$. So,

$$T_{\Psi^Q(a)}(f_1 \circ h_1, v) \geq 0 = \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$$

Case 4: If $h_1 \in H$ and $f_1 \notin H$. It has a similar argument with case 3. This complete the proof.

Theorem 3.6. *If* $(\Lambda^Q, \mathfrak{A})$ *is a* $Q - NS\hat{G}$ *over* $\hat{G}$*. Then,*

(1) $T_{\Psi^Q(a)}(f_1, v) = \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha)\}$
(2) $I_{\Psi^Q(a)}(f_1, v) = \inf\{\beta \in [0, 1] : f_1 \in I(\Lambda^Q, \beta)\}$ *and*
(3) $F_{\Psi^Q(a)}(f_1, v) = \inf\{\gamma \in [0, 1] : f_1 \in L(\Lambda^Q, \gamma)\}$

for all $f_1 \in \hat{G}$ *and* $v \in Q$

Proof:

(1) Given $\epsilon > 0$, let $\delta = \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha)\}$. Then, $\delta - \epsilon < \alpha$ for some $\alpha \in [0, 1]$. This implies that $\delta - \epsilon < T_{\Psi^Q(a)}(f_1, v)$ so that $\delta \leq T_{\Psi^Q(a)}(f_1, v)$ for every an arbitrary ϵ and for all $v \in Q$ and $f_1 \in \hat{G}$.

Next, we show that $T_{\Psi^Q(a)}(f_1, v) \leq \delta$. If $T_{\Psi^Q(a)}(f_1, v) = \alpha_1$, then $f_1 \in U(\Lambda^Q, \alpha_1)$ and so

$$\alpha_1 \in \{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha), v \in Q\}$$

Hence,

$$T_{\Psi^Q(a)}(f_1, v) = \alpha_1 \leq \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha), v \in Q\} = \delta$$

Therefore,

$$T_{\Psi^Q(a)}(f_1, v) = \delta = \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha), v \in Q\}$$

(2) Let $\tau = \inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\}$. Then, $\inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\} < \tau + \epsilon$. For any $\epsilon > 0$, we show that $\beta < \tau + \epsilon$ for some $\beta \in [0,1]$ with $f_1 \in I(\Lambda^Q, \beta)$. Since ϵ is an arbitrary element, we have $I_{\Psi^Q(a)}(f_1, v) \leq \beta$ for any $v \in Q$. This implies that

$$I_{\Psi^Q(a)}(f_1, v) \leq \tau$$

.

To show that $I_{\Psi^Q(a)}(f_1, v) \geq \tau$, let $I_{\Psi^Q(a)}(f_1, v) = \beta_1$.
Then, $f_1 \in I(\Lambda^Q, \beta)$ and thus, $\beta_1 \in \{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\}$.
Hence,

$$\inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\} \leq \tau$$

That is $\tau \leq \beta_1 = I_{\Psi^Q(a)}(f_1, v)$ for any $v \in Q$. Consequently,

$$I_{\Psi^Q(a)}(f_1, v) = \tau = \inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta) \ \forall \ v \in Q\}$$

(3) The argument is similar with 2 above.

Theorem 3.7. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over a $(\hat{G}, \circ)$. The following hold*

(1) $T_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = T_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$, $I_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = I_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$ *and* $F_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = F_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$
(2) $T_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = T_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$, $I_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = I_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$ *and* $F_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = F_{\Psi^Q(a)}(hf_1 \circ f_1, v)$
(3) $T_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = T_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$, $I_{\Psi^Q(a)}(f_1^2 \circ h, v) = I_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = F_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$

Proof: Let $(\Phi^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over a quasigroup $(\hat{G}, \circ)$. For all $f_1, h_1 \in \hat{G}, v \in Q$ and $a \in \mathfrak{A}$, we have

(1) Considering the LHS.

$$\begin{aligned} T_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) &\geq \min\{T_{\Psi^Q(a)}(f_1 \circ h_1, v), T_{\Psi^Q(a)}(f_1, v)\} \\ &= \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1 \circ h_1, v)\} \\ &\geq \min\left\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}\right\} \\ &= \min\left\{\min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\right\} \\ &= \min\left\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\right\} \end{aligned} \tag{4}$$

Considering the RHS.

$$\begin{aligned}
T_{\Psi^{Q}(a)}(f_1 \circ h_1 f_1, v) &\geq \min\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(h_1 f_1, v)\} \\
&= \min\{T_{\Psi^{Q}(a)}(h_1 f_1, v), T_{\Psi^{Q}(a)}(f_1, v)\} \\
&\geq \min\Big\{ \min\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\}, T_{\Psi^{Q}(a)}(f_1, v)\Big\} \\
&= \min\Big\{T_{\Psi^{Q}(a)}(h_1, v), \min\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(f_1, v)\}\Big\} \\
&= \min\Big\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\Big\} \qquad (5)
\end{aligned}$$

Therefore, $\min\Big\{T_{\Psi^{Q}(a)}(h, v), T_{\Psi^{Q}(a)}(f_1, v)\Big\} = \min\Big\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(h, v)\Big\}$. Thus, $T_{\Psi^{Q}(a)}(f_1 h \circ f_1, v) = T_{\Psi^{Q}(a)}(f_1 \circ h f_1, v)$

$$\begin{aligned}
I_{\Psi^{Q}(a)}(f_1 h_1 \circ f_1, v) &\leq \max\{I_{\Psi^{Q}(a)}(f_1 \circ h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\} \\
&= \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(f_1 \circ h_1, v)\} \\
&\leq \max\Big\{I_{\Psi^{Q}(a)}(f_1, v), \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(h_1, v)\}\Big\} \\
&= \max\Big\{ \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(f_1, v)\}, I_{\Psi^{Q}(a)}(h_1, v)\Big\} \\
&= \max\Big\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(h_1, v)\Big\} \qquad (6)
\end{aligned}$$

Considering the RHS.

$$\begin{aligned}
I_{\Psi^{Q}(a)}(f_1 \circ h_1 f_1, v) &\leq \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(h_1 f_1, v)\} \\
&= \max\{I_{\Psi^{Q}(a)}(h_1 f_1, v), I_{\Psi^{Q}(a)}(f_1, v)\} \\
&\leq \max\Big\{ \max\{I_{\Psi^{Q}(a)}(h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\}, I_{\Psi^{Q}(a)}(f_1, v)\Big\} \\
&= \max\Big\{I_{\Psi^{Q}(a)}(h_1, v), \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}}(a)(f_1, v)\}\Big\} \\
&= \max\Big\{I_{\Psi^{Q}(a)}(h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\Big\} \qquad (7)
\end{aligned}$$

Therefore, $\max\Big\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\Big\} = \max\Big\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(h_1, v)\Big\}$. Thus, $I_{\Psi^{Q}(a)}(f_1 h_1 \circ f_1, v) = I_{\Psi^{Q}(a)}(f_1 \circ h_1 f_1, v)$. The result for falsity membership is obtain in similar procedure.

(2) Let $f_1, h_1 \in \hat{G}, a \in \mathfrak{A}$ and $v \in Q$, we want show that $T_{\Psi^{Q}(a)}(h_1 \circ f_1^2, v) = T_{\Psi^{Q}(a)}(h_1 f_1 \circ f_1, v)$ for true membership.

Considering the RHS,

$$T_{\Psi^Q(a)}(h_1f_1 \circ f_1, v) \geq \min\{T_{\Psi^Q(a)}(h_1f_1, v), T_{\Psi^Q(a)}(f_1, v)\}$$
$$\geq \min\left\{\min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} \tag{8}$$

Considering the LHS.

$$T_{\Psi^Q(a)}(h_1 \circ f_1^2, v) \geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1^2, v)\}$$
$$= \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1 \circ f_1, v)\}$$
$$\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} \tag{9}$$

That is, $\min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} = \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\}$

(3) Follows in the similar result of 2.

Corollary 3.8. *Let $(\Phi^Q, \mathfrak{A})$ be a $Q-$NSS over a quasigroup $(\hat{G}, \circ)$. Then, the following are equivalent.*

(1) $(\Phi^Q, \mathfrak{A})$ *is a* $Q - NS\hat{G}$

(2) $T_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = T_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$, $I_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = I_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$ *and* $F_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = F_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$

(3) $T_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = T_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$, $I_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = I_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$ *and* $F_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = F_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$

(4) $T_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = T_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$, $I_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = I_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = F_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$

Proof: It following from Theorem 3.7.

Definition 3.9. Let $(\Psi^Q, \mathfrak{A})$ be a $Q-$NSS defined over a loop $(\hat{L}, \circ, /, \backslash)$. Then $(\Psi^Q, \mathfrak{A})$ is called a $Q-$neutrosphis soft loop $(Q - NS\hat{L})$ over $\hat{L}$ if for all $a \in \mathfrak{A}, f_1, h_1 \in \hat{L}$, and $v \in Q$ satisfies the following conditions

(1) $T_{\Psi^Q(a)}(f_1 * h_1, v) \geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$,
$I_{\Psi^Q(a)}((f_1 * h_1), v) \leq \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$
$F_{\Psi^Q(a)}(f_1 * h_1, v) \leq \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$

(2) $T_{\Psi^Q(a)}(f_1^{-1}, v) \geq T_{\Psi^Q(a)}(f_1, v)$,
$I_{\Psi^Q(a)}(f_1^{-1}, v) \leq I_{\Psi^Q(a)}(f_1, v)$,
$F_{\Psi^Q(a)}(f_1^{-1}, v) \leq F_{\Psi^Q(a)}(f_1, v)$

(3) $T_{\Psi^Q(a)}(^{1-}f_1, q) \geq T_{\Psi^Q(a)}(f_1, v)$,
$I_{\Psi^Q(a)}(^{1-}f_1, v) \leq I_{\Psi^Q(a)}(f_1, v)$,
$F_{\Psi^Q(a)}(^{1-}f_1, v) \leq F_{\Psi^Q(a)}(f_1, v)$

where f^{-1} and ^{1-}f are right inverse and left inverse in $\hat{L}$ and $* \in \{\circ, /, \backslash\}$.

Theorem 3.10. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q-$NSS over a loop $\hat{L}$. Then, $(\Lambda^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft subloop of $\hat{L}$ if and only if the nonempty $Q-$level soft set $(\Lambda_{Q(\alpha,\beta,\gamma)}, \mathfrak{A})$ is a soft subloop for all $\alpha, \beta, \gamma \in [0, 1]$ and $a \in \mathfrak{A}$*

Proof: The proof is follow from Theorem 3.4 with definition 3.9.

Lemma 3.11. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q - NS\hat{L}$ over a loop $(\hat{L}, \circ)$. Then, for all $f_1 \in \hat{L}, v \in Q$ the following hold*

(1) $T_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) = T_{\Psi^Q(a)}(f_1, v)$, $I_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) = I_{\Psi^Q(a)}(f_1, v)$ $aF_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) = F_{\Psi^Q(a)}(f_1, v)$

Proof. Follows from Definition 3.9. □

Theorem 3.12. *Let $(\Psi^Q, \mathbf{A})$ be a $Q - NS\hat{L}$ over a loop $(\hat{L}, \circ, /, \backslash)$. Then, for all $a \in \mathfrak{A}, f \in \hat{L}, v \in Q$ the following hold*

(1) $T_{\Psi^Q(a)}(f_1^{-1}, v) \geq T_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1^{-1}, v) \leq I_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(f_1^{-1}, v) \leq F_{\Psi^Q(a)}(f_1, v)$

(2) $T_{\Psi^Q(a)}(^{1-}f_1, v) \geq T_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(^{1-}f_1, v) \leq I_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(^{1-}f_1, v) \leq F_{\Psi^Q(a)}(f_1, v)$

(3) $T_{\Psi^Q(a)}(e, q) \geq T_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(e, v) \leq I_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(e, v) \leq F_{\Psi^Q(a)}(f_1, v)$

Proof:

(1) Let $(\Psi^Q,)$ be a $Q - NS\hat{L}$ over loop $(G, \circ, /, \backslash)$, then for all $a \in \mathfrak{A}, f_1 \in \hat{L}, v \in Q$ we have

$$T_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) \geq T_{\Phi_{Q(a)}}(f_1, v)$$
$$I_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) \leq I_{\Phi_{Q(a)}}(f_1, v)$$
$$F_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) \leq F_{\Phi_{Q(a)}}(f_1, v)$$

(2) it is similar with (1)

(3) Let $(\Psi^Q, \mathfrak{A})$ be a $Q - NS\hat{L}$ over loop $(\hat{L}, \circ)$ with an identith element $e \in \hat{L}$. Then, for all $f_1 \in \hat{L}, v \in Q$, we have

$$
\begin{aligned}
T_{\Psi^Q(a)}(e, v) &= T_{\Psi^Q(a)}((f_1^{-1} \circ f_1, v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1^{-1}, v), T_{\Phi_{Q(a)}}(f_1, v)\} \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1, v), T_{\Phi_{Q(a)}}(f_1, v)\} \\
&= T_{\Phi_{Q(a)}}(f_1, v) \\
I_{\Psi^Q(a)}(e, v) &= I_{\Psi^Q(a)}((f_1^{-1} \circ f_1, v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1, v)\} \\
&= I_{\Phi_{Q(a)}}(f_1, v) \\
F_{\Psi^Q(a)}(e, v) &= F_{\Psi^Q(a)}((f_1^{-1} \circ f_1, v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1, v)\} \\
&= F_{\Phi_{Q(a)}}(f_1, v)
\end{aligned}
$$

$$
\begin{aligned}
T_{\Psi^Q(a)}(e, v) &= T_{\Psi^Q(a)}((f_1 \circ f_1^{-1}, v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1, v), T_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1, v), T_{\Phi_{Q(a)}}(f_1, v)\} \\
&= T_{\Phi_{Q(a)}}(f_1, v) \\
I_{\Psi^Q(a)}(e, v) &= I_{\Psi^Q(a)}((f_1 \circ f_1^{-1}, v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1, v)\} \\
&= I_{\Phi_{Q(a)}}(f_1, v) \\
F_{\Psi^Q(a)}(e, v) &= F_{\Psi^Q(a)}((f_1 \circ f_1^{-1}, v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1, v)\} \\
&= F_{\Phi_{Q(a)}}(f_1, v)
\end{aligned}
$$

$$\begin{aligned}
T_{\Psi^Q(a)}(e,v) &= T_{\Psi^Q(a)}((f_1/f_1,v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1,v), T_{\Phi_{Q(a)}}(f_1,v)\} \\
&= T_{\Phi_{Q(a)}}(f_1,v) \\
I_{\Psi^Q(a)}(e,v) &= I_{\Psi^Q(a)}((f_1/f_1,v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1,v), I_{\Phi_{Q(a)}}(f_1,v)\} \\
&= I_{\Phi_{Q(a)}}(f_1,v) \\
F_{\Psi^Q(a)}(e,v) &= F_{\Psi^Q(a)}((f_1/f_1,v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1,v), F_{\Phi_{Q(a)}}(f_1,v)\} \\
&= F_{\Phi_{Q(a)}}(f_1,v)
\end{aligned}$$

$$\begin{aligned}
T_{\Psi^Q(a)}(e,v) &= T_{\Psi^Q(a)}((f_1\backslash f_1,v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1,v), T_{\Phi_{Q(a)}}(f_1,v)\} \\
&= T_{\Phi_{Q(a)}}(f_1,v) \\
I_{\Psi^Q(a)}(e,v) &= I_{\Psi^Q(a)}((f_1\backslash f_1,v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1,v), I_{\Phi_{Q(a)}}(f_1,v)\} \\
&= I_{\Phi_{Q(a)}}(f_1,v) \\
F_{\Psi^Q(a)}(e,v) &= F_{\Psi^Q(a)}((f_1\backslash f_1,v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1,v), F_{\Phi_{Q(a)}}(f_1,v)\} \\
&= F_{\Phi_{Q(a)}}(f_1,v)
\end{aligned}$$

The proof is compete.

Theorem 3.13. *Let* $(\Psi^Q, \mathfrak{A})$ *be a* $Q-NS\hat{L}$ *over a loop* $(\hat{L}, \circ, /, \backslash)$*. Then, for all* $a \in \mathfrak{A}, f_1 \in \hat{L}, v \in Q$ *the following hold*

(1) $T_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = T_{\Psi^Q(a)}(h_1, v)$, $I_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = I_{\Psi^Q(a)}(h_1, v)$, *and* $F_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = F_{\Psi^Q(a)}(h_1, v)$,

(2) $T_{\Psi^Q(a)}(f_1^{-1} \circ f_1 h_1, v) = T_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1^{-1} \circ f_1 h_1, v) = I_{\Psi^Q(a)}(h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^{-1} \circ f_1 h_1, v) = F_{\Psi^Q(a)}(h_1, v)$

(3) $T_{\Psi^Q(a)}(f_1^{-1} \circ h_1 f_1, v) = T_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1^{-1} \circ h_1 f_1, v) = I_{\Psi^Q(a)}(h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^{-1} \circ h_1 f_1, v) = F_{\Psi^Q(a)}(h_1, v)$

(4) $T_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) = T_{\Psi^Q(a)}(f_1^{-1}, v), I_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) = I_{\Psi^Q(a)}(f_1^{-1}, v)$ *and* $F_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) = F_{\Psi^Q(a)}(f_1^{-1}, v)$

Proof:

(1) Let $(\Psi^Q, \mathfrak{A})$ be $Q-NS\hat{L}$ over a loop $(\hat{L}, \circ, /, \backslash)$. Then, we shall show that for all $a \in \mathfrak{A}, f_1, h_1 \in \hat{L}, v \in Q$

$$\begin{aligned}
T_{\Psi^Q(a)}(h_1 f \circ f^{-1}, v) &= T_{\Psi^Q(a)}(h_1, v) \\
\Rightarrow T_{\Psi^Q(a)}(h_1 f_1 \circ f^{-1}, v) &\geq \min\{T_{\Psi^Q(a)}(h_1 \circ f_1, v), T_{\Psi^Q(a)}(f_1^{-1}, v)\} \\
&\geq \min\{T_{\Psi^Q(a)}(h_1 \circ f, v), T_{\Psi^Q(a)}(f_1, v)\} \\
&\geq \min\left\{\min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\right\} \\
&= \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\} \\
&\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} \qquad (10)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
T_{\Psi^Q(a)}(h_1, v) &= T_{\Psi^Q(a)}((h_1/f_1) \circ f, v) \\
&\geq \min\{T_{\Psi^Q(a)}((h_1/f_1), v), T_{\Psi^Q(a)}(f_1, v)\} \\
&\geq \min\left\{\min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\right\} \\
&= \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\} \\
&\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} \qquad (11)
\end{aligned}$$

And

$$\begin{aligned}
I_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) &= I_{\Psi^Q(a)}(h_1, v) \\
\Rightarrow I_{\Psi^Q(a)}(h_1 f_1 \circ f^{-1}, v) &\leq \max\{I_{\Psi^Q(a)}(h_1 \circ f_1, v), I_{\Psi^Q(a)}(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Psi^Q(a)}(h_1 \circ f_1, v), I_{\Psi^Q(a)}(f_1, v)\} \\
&\leq \max\left\{\max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\}, I_{\Psi^Q(a)}(f_1, v)\right\} \\
&= \max\left\{I_{\Psi^Q(a)}(h_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1, v)\}\right\} \\
&\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\right\} \qquad (12)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
I_{\Psi^Q(a)}(h_1,v) &= I_{\Psi^Q(a)}((h_1/f_1)\circ f_1,v)\\
&\leq \max\{I_{\Psi^Q(a)}((h_1/f_1),v), I_{\Psi^Q(a)}(f_1,v)\}\\
&\leq \max\Big\{\max\{I_{\Psi^Q(a)}(h_1,v), I_{\Psi^Q(a)}(f_1,v)\}, I_{\Psi^Q(a)}(f_1,v)\Big\}\\
&= \max\Big\{I_{\Psi^Q(a)}(h_1,v), \max\{I_{\Psi^Q(a)}(f_1,v), I_{\Psi^Q(a)}(f_1,v)\}\Big\}\\
&\leq \max\Big\{I_{\Psi^Q(a)}(h_1,v), I_{\Psi^Q(a)}(f_1,v)\Big\} \qquad (13)
\end{aligned}
$$

Similarly, we can use the identity $T_{\Psi^Q(a)}(h_1,v) = T_{\Psi^Q(a)}((f_1\circ f_1\backslash h_1),v)$. Result for falsity is argued the same way.

(2) Use the same argument of 1

(3) Similar argument with 2

(4) We shall show that $T_{\Psi^Q(a)}(h_1\circ (f_1h_1)^{-1},v) = T_{\Psi^Q(a)}(f^{-1},v)$.
Considering the LHS,

$$
\begin{aligned}
T_{\Psi^Q(a)}(h_1\circ (fh_1)^{-1},v) &\geq \min\{T_{\Psi^Q(a)}(h_1,v), T_{\Psi^Q(a)}((fh_1)^{-1},v)\\
&\geq \min\Big\{T_{\Psi^Q(a)}(h_1,v), \min\underbrace{\{T_{\Psi^Q(a)}(f^{-1},v), T_{\Psi^Q(a)}(h_1^{-1},v)\}}_{AIP}\Big\}\\
&\geq \min\Big\{T_{\Psi^Q(a)}(h_1,v), \min\{T_{\Psi^Q(a)}(f_1,v), T_{\Psi^Q(a)}(h_1,v)\}\Big\}\\
&= \min\Big\{\min\{T_{\Psi^Q(a)}(f_1,v), T_{\Psi^Q(a)}(h_1,v)\}, T_{\Psi^Q(a)}(h_1,v)\Big\}\\
&= \min\Big\{T_{\Psi^Q(a)}(f_1,v), \min\{T_{\Psi^Q(a)}(h_1,v), T_{\Psi^Q(a)}(h_1,v)\}\Big\}\\
&\geq \min\Big\{T_{\Psi^Q(a)}(f_1,v), T_{\Psi^Q(a)}(h_1,v)\Big\} \qquad (14)
\end{aligned}
$$

On the other hand,

$$T_{\Psi^Q(a)}(f_1^{-1},v) \geq T_{\Psi^Q(a)}(f_1,v) \qquad (15)$$

Note that $T_{\Psi^Q(a)}(f_1, v) = T_{\Psi^Q(a)}((f_1/h_1) \circ h_1, v)$. Then, using the last equality in (18), we get

$$T_{\Psi^Q(a)}(f_1, v) \geq \min\{T_{\Psi^Q(a)}((f_1/h_1), v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$\geq \min\left\{ \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(h_1, v)\}\right\}$$
$$\geq \min\left\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\right\} \tag{16}$$

Considering the LHS, for the indeterminate membership

$$I_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) \leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}((f_1 h_1)^{-1}, v)$$
$$\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), \underbrace{\max\{I_{\Psi^Q(a)}(f_1^{-1}, v), I_{\Psi^Q(a)}(h_1^{-1}, v)\}}_{AIP}\right\}$$
$$\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}\right\}$$
$$= \max\left\{ \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\right\}$$
$$= \max\left\{I_{\Psi^Q(a)}(f_1, v), \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(h_1, v)\}\right\}$$
$$\leq \max\left\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\right\} \tag{17}$$

On the other hand,

$$I_{\Psi^Q(a)}(f_1^{-1}, v) \leq I_{\Psi^Q(a)}(f_1, v) \tag{18}$$

Note that $I_{\Psi^Q(a)}(f_1, v) = I_{\Psi^Q(a)}((f_1/h_1) \circ h_1, v)$. Then, using the last equalith in (18), we get

$$I_{\Psi^Q(a)}(f_1, v) \leq \max\{I_{\Psi^Q(a)}((f_1/h_1), v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$\leq \max\left\{ \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\right\}$$
$$= \max\left\{I_{\Psi^Q(a)}(f_1, v), \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(h_1, v)\}\right\}$$
$$\leq \max\left\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\right\} \tag{19}$$

The result for falsity membership is similar with the result for indeterminate membership obtained.

Theorem 3.14. *Let $(\Psi^Q, \mathfrak{A})$ be $Q - NS\hat{L}$ over a loop $(\hat{L}, \circ)$. Then, for all $a \in \mathfrak{A}, f_1, h_1, z_1 \in \hat{L}, v \in Q$ the following hold:*

(1) $T_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}, v) = T_{\Psi^Q(a)}(h_1^{-1} \circ f_1^{-1}, v)$, $I_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}, v) = I_{\Psi^Q(a)}(h_1^{-1} \circ f_1^{-1}, v)$, *and* $F_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}, v) = F_{\Psi^Q(a)}(h_1^{-1} \circ f_1^{-1}, v)$,

(2) $T_{\Psi^Q(a)}((f_1 h_1 \circ f_1)^{-1}, v) = T_{\Psi^Q(a)}(f_1^{-1} h_1^{-1} \circ f_1^{-1}, v), I_{\Psi^Q(a)}((f_1 h_1 \circ f_1)^{-1}, v) = I_{\Psi^Q(a)}(f_1^{-1} h_1^{-1} \circ f_1^{-1}, v)$ *and* $F_{\Psi^Q(a)}((f_1 h_1 \circ f_1)^{-1}, v) = F_{\Psi^Q(a)}(f_1^{-1} h_1^{-1} \circ f_1^{-1}, v)$

(3) $T_{\Psi^Q(a)}((f_1 \circ h_1 z)^{-1}, v) = T_{\Psi^Q(a)}(z^{-1} \circ h_1^{-1} f_1^{-1}, v), I_{\Psi^Q(a)}((f_1 \circ h_1 z)^{-1}, v) = I_{\Psi^Q(a)}(z^{-1} \circ h_1^{-1} f_1^{-1}, v)$ *and* $F_{\Psi^Q(a)}((f_1 \circ h_1 z)^{-1}, v) = F_{\Psi^Q(a)}(z^{-1} \circ h_1^{-1} f_1^{-1}, v)$

Proof:

(1) Let $(\Psi^Q, \mathfrak{A})$ be $Q-$neutrosophic soft loop over a loop $(\hat{L}, \circ)$. Then, we shall show that for all $a \in \mathfrak{A}, f_1, h_1 \in \hat{L}, v \in Q$

$$\begin{aligned} T_{\Psi^Q(a)}(f_1 \circ h_1)^{-1}, v) &= T_{\Psi^Q(a)}(f_1^{-1} \circ h_1^{-1}, v) \\ &\geq \min\{T_{\Psi^Q(a)}(f_1^{-1}, v), T_{\Psi^Q(a)}(h_1^{-1}, v)\} \\ &\geq \min\Big\{T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\Big\} \end{aligned} \tag{20}$$

On the other hand

$$\begin{aligned} T_{\Psi^Q(a)}(h_1^{-1} \circ f_1^{-1}, v) &\geq \min\{T_{\Psi^Q(a)}(h_1^{-1}, v), T_{\Psi^Q(a)}(f_1^{-1}, v)\} \\ &\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\} \end{aligned} \tag{21}$$

$$\begin{aligned} I_{\Psi^Q(a)}(f_1 \circ h_1)^{-1}, v) &= I_{\Psi^Q(a)}(f_1^{-1} \circ h_1^{-1}, v) \\ &\leq \max\{I_{\Psi^Q(a)}(f_1^{-1}, v), I_{\Psi^Q(a)}(h_1^{-1}, v)\} \\ &\leq \max\Big\{I_{\Psi^Q(a)}(f_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\Big\} \end{aligned} \tag{22}$$

RHS

$$\begin{aligned} I_{\Psi^Q(a)}(h_1^{-1} \circ f_1^{-1}, v) &\geq \max\{I_{\Psi^Q(a)}(h_1^{-1}, v), I_{\Psi^Q(a)}(f_1^{-1}, v)\} \\ &\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\} \end{aligned} \tag{23}$$

The result for falsity membership is similar with the result of indeterminate membership.

(2) For the true membership,

$$
\begin{aligned}
T_{\Psi^Q(a)}((f_1h_1 \circ f_1)^{-1}, v) &= \underbrace{T_{\Psi^Q(a)}((f_1 \circ h_1)^{-1} \circ f_1^{-1}), v)}_{AIP} \\
&\geq \min\{T_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}), v), T_{\Psi^Q(a)}(f_1^{-1}, v)\} \\
&\geq \min\{\min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\} \\
&= \min\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}\} \\
&= \min\{\min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\} \\
&\geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\} \qquad (24)
\end{aligned}
$$

Similarly, we obtain

$$
T_{\Psi^Q(a)}(f_1^{-1}h_1^{-1} \circ f_1^{-1}, v) \geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\} \qquad (25)
$$

$$
\begin{aligned}
I_{\Psi^Q(a)}((f_1h_1 \circ f_1)^{-1}, v) &= \underbrace{I_{\Psi^Q(a)}((f_1 \circ h_1)^{-1} \circ f_1^{-1}), v)}_{AIP} \\
&\leq \max\{I_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}), v), I_{\Psi^Q(a)}(f_1^{-1}, v)\} \\
&\leq \max\{\max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}, I_{\Psi^Q(a)}(f_1, v)\} \\
&= \max\{I_{\Psi^Q(a)}(f_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}\} \\
&= \max\{\max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\} \\
&\leq \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\} \qquad (26)
\end{aligned}
$$

Also, we obtain the indeterminate membership for the RHS

$$
I_{\Psi^Q(a)}(f_1^{-1}h_1^{-1} \circ f_1^{-1}, v) \geq \min\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\} \qquad (27)
$$

Using similar approach to obtain result for indeterminate membership.

(3) The proof is similar with the result obtained for 2

Theorem 3.15. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q-$NSS over a loop $(\hat{L}, \circ, /, \backslash)$. Then, $(\Psi^Q, \mathfrak{A})$ is $Q - NS\hat{L}$ if and if only for all $f_1, h_1 \in \hat{L}, v \in Q$*

(1) $T_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\}$
$I_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1, v)\}$,
$F_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1, v)\}$

(2) $T_{\Psi^Q(a)}(^{1-}f_1 * h_1, v) \geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q}(a)(h_1, v)\}$,
$I_{\Psi^Q(a)}(^{1-}f_1 * h_1, v) \leq \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q}(a)(h_1, v)\}$
$F_{\Psi^Q(a)}(^{1-}f_1 * h_1, v) \leq \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q}(a)(h_1, v)\}$

(1) Suppose that $(\Psi^Q, \mathfrak{A})$ is a $Q-NS\hat{L}$ over $(\hat{L}, \circ, /, \backslash)$. Let $* \in \{\circ, /, \backslash\}$ then we show that $(\Psi^Q, \mathfrak{A})$ satisfies 3.9 for all $f_1, h_1 \in \hat{L}$, and $v \in Q$, we have

$$\begin{aligned}
T_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) &\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1^{-1}, v)\} \\
&\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\} \\
I_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) &\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1, v)\} \\
F_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) &\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1^{-1}, v)\} \\
&\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1, v)\}
\end{aligned}$$

Conversely, suppose equality (1) hold, then for all $f_1, h_1, \in G, v \in Q$, and $a \in \mathfrak{A}$, we show $(\Psi^Q, \mathfrak{A})$ is $Q-$neutrosophic soft subquasigroup over quasigroup $(\hat{L}, \circ, /, \backslash)$. Thus,

$$\begin{aligned}
T_{\Psi^Q(a)}(h_1 * f_1, v) &= T_{\Psi^Q(a)}(h_1 * (f_1^{-1})^{-1}, v) \\
&\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1^{-1}, v) \\
&\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1^{-1}, v)\} \\
&\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\}
\end{aligned}$$

Next:

$$\begin{aligned}
I_{\Psi^Q(a)}(h_1 * f_1, v) &= I_{\Psi^Q(a)}(h_1 * (f_1^{-1})^{-1}, v) \\
&\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1, v)\}
\end{aligned}$$

Finally:

$$\begin{aligned}
F_{\Psi^Q(a)}(h_1 * f_1, v) &= F_{\Psi^Q(a)}(h_1 * (f_1^{-1})^{-1}, v) \\
&\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1^{-1}, v)\} \\
&\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1, v)\}
\end{aligned}$$

(2) it is similar to (1)

4. **Conclusion**

In this study, it was found that $Q-NS\hat{G}$ obeys LIP, RIP, LAP, RAP and flexible law. With the help AIP, it was shown that $Q-NS\hat{G}$ obey AAIP, SAIP, SAAIP. $Q-NS\hat{L}$ were also defined, and the definition was used to shown when is $Q-$NSS under loop said to be $Q-NS\hat{G}$. Furthermore, this research revealed that left and right inverse elements of $Q-NS\hat{L}$ coincided. In future research, Definitions 3.1 and 3.9 will be use to study the structure of isotopy theory of quasigroup.

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References

1. M. Abu Qamar and N. Hassan, $Q-$neutrosophic soft relation and its application in decision making, Entropy, 20, 172, pp. 1-14, (2018)
2. L. A. Zadeh, Fuzzy Sets, Inform. Control, 8, pp. 338-353, (1965)
3. D. Molodtsov. Soft set theory-first results, Comput. Math. App., 37, pp. 19–31, (1999)
4. K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets Syst. 20, 87–96,(1986)
5. A. Rosenfeld, fuzzy groups, J. math. Anal.Appl. 35, PP. 512-517, (1971)
6. K. T. Atanassov, Intuitionistic Fuzzy Sets, Physica-Verlag, Heidelberg, N.Y., 1999.
7. P. K. Maji, Neutrosophic soft set, Ann. Fuzzy Math. Inform., 5, pp.157–168, (2013)
8. M. Jdid, F. Smarandache and S. Broumi, Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic, Neutrosophic Systems with Applications, 1, 4–13,(2023) (Doi: https://doi.org/10.5281/zenodo.8171135).
9. S. Dey and G. C. Ray, Separation Axioms in Neutrosophic Topological Spaces, Neutrosophic Systems with Applications, 2(2023), 38–54. (Doi: https://doi.org/10.5281/zenodo.8195851);
10. M. Jdid and F. Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, Neutrosophic Systems with Applications, 3 (2023), 1–16. (Doi: https://doi.org/10.5281/zenodo.8196397);
11. M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number, Applied Soft Computing, 77, pp.438-452, (2019)
12. R. H. Bruck, Contribution to the theory of quasigroups, Trans. Amer. Math. Soc., Vol. 60, pp. 245-354, (1946)
13. M. Abdel-Baset and V. Chang, and A. Gamal, Evaluation of the green supply chain management practices: A novel neutrosophic approach, Computers in Industry, 108, pp. 210-220, (2019)
14. F. Smarandache, Neutrosophy. Neutrosophic Probability, Set and Logic, American Research Press: Rehoboth, IL, USA, (1998).
15. F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. Int. J. Pure Appl. Math., 24, pp. 287–297, (2005)
16. A. Solairaju and R. Nagarajan, A new structure and construction of Q-fuzzy groups, Advances in Fuzzy Mathematics, 4, pp.23-29, (2009)
17. S. Thiruveni and A. Solairaju, Neutrosophic Q-fuzzy subgroups, Int. J. Math. And Appl., 6, pp. 859-866, (2018)
18. M. Abu-Qamar, and N. Hassan, Characterization of group theory under Q-Neutrosophic soft Environment, Neutrosophic Sets and system, Vol. 27, pp. 114-131, (2019).
19. Muhammad Shabi, Mumtaz Ali, Munazza Naz, and Florentin Smarandache, Soft Neutrosophic Group, Neutrosophic Sets and Systems, Vol. 1, pp. 13-25, (2013)
20. M. Ali, F. Smarandache, M. Shabir, Soft Neutrosophic Groupoids and Their Generalization, Neutrosophic Sets and Systems, Vol. 6, pp. 62-81, (2014)
21. B. V. N. Prasad and J. Venkateswara Rao, Characterization of Quasigroups and Loops. International Journal of Scientific and Innovative Mathematical Research (IJSIMR) 1, 2, pp. 95-102, (2013)

22. Davvaz .B, Dudek W. A and Yun. Y. B, Intuitionistic fuzzy sub hyper quasi groups of hyper quasi groups, Information sciences, 170, pp. 251-262, (2005)
23. W. A. Dudek, Fuzzy subquasigroups, Quasigroups and Related Systems 5, pp. 81-98, (1998)
24. W. A. Dudek and Y. B. Jun, Fuzzy subquasigroups over a t-norm, Quasigroups and Related Systems 6, pp. 87-89, (1999)
25. Y. T.Oyebo, B. Osoba and A. O. Abdulkareem, Distributive Properties of $Q-$neutrosophic Soft Quasigroups, Neutrosophic Set Systems.58, pp. 448-524, (2023)
26. B. Osoba and Y. T. Oyebo, *On the Core of Second Smarandach Bol Loops*, International Journal of Mathematical Combinatorics, 2, pp. 18–30, (2023). http://doi.org/10.5281/zenodo.32303.
27. K. H, W. A. Dudek and Y. B. Jun, Intuitionistic Fuzzy subquasigroups of quasigroups, Quasigroups and Related Systems 7, pp. 15-28,(2000)
28. W. A. Dudek, Intuitionistic Fuzzy approach to n ary systems, Quasigroups and Related Systems 13, pp. 213-228, (2005)
29. Oyem. A, Olaluru. J. O, Jaiyeola. T. G and Akewe. H, Soft quasigroup, International Journal of mathematical Sci. Opt.: Theory and appl. 2, pp. 834-846,(2020)
30. A. Oyem, T. G. Jaiyeola, J. O. Olaleru and B. Osoba, Soft Neutrosophic quasigroups, Neutrosophic Set Systems. 50, 488–503, (2022)
31. A. I. Mal'tsev, Algebraic Systems. Nauka, Moscow, 1976 (in Russian)
32. O. Hala. Pflugfelder, Quasigroups and loops: introduction, Sigma Series in Pure Mathematics Volume 7, (1990)
33. Jaiyeola, Temitope Gbolahan; Kehinde Adam Olurode; and Benard Osoba, Some Neutrosophic Triplet Subgroup Properties and Homomorphism Theorems in Singular Weak Commutative Neutrosophic Extended Triplet Group, Neutrosophic Sets and Systems 45, 1, (2021).

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