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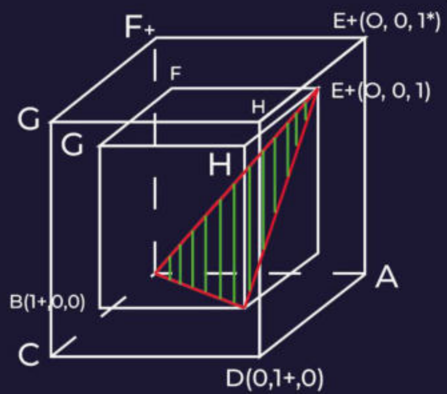

NEUTROSOPHIC SETS AND SYSTEMS

Special Issue on Neutrosophic Algebraic Structures, NeutroAlgebra & AntiAlgebra and SuperHyperAlgebra & Neutrosophic SuperHyperAlgebra Contributions of Researchers from the Arab World

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Information for Authors and Suscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their inter actions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).
According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of]-0, 1+[.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.
Neutrosophic Statistics is a generalization of the classical statistics.
What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, igno- rance, imprecision, etc.

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Preface of The Special Issue on Neutrosophic Algebraic Structures, NeutroAlgebra & AntiAlgebra and SuperHyperAlgebra & Neutrosophic SuperHyperAlgebra - Contributions of Researchers from the Arab World

It is our pleasure to introduce this special issue on Neutrosophic Algebraic Structures. The study of algebraic structures has always been at the forefront of mathematical research, providing a framework for understanding the properties of mathematical objects and the relationships between them. In recent years, the study of Neutrosophic Algebraic Structures has emerged as a new and promising area of research that combines the traditional algebraic structures with the concept of Neutrosophy, which deals with the study of indeterminacy, contradiction, and incomplete information.

The papers in this special issue cover a wide range of topics related to Neutrosophic Algebraic Structures, including the development of new algebraic structures, the study of their properties and applications, and the implementation of these structures in various real-world problems. The papers also demonstrate the active and vibrant research community in Arab countries and their contributions to the field.

We would like to express our sincere gratitude to the authors who have contributed their research to this special issue, as well as the reviewers who have provided valuable feedback and suggestions. We hope that this special issue will inspire further research and collaboration in the field of Neutrosophic Algebraic Structures and contribute to the advancement of science and technology in Arab countries and beyond.

Finally, we would like to thank the editorial board of Neutrosophic Sets and Systems for providing us with the opportunity to publish this special issue and for their continued support.

Mohammad Abobala

Guest Editor

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On The 3-Refined Neutrosophic Analytical Structures and Number Theoretical Concepts

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Abstract:

n-refined neutrosophic structures are considered as generalizations of classical structures, and neutrosophic structures.

The main goal of this paper, is to study several structures generated by using 3-refined neutrosophic numbers, where we find the mathematical formulas of 3-refined neutrosophic real functions. Also, the inner products over 3-refined neutrosophic vector spaces and orthogonal properties. In addition, we present the foundations of 3-refined neutrosophic number theory, especially division, congruencies, and some related equations.

Keywords: 3-refined neutrosophic real function, 3-refined neutrosophic inner product, 3-refined neutrosophic vector space, 3-refined neutrosophic number theory

Introduction and basic concepts

The concept of neutrosophic structures plays an important role in the theory of algebraic structures and analysis. Many concepts and structures were defined previously, such as neutrosophic vector spaces, neutrosophic matrices, and algebraic rings [1-3, 5-7, 14-16].

Laterally, refined neutrosophic structures were defined to generalize the neutrosophic structures, where refined neutrosophic rings, modules, and other structures were presented [4, 8-11, 24-28].

The concept of n-refined neutrosophic structure is considered as a generalization of refined structure [12, 29]. For each value of the integer n, we get a generalized structure.

This work will study some of 3-refined neutrosophic structures, where we present the formulas of 3-refined neutrosophic real functions, 3-refined neutrosophic inner products defined over 3-refined neutrosophic vector spaces, and 3-refined number theoretical concepts.

First, we recall some basic concepts.

Definition:

Let (R,+,×) be a ring, $R(I) = \{a + bI \; ; \; a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Definition:

Let (R,+,×) be a ring, (R(I_1, I_2) , + ,×) is called a refined neutrosophic ring generated by R , I_1, I_2.

Definition:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)=$\{a_0 + a_1 I + \cdots + a_n I_n \; ; \; a_i \in R\}$ to be n-refined neutrosophic ring.

Addition and multiplication on R_n(I) are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i$,$\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

For n=3, we get the 3-refined neutrosophic ring.

Main Discussion

Definition.

Let $R_3(I)$ be the 3-refined neutrosophic ring of reals, $f: R_3(I) \to R_3(I); f = f(X); X \in R_3(I)$. f is called 3-refined neutrosophic real function with one variable.

Theorem.

$R_3(I) \cong R^4$.

Proof.

We define $g: R_3(I) \to R^4; g(a + bI_1 + cI_2 + dI_3) = (a, a + b + c + d, a + c + d, a + d)$.

It is clear that g is well defined function.

$ker(g) = \{0\}$, thus g is injective.

$Im(g) = R^4$, thus g is surjective, so that g is one-to-one.

Now, let $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3, B = b_0 + b_1 I_1 + b_2 I_2 + b_3 I_3$,

$A + B = (a_0 + b_0) + (a_1 + b_1)I_1 + (a_2 + b_2)I_2 + (a_3 + b_3)I_3$

$g(A + B) = g(A) + g(B)$.

$$A.B = a_0 b_0 + [(a_0 + a_1 + a_2 + a_3)(b_0 + b_1 + b_2 + b_3) - (a_0 + a_2 + a_3)(b_0 + b_2 + b_3)]I_1 + [(a_0 + a_2 + a_3)(b_0 + b_2 + b_3) - (a_0 + a_3)(b_0 + b_3)]I_2 + [(a_0 + a_3)(b_0 + b_3) - a_0 b_0]I_3$$

$g(A.B) = g(A).g(B)$, hence g is a ring isomorphism.

Remark.

Let $f: R_3(I) \to R_3(I)$ be a 3-refined neutrosophic real function with one variable, then f can be represented by four classical real functions by taking the direct isomorphic image $g(f(X))$.

Example.

Take $f(X) = (1 + I_1)X^2 + (2 - I_2 - I_3)X + 1 + 2I_1 + I_2 + I_3$, f can be represented as follows:

$$g(f(X)) = g(1 + I_1)(g(X))^2 + g(2 - I_2 - I_3)g(X) + g(1 + 2I_1 + I_2 + I_3)$$

$$g(f(X)) = (1,2,1,1)({x_0}^2, (x_0 + x_1 + x_2 + x_3)^2, (x_0 + x_2 + x_3)^2, (x_0 + x_3)^2) + (2,0,0,1)(x_0, x_0 + x_1 + x_2 + x_3, x_0 + x_2 + x_3, x_0 + x_3) + (1,5,3,2)$$

$$g(f(X)) = ({x_0}^2 + 2x_0 + 1, 2(x_0 + x_1 + x_2 + x_3)^2 + 5, (x_0 + x_2 + x_3)^2 + 3, (x_0 + x_3)^2 + (x_0 + x_3) + 2)$$

The four classical functions that represent (f) are:

$f_1: R \to R;\ f_1(x_0) = {x_0}^2 + 2x_0 + 1$

$f_2: R \to R;\ f_2(x_0 + x_1 + x_2 + x_3) = 2(x_0 + x_1 + x_2 + x_3)^2 + 5$

$f_3: R \to R;\ f_3(x_0 + x_2 + x_3) = (x_0 + x_2 + x_3)^2 + 3$

$f_4: R \to R;\ f_4(x_0 + x_3) = (x_0 + x_3)^2 + (x_0 + x_3) + 2$

Theorem.

Let $g: R_3(I) \to R^4$ be the isomorphism defined above, then:

$g^{-1}: R^4 \to R_3(I); g^{-1}(a, b, c, d) = a + (b - c)I_1 + (c - d)I_2 + (d - a)I_3$.

The proof is easy.

Remark.

To find the formula of a 3-refined neutrosophic real function $f: R_3(I) \to R_3(I)$, we went compute:

$g^{-1}\left(g\big(f(X)\big)\right)$.

Example.

For the function $f(X) = (1 + I_1)X^2 + (2 - I_2 - I_3)X + 1 + 2I_1 + I_2 + I_3$, we compute:

$$g^{-1}\left(g\big(f(X)\big)\right) = (x_0{}^2 + 2x_0 + 1) + [2(x_0 + x_1 + x_2 + x_3)^2 + 5 - (x_0 + x_2 + x_3)^2 - 3]I_1 + [(x_0 + x_2 + x_3)^2 + 3 - (x_0 + x_3)^2 - (x_0 + x_3) - 2]I_2 + [(x_0 + x_3)^2 + (x_0 + x_3) + 2 - x_0{}^2 - 2x_0 - 1]I_3$$

$$= x_0{}^2 + 2x_0 + 1 + [2(x_0 + x_1 + x_2 + x_3)^2 - (x_0 + x_2 + x_3)^2 + 2]I_1 + [(x_0 + x_2 + x_3)^2 - (x_0 + x_3)^2 - (x_0 + x_3) + 1]I_2 + [(x_0 + x_3)^2 + (x_0 + x_3) - x_0{}^2 - 2x_0 + 1]I_3$$

Definition.

Let $f: R_3(I) \to R_3(I)$ be a 3-refined neutrosophic real function, and $g\big(f(X)\big) = (f_1, f_2, f_3, f_4)$, with $f_i: R \to R; 1 \le i \le 4$, we say:

a). f is differentiable if and only if f_i are differentiable.

b). f is integrable if and only if f_i are integrable.

We mean by differentiable/integrable on all R not only for sub-domains $]a, b[\subseteq R$.

Example on famous functions.

1. $f: R_3(I) \to R_3(I), f(X) = sin(X)$.

It's formula is $f(X) = g^{-1}\left(g\big(f(X)\big)\right) = sin(x_0) + [sin(x_0 + x_1 + x_2 + x_3) - sin(x_0 + x_2 + x_3)]I_1 + [sin(x_0 + x_2 + x_3) - sin(x_0 + x_3)]I_2 + [sin(x_0 + x_3) - sin(x_0)]I_3$.

2. $f(X) = cos(X) = g^{-1}\left(g\big(f(X)\big)\right) = cos(x_0) + [cos(x_0 + x_1 + x_2 + x_3) - cos(x_0 + x_2 + x_3)]I_1 + [cos(x_0 + x_2 + x_3) - cos(x_0 + x_3)]I_2 + [cos(x_0 + x_3) - cos(x_0)]I_3$.

3. $f(X) = tan(X) = tan(x_0) + [tan(x_0 + x_1 + x_2 + x_3) - tan(x_0 + x_2 + x_3)]I_1 + [tan(x_0 + x_2 + x_3) - tan(x_0 + x_3)]I_2 + [tan(x_0 + x_3) - tan(x_0)]I_3$

4. $f(X) = cot(X) = cot(x_0) + [cot(x_0 + x_1 + x_2 + x_3) - cot(x_0 + x_2 + x_3)]I_1 + [cot(x_0 + x_2 + x_3) - cot(x_0 + x_3)]I_2 + [cot(x_0 + x_3) - cot(x_0)]I_3$

5. $f(X) = e^X = e^{x_0} + [e^{x_0+x_1+x_2+x_3} - e^{x_0+x_2+x_3}]I_1 + [e^{x_0+x_2+x_3} - e^{x_0+x_3}]I_2 + [e^{x_0+x_3} - e^{x_0}]I_3$

6. $f(X) = ln(X) = ln(x_0) + [ln(x_0 + x_1 + x_2 + x_3) - ln(x_0 + x_2 + x_3)]I_1 + [ln(x_0 + x_2 + x_3) - ln(x_0 + x_3)]I_2 + [ln(x_0 + x_3) - ln(x_0)]I_3$, with $X > 0$.
7. $f(X) = X^n = {x_0}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - {x_0}^n]I_3; n \in N.$

Definition.

Let V be vector space over R, the 3-refined neutrosophic vector space is defined as follows:

$$V_3(I) = V + VI_1 + VI_2 + VI_3 = \{x + yI_1 + zI_2 + tI_3\ ; x, y, z, t \in R\}$$

Remark.

Addition on $V_3(I)$ is defined:

$$(x_0 + y_0I_1 + z_0I_2 + t_0I_3) + (x_1 + y_1I_1 + z_1I_2 + t_1I_3) = (x_0 + x_1) + (y_0 + y_1)I_1 + (z_0 + z_1)I_2 + (t_0 + t_1)I_3$$

Where $x_i, y_i, z_i, t_i \in V\ ; 0 \le i \le 1$.

Multiplication on $V_3(I)$ is defined:

$$(a + bI_1 + cI_2 + dI_3).(x + yI_1 + zI_2 + tI_3) = (a.x) + (ay + bx + by + bz + bt + cy + dy)I_1 + (az + cx + cz + ct + dz)I_2 + (at + dx + dt)I_3$$

Where $a, b, c, d \in R\ , x, y, z, t \in V$

Remark.

$(V_3(I), +, .)$ Is a module over $R_3(I)$.

Definition.

Let $f: V_3(I) \times V_3(I) \to R_3(I)$ be a well defined mapping, we call f a 3-refined neutrosophic real inner product if and only if the following conditions hold:

1). $f(X, X) \ge 0\ ;\ \forall X \in V_3(I)$.

2). $f(X, X) = 0 \Leftrightarrow X = 0$.

3). $f(X, Y) = f(Y, X)\ ;\ \forall X, Y \in V_3(I)$.

4). $f(\alpha X + \beta Y, Z) = \alpha f(X, Z) + \beta f(X, Z)\ ;\ \forall X, Y, Z \in V_3(I)\ , \alpha, \beta \in R_3(I)$.

Theorem.

Let $g: V \times V \to R$be an inner production on V, then for $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3 \in V_3(I)$, the mapping $f: V_3(I) \times V_3(I) \to R_3(I)$such that:

$$f(X,Y)=g(x_0,y_0)+\big(g(x_0+x_1+x_2+x_3,y_0+y_1+y_2+y_3)-g(x_0+x_2+x_3,y_0+y_2+y_3)\big)I_1+\big(g(x_0+x_2+x_3,y_0+y_2+y_3)-g(x_0+x_3,y_0+y_3)\big)I_2+\big(g(x_0+x_3,y_0+y_3)-g(x_0,y_0)\big)I_3$$

Is a 3-refined neutrosophic inner product.

Proof.

$$f(X,X)=g(x_0,x_0)+\big(g(x_0+x_1+x_2+x_3,x_0+x_1+x_2+x_3)-g(x_0+x_2+x_3,x_0+x_2+x_3)\big)I_1+\big(g(x_0+x_2+x_3,x_0+x_2+x_3)-g(x_0+x_3,x_0+x_3)\big)I_2+\big(g(x_0+x_3,x_0+x_3)-g(x_0,x_0)\big)I_3$$

$$=\|x_0\|^2+(\|x_0+x_1+x_2+x_3\|^2-\|x_0+x_2+x_3\|^2)I_1+(\|x_0+x_2+x_3\|^2-\|x_0+x_3\|^2)I_2+(\|x_0+x_3\|^2-\|x_0\|^2)I_3\geq 0$$

According to the concept of partial ordering on $R_3(I)$.

$f(X,X)=0\Leftrightarrow \|x_0\|^2=\|x_0+x_1+x_2+x_3\|^2=\|x_0+x_2+x_3\|^2=\|x_0+x_3\|^2=0$

Thus $x_0=x_1=x_2=x_3=0$and $X=0$.

It is clear that $f(X,Y)=f(Y,X)$.

Now, let $A=a_0+a_1I_1+a_2I_2+a_3I_3, B=b_0+b_1I_1+b_2I_2+b_3I_3\in R_3(I)$ and $Z=z_0+z_1I_1+z_2I_2+z_3I_3\in V_3(I)$, we have:

$AX+BY=(a_0x_0+b_0y_0)+\big((a_0+a_1+a_2+a_3)(x_0+x_1+x_2+x_3)-(a_0+a_2+a_3)(x_0+x_2+x_3)+(b_0+b_1+b_2+b_3)(y_0+y_1+y_2+y_3)-(b_0+b_2+b_3)(y_0+y_2+y_3)\big)I_1+\big((a_0+a_2+a_3)(x_0+x_2+x_3)-(a_0+a_3)(x_0+x_3)+(b_0+b_2+b_3)(y_0+y_2+y_3)-(b_0+b_3)(y_0+y_3)\big)I_2+\big((a_0+a_3)(x_0+x_3)-a_0x_0+(b_0+b_3)(y_0+y_3)-b_0y_0\big)I_3$

$$f(AX+BY,Z)=g(a_0x_0+b_0y_0,z_0)$$
$$+\Big(g\big((a_0+a_1+a_2+a_3)(x_0+x_1+x_2+x_3)$$
$$+(b_0+b_1+b_2+b_3)(y_0+y_1+y_2+y_3),z_0+z_1+z_2+z_3\big)$$
$$-g\big((a_0+a_2+a_3)(x_0+x_2+x_3)+(b_0+b_2+b_3)(y_0+y_2+y_3),z_0+z_2$$
$$+z_3\big)\Big)I_1$$
$$+\Big(g\big((a_0+a_2+a_3)(x_0+x_2+x_3)+(b_0+b_2+b_3)(y_0+y_2+y_3),z_0+z_2$$
$$+z_3\big)-g\big((a_0+a_3)(x_0+x_3)+(b_0+b_3)(y_0+y_3),z_0+z_3\big)\Big)I_2$$
$$+\Big(g\big((a_0+a_3)(x_0+x_3)+(b_0+b_3)(y_0+y_3),z_0+z_3\big)$$
$$-g(a_0x_0+b_0y_0,z_0)\Big)I_3$$
$$=(a_0+a_1I_1+a_2I_2+a_3I_3)f(X,Z)+(b_0+b_1I_1+b_2I_2+b_3I_3)f(X,Z)$$

Theorem.

Let $f:V_3(I)\times V_3(I)\to R_3(I)$ be a 3-refined neutrosophic real inner product, then $g:V\times V\to R$ such that:

$g(x,y)=f(x+0I_1+0I_2+0I_3,y+0I_1+0I_2+0I_3)$ is a classical inner product on V.

The proof is clear.

Definition.

Let $X=x_0+x_1I_1+x_2I_2+x_3I_3\in V_3(I)$ and $f:V_3(I)\times V_3(I)\to R_3(I)$ be a 3-refined neutrosophic real inner product, then:

1. If $Y=y_0+y_1I_1+y_2I_2+y_3I_3\in V_3(I)$, then $X\perp Y$ if and only if $f(X,Y)=0$.
2. $\|X\|^2=f(X,X)$.

Theorem.

Let f be a 3-refined neutrosophic real inner product on $V_3(I)$ and $X=x_0+x_1I_1+x_2I_2+x_3I_3, Y=y_0+y_1I_1+y_2I_2+y_3I_3\in V_3(I)$, then:

1). $X\perp Y$ if and only if $\begin{cases} x_0\perp y_0, x_0+x_3\perp y_0+y_3\\ x_0+x_2+x_3\perp y_0+y_2+y_3\\ x_0+x_1+x_2+x_3\perp y_0+y_1+y_2+y_3\end{cases}$

2). $\|X\|=\|x_0\|+(\|x_0+x_1+x_2+x_3\|-\|x_0+x_2+x_3\|)I_1+(\|x_0+x_2+x_3\|-\|x_0+x_3\|)I_2+(\|x_0+x_3\|-\|x_0\|)I_3$

Proof.

1). $X\perp Y\Leftrightarrow f(X,Y)=0\Leftrightarrow g(x_0,y_0)=g(x_0+x_3,y_0+y_3)=g(x_0+x_1+x_2+x_3,y_0+y_1+y_2+y_3)=g(x_0+x_2+x_3,y_0+y_2+y_3)=0$, hence the proof holds.

2). We put $T = \|x_0\| + (\|x_0 + x_1 + x_2 + x_3\| - \|x_0 + x_2 + x_3\|)I_1 + (\|x_0 + x_2 + x_3\| - \|x_0 + x_3\|)I_2 + (\|x_0 + x_3\| - \|x_0\|)I_3$

We compute $T^2 = \|x_0\|^2 + (\|x_0 + x_1 + x_2 + x_3\|^2 - \|x_0 + x_2 + x_3\|^2)I_1 + (\|x_0 + x_2 + x_3\|^2 - \|x_0 + x_3\|^2)I_2 + (\|x_0 + x_3\|^2 - \|x_0\|^2)I_3 = f(X, X) = \|X\|^2$, thus $T = \|X\|$.

Example.

Let $X = 3 + 2I_1 - I_2 - I_3,\ x_0 = 3, x_1 = 2, x_2 = -1, x_3 = -1$, then:

$\|X\| = |3| + (|3| - |1|)I_1 + (|1| - |2|)I_2 + (|2| - |3|)I_3 = 3 + 2I_1 - I_2 - I_3$

Example.

Let $V = R^2,\ V_3(I) = R_3{}^2(I)$, take $X = (1,1) + (2,1)I_1 + (3,-1)I_2 + (-1,4)I_3$.

$x_0 = (1,1), \|x_0\| = \sqrt{2}$, $x_0 + x_3 = (0,5), \|x_0 + x_3\| = 5$, $x_0 + x_2 + x_3 = (3,4), \|x_0 + x_2 + x_3\| = 5,\ +x_1 + x_2 x_0 + x_3 = (5,5), \|x_0 + x_1 + x_2 + x_3\| = 5\sqrt{2}$

$\|X\| = \sqrt{2} + (5\sqrt{2} - 5)I_1 + (5 - 5)I_2 + (5 - \sqrt{2})I_3 = \sqrt{2} + (5\sqrt{2} - 5)I_1 + (5 - \sqrt{2})I_3$

Remark.

$\forall X, Y \in V_3(I)$, then: $\|X\| \geq 0,\ \|X + Y\| \leq \|X\| + \|Y\|$.

Theorem.

Let $X, Y \in V_3(I)$, then $|f(X, Y)| \leq \|X\|.\|Y\|$.

Proof.

$$\begin{aligned} f(X,Y) = g(x_0, y_0) & \\ &+ \big(g(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) - g(x_0 + x_2 + x_3, y_0 + y_2 + y_3)\big)I_1 \\ &+ \big(g(x_0 + x_2 + x_3, y_0 + y_2 + y_3) - g(x_0 + x_3, y_0 + y_3)\big)I_2 \\ &+ \big(g(x_0 + x_3, y_0 + y_3) - g(x_0, y_0)\big)I_3 \end{aligned}$$

According to Cauchy-Shwartz inequality on the space V, we have:

$|g(x_0, y_0)| \leq \|x_0\|.\|y_0\|,\ |g(x_0 + x_2 + x_3, y_0 + y_2 + y_3)| \leq \|x_0 + x_2 + x_3\|.\|y_0 + y_2 + y_3\|$

$|g(x_0 + x_3, y_0 + y_3)| \leq \|x_0 + x_3\|.\|y_0 + y_3\|$

$|g(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3)| \leq \|x_0 + x_1 + x_2 + x_3\|.\|y_0 + y_1 + y_2 + y_3\|$

Thus, $|f(X, Y)| \leq \|X\|.\|Y\|$, according to the definition of partial order relation.

Example.

Take $V_3(I) = R_3{}^2(I), X = (1,1) + (1,0)I_1 + (0,1)I_2, Y = (2,0) + (0,3)I_1 + (1,0)I_2 + (0,1)I_3$

$x_0 = (1,1), y_0 = (2,0), g(x_0, y_0) = 2, \|x_0\| = \sqrt{2}, \|y_0\| = 2$, $x_0 + x_3 = (1,1), \|x_0 + x_3\| = \sqrt{2}, y_0 + y_3 = (2,1), \|y_0 + y_3\| = \sqrt{5}, g(x_0 + x_3, y_0 + y_3) = 3$, $x_0 + x_2 + x_3 = (1,2), \|x_0 + x_2 + x_3\| = \sqrt{3}, y_0 + y_2 + y_3 = (3,1), \|y_0 + y_2 + y_3\| = \sqrt{10},$, $x_0 + x_1 + x_2 + x_3 =$

$(2,2), \|x_0 + x_1 + x_2 + x_3\| = 2\sqrt{2}, y_0 + y_1 + y_2 + y_3 = (3,4), \|y_0 + y_1 + y_2 + y_3\| = 5$,

$g(x_0 + x_2 + x_3, y_0 + y_2 + y_3) = 5,\ g(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = 14.$

$f(X,Y) = 2 + (14 - 5)I_1 + (5 - 3)I_2 + (3 - 2)I_3 = 2 + 9I_1 + 2I_2 + I_3$

$|f(X,Y)| = 2 + 9I_1 + 2I_2 + I_3$

$\|X\| = \sqrt{2} + (2\sqrt{2} - \sqrt{5})I_1 + (\sqrt{5} - \sqrt{2})I_2 + (\sqrt{2} - \sqrt{2})I_3$

$= \sqrt{2} + (2\sqrt{2} - \sqrt{5})I_1 + (\sqrt{5} - \sqrt{2})I_2$

$\|X\| = 2 + (5 - \sqrt{10})I_1 + (\sqrt{10} - \sqrt{5})I_2 + (\sqrt{5} - \sqrt{2})I_3$

$\|X\|.\|Y\| = 2\sqrt{2} + (10\sqrt{2} - 5\sqrt{2})I_1 + (5\sqrt{2} - \sqrt{10})I_2 + (\sqrt{10} - 2\sqrt{2})I_3$

$= 2\sqrt{2} + 5\sqrt{2}I_1 + (5\sqrt{2} - \sqrt{10})I_2 + (\sqrt{10} - 2\sqrt{2})I_3$

On the other hand, we have:

$2 \le 2\sqrt{2}, 2 + 1 = 3 \le \sqrt{10}, 5 \le 5\sqrt{2}, 14 \le 10\sqrt{2}$, hence $|f(X,Y)| \le \|X\|.\|Y\|$

The Foundations 3-Refined Number Theory

Definition.

Let $Z_3(I) = \{a + bI_1 + cI_2 + dI_3; a,b,c,d \in Z\}$ be a set. It is called the ring of 3-refined neutrosophic integers if $I_i.I_j = I_{min(i,j)},\ I_i^2 = I_i; 1 \le i \le 3.$

It is a special case of the n-refined neutrosophic ring with $n = 3$.

Definition.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3, Z = z_0 + z_1I_1 + z_2I_2 + z_3I_3 \in Z_3(I)$, we define:

1). $X \setminus Y$ if there exists $T = t_0 + t_1I_1 + t_2I_2 + t_3I_3 \in Z_3(I)$ such that $T.X = Y$.

2). $X \equiv Y(mod\ Z)$ if and only if $Z \setminus X - Y$.

3). $X \ge Y$ if and only if $\begin{cases} x_0 \ge y_0 \\ x_0 + x_1 + x_2 + x_3 \ge y_0 + y_1 + y_2 + y_3 \\ x_0 + x_2 + x_3 \ge y_0 + y_2 + y_3 \\ x_0 + x_3 \ge y_0 + y_3 \end{cases}$

Theorem.

Let X, Y, Z be the previous 3-refined neutrosophic integers, then:

1. $X \setminus Y$ if and only if $x_0 \setminus y_0, x_0 + x_1 + x_2 + x_3 \setminus y_0 + y_1 + y_2 + y_3, x_0 + x_2 + x_3 \setminus y_0 + y_2 + y_3, x_0 + x_3 \setminus y_0 + y_3$.
2. If $X \setminus Y$, then $X \le Y$.

3. $X \equiv Y(mod\, Z)$ if and only if

$$\begin{cases} x_0 \equiv y_0(mod\, z_0) \\ x_0 + x_1 + x_2 + x_3 \equiv y_0 + y_1 + y_2 + y_3(mod\, z_0 + z_1 + z_2 + z_3) \\ x_0 + x_2 + x_3 \equiv y_0 + y_2 + y_3(mod\, z_0 + z_2 + z_3) \\ x_0 + x_3 \equiv y_0 + y_3(mod\, z_0 + z_3) \end{cases}$$

Proof.

1. Assume that $X \setminus Y$, this is true if and only if there exists $T = t_0 + t_1I_1 + t_2I_2 + t_3I_3 \in Z_3(I)$ such that $Y = X$.T.

We have:

$$X.T = (x_0 + x_1I_1 + x_2I_2 + x_3I_3)(t_0 + t_1I_1 + t_2I_2 + t_3I_3)$$
$$= x_0t_0 + (x_0t_1 + x_1t_0 + x_1t_1 + x_1t_2 + x_1t_3 + x_2t_1 + x_3t_1)I_1$$
$$+ (x_0t_2 + x_2t_0 + x_2t_2 + x_3t_2)I_2 + (x_0t_3 + x_3t_0 + x_3t_3)I_3$$
$$= y_0 + y_1I_1 + y_2I_2 + y_3I_3$$

Thus:

$$\begin{cases} y_0 = x_0t_0 \dots (1) \\ y_1 = x_0t_1 + x_1t_0 + x_1t_1 + x_1t_2 + x_1t_3 + x_2t_1 + x_3t_1 \dots (2) \\ y_2 = x_0t_2 + x_2t_0 + x_2t_2 + x_3t_2 \dots (3) \\ y_3 = x_0t_3 + x_3t_0 + x_3t_3 \dots (4) \end{cases}$$

We add (1) to (4), (1) to (2) to (4), (1) to (2) to (3) to (4).

$$\begin{cases} y_0 = x_0t_0 \\ y_0 + y_3 = (x_0 + x_3)(t_0 + t_3) \\ y_0 + y_2 + y_3 = (x_0 + x_2 + x_3)(t_0 + t_2 + t_3) \\ y_0 + y_1 + y_2 + y_3 = (x_0 + x_1 + x_2 + x_3)(t_0 + t_1 + t_2 + t_3) \end{cases}$$

Thus, the proof of (1) is complete.

2. If $X \setminus Y$,then $x_0 \setminus y_0$, so $x_0 \le y_0$.

Also:

$$\begin{cases} x_0 + x_3 \setminus y_0 + y_3, \text{so } x_0 + x_3 \le y_0 + y_3 \\ x_0 + x_2 + x_3 \setminus y_0 + y_2 + y_3, \text{so } x_0 + x_2 + x_3 \le y_0 + y_2 + y_3 \\ x_0 + x_1 + x_2 + x_3 \setminus y_0 + y_1 + y_2 + y_3, \text{so } x_0 + x_1 + x_2 + x_3 \le y_0 + y_1 + y_2 + y_3 \end{cases}$$

Thus $X \le Y$.

3. $X \equiv Y(mod\, Z)$ if and only if $Z \setminus X - Y$, thus:

$$\begin{cases} z_0 \setminus x_0 - y_0 \\ z_0 + z_3 \setminus (x_0 + x_3) - (y_0 + y_3) \\ z_0 + z_2 + z_3 \setminus (x_0 + x_2 + x_3) - (y_0 + y_2 + y_3) \\ z_0 + z_1 + z_2 + z_3 \setminus (x_0 + x_1 + x_2 + x_3) - (y_0 + y_1 + y_2 + y_3) \end{cases}$$

Thus $x_0 \equiv y_0(mod\, z_0), x_0 + x_3 \equiv y_0 + y_3(mod\, z_0 + z_3), x_0 + x_2 + x_3 \equiv y_0 + y_2 + y_3(mod\, z_0 + z_2 + z_3), x_0 + x_1 + x_2 + x_3 \equiv y_0 + y_1 + y_2 + y_3(mod\, z_0 + z_1 + z_2 + z_3)$.

Example.

Take $X = 3 + 2I_1 + I_2 - I_3, Y = 3 + 4I_1 + 2I_2 + I_3$, we have $X \setminus Y$ that is because:

$3 \setminus 3, 3 - 1 = 2 \setminus 3 + 1 = 4, 3 + 1 - 1 = 3 \setminus 3 + 2 + 1 = 6, 3 + 2 + 1 - 1 \setminus 3 + 4 + 2 + 1 = 10$.

Example.

Take $X = 7 + 3I_1 + I_2 + 5I_3, Y = 4 + I_1 + I_2 + I_3, Z = 3 + 2I_1 + I_3$, we have $7 \equiv 4(mod\ 3)$, $7 + 5 = 12 \equiv 4 + 1(mod\ 3 + 4)$, $7 + 1 + 5 = 13 \equiv 4 + 1 + 1(mod\ 3 + 0 + 4)$, $7 + 3 + 1 + 5 = 16 \equiv 4 + 1 + 1 + 1\ (mod 9)$ thus, $X \equiv Y(mod\ Z)$.

Theorem.

The relation $(\leq)$ is a partial order relation.

Proof.

$X \leq Y$ clearly.

If $X \leq Y$ and $Y \leq Z$, then $x_0 \leq y_0 \leq z_0$, $x_0 + x_3 \leq y_0 + y_3 \leq z_0 + z_3$, $x_0 + x_2 + x_3 \leq y_0 + y_2 + y_3 \leq z_0 + z_2 + z_3$, $x_0 + x_1 + x_2 + x_3 \leq y_0 + y_1 + y_2 + y_3 \leq z_0 + z_1 + z_2 + z_3$.

Thus $X \leq Z$.

If $X \leq Y$ and $Y \leq X$, then $x_0 = y_0$, $x_0 + x_3 = y_0 + y_3$, $x_0 + x_2 + x_3 = y_0 + y_2 + y_3$, $x_0 + x_1 + x_2 + x_3 = y_0 + y_1 + y_2 + y_3$, thus $X = Y$.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3, Z = z_0 + z_1I_1 + z_2I_2 + z_3I_3, T = t_0 + t_1I_1 + t_2I_2 + t_3I_3, S = s_0 + s_1I_1 + s_2I_2 + s_3I_3 \in Z_3(I)$, then:

1). If $X \equiv Y(mod\ Z)$, $T \equiv S(mod\ Z)$,then $X + T \equiv Y + S(mod\ Z)$ and $X - T \equiv Y - S(mod\ Z), X.T \equiv Y.S(mod\ Z)$.

2). $X^n = {x_0}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - {x_0}^n]I_3\ ; n \in N$.

3). $X^n \equiv Y^n(mod\ Z^n)\ ; n \in N$.

Proof.

1). $X + T = (x_0 + t_0) + (x_1 + t_1)I_1 + (x_2 + t_2)I_2 + (x_3 + t_3)I_3$.

$Y + S = (y_0 + s_0) + (y_1 + s_1)I_1 + (y_2 + s_2)I_2 + (y_3 + s_3)I_3$.

Since $z_0 \setminus x_0 - y_0$, $z_0 \setminus t_0 - s_0$, then $z_0 \setminus (x_0 + t_0) - (y_0 + s_0)$ and $x_0 + t_0 \equiv y_0 + s_0(\ mod\ z_0)$.

$z_0 + z_3 \setminus (x_0 + x_3) - (y_0 + y_3), z_0 \setminus (t_0 + t_3) - (s_0 + s_3)$, then:

$z_0 + z_3 \setminus (x_0 + x_3 + t_0 + t_3) - (y_0 + y_3 + s_0 + s_3)$, thus:

$(x_0 + x_3) + (t_0 + t_3) \equiv (y_0 + y_3) + (s_0 + s_3)(\, mod\, z_0 + z_3)$

By a similar discussion, we get:

$$z_0 + z_2 + z_3 \setminus (x_0 + x_2 + x_3 + t_0 + t_2 + t_3) - (y_0 + y_2 + y_3 + s_0 + s_2 + s_3)$$

$$z_0 + z_1 + z_2 + z_3 \setminus (x_0 + x_1 + x_2 + x_3 + t_0 + t_1 + t_2 + t_3) \\ - (y_0 + y_1 + y_2 + y_3 + s_0 + s_1 + s_2 + s_3)$$

So that $X + T \equiv Y + S(mod\, Z)$.

It is easy check that $X - T \equiv Y - S(mod\, Z), X.T \equiv Y.S(mod\, Z)$.

2). For $n = 1$ it is true clearly.

Assume that it is true for $n = k$, we must prove it for $n = k + 1$.

$$X^{k+1} = X.X^k = [x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3][x_0{}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 \\ + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - x_0{}^n]I_3]$$

$$= x_0{}^{n+1} + [x_0(x_0 + x_1 + x_2 + x_3)^n - x_0(x_0 + x_2 + x_3)^n + x_1(x_0 + x_1 + x_2 + x_3)^n \\ - x_1(x_0 + x_2 + x_3)^n + x_1 x_0{}^n + x_1(x_0 + x_2 + x_3)^n - x_1(x_0 + x_3)^n \\ + x_1(x_0 + x_3)^n - x_1 x_0{}^n + x_2(x_0 + x_1 + x_2 + x_3)^n - x_2(x_0 + x_2 + x_3)^n \\ + x_3(x_0 + x_1 + x_2 + x_3)^n - x_3(x_0 + x_2 + x_3)^n]I_1 \\ + [x_0(x_0 + x_2 + x_3)^n - x_0(x_0 + x_3)^n + x_2 x_0{}^n + x_2(x_0 + x_2 + x_3)^n \\ - x_2(x_0 + x_3)^n + x_2(x_0 + x_3)^n - x_2 x_0{}^n + x_3(x_0 + x_2 + x_3)^n - x_3(x_0 + x_3)^n]I_2 \\ + [x_0(x_0 + x_3)^n - x_0{}^{n+1} + x_3 x_0{}^n + x_3(x_0 + x_3)^n - x_3 x_0{}^n]I_3$$

$$= x_0{}^{n+1} + [(x_0 + x_1 + x_2 + x_3)^{n+1} - (x_0 + x_2 + x_3)^{n+1}]I_1 \\ + [(x_0 + x_2 + x_3)^{n+1} - (x_0 + x_3)^{n+1}]I_2 + [(x_0 + x_3)^{n+1} - x_0{}^{n+1}]I_3$$

This implies that is true by induction.

3). It holds directly from (1) and (2).

Example.

Take $X = 1 + 2I_1 - I_2 + I_3, n = 2$, then:

$X^2 = 1 + [(3)^2 - 1]I_1 + [1 - (2)^2]I_2 + [(2)^2 - 1]I_3 = 1 + 8I_1 - 3I_2 + 3I_3$

Theorem.

Let $X, Y \in Z_3(I)$, then $gcd(X,Y) = gcd(x_0, y_0) + [gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) - gcd(x_0 + x_2 + x_3, y_0 + y_2 + y_3)]I_1 + [gcd(x_0 + x_2 + x_3, y_0 + y_2 + y_3) - gcd(x_0 + x_3, y_0 + y_3)]I_2 + [gcd(x_0 + x_3, y_0 + y_3) - gcd(x_0, y_0)]I_3$

Example.

Take $X = 4 + 3I_1 + 5I_2 - I_3, Y = 7 + I_1 + I_2 + 3I_3$.

$gcd(4,7) = 1, gcd(11,13) = 1, gcd(8,11) = 1, gcd(3,10) = 1$, thus $gcd(X, Y) = 1$.

Remark.

X, Y are called coprime (relatively prime) if and only if $gcd(X, Y) = 1$, which is equivalent to:

$gcd(x_0, y_0) = gcd(x_0 + x_3, y_0 + y_3) = gcd(x_0 + x_2 + x_3, y_0 + y_2 + y_3) = gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = 1$.

Definition.

Let $X = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 \in Z_3(I)$, with $X > 0$, we define:

$$\varphi_s(X) = \varphi(x_0) + [\varphi(x_0 + x_1 + x_2 + x_3) - \varphi(x_0 + x_2 + x_3)]I_1 + [\varphi(x_0 + x_2 + x_3) - \varphi(x_0 + x_3)]I_2 + [\varphi(x_0 + x_3) - \varphi(x_0)]I_3$$

where φ is the ordinary Euler's function, φ_s is called the special 3-refined neutrosophic Euler's function.

Example.

Take $X = 3 + I_1 + I_2 + I_3 > 0\,; x_0 = 3, x_1 = 1, x_2 = x_3 = 1$.

$\varphi(x_0) = 2, \varphi(x_0 + x_1 + x_2 + x_3) = 2, \varphi(x_0 + x_2 + x_3) = 4, \varphi(x_0 + x_3) = 2$.

Thus

$\varphi_s(X) = 2 + [2 - 4]I_1 + [4 - 2]I_2 + [2 - 2]I_3 = 2 - 2I_1 + 2I_2 + 0I_3$.

It is clear that $\varphi_s(X) > 0;\ \forall X > 0$.

Theorem.

Let $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3, M = m_0 + m_1 I_1 + m_2 I_2 + m_3 I_3 \in Z_3(I)$, such that:

$A > 0, M > 0$ and $gcd(A, M) = 1$, then:

1). $A^{\varphi_s(M)} \equiv 1\ (mod\ M)$.

2). $A^{-1}\ (mod\ M) \equiv {a_0}^{-1}\ (mod m_0) + [(a_0 + a_1 + a_2 + a_3)^{-1}(mod\ m_0 + m_1 + m_2 + m_3) - (a_0 + a_2 + a_3)^{-1}(mod\ m_0 + m_2 + m_3)]I_1 + [(a_0 + a_2 + a_3)^{-1}(mod\ m_0 + m_2 + m_3) - (a_0 + a_3)^{-1}(mod\ m_0 + m_3)]I_2 + [(a_0 + a_3)^{-1}(mod\ m_0 + m_3) - {a_0}^{-1}\ (mod m_0)]I_3$.

Proof.

1). $A^{\varphi_s(M)} = {a_0}^{\varphi(m_0)} + \left[(a_0 + a_1 + a_2 + a_3)^{\varphi(m_0+m_1+m_2+m_3)} - (a_0 + a_2 + a_3)^{\varphi(m_0+m_2+m_3)}\right]I_1 + \left[(a_0 + a_2 + a_3)^{\varphi(m_0+m_2+m_3)} - (a_0 + a_3)^{\varphi(m_0+m_3)}\right]I_2 + \left[(a_0 + a_3)^{\varphi(m_0+m_3)} - {a_0}^{\varphi(m_0)}\right]I_3 \equiv 1\ (mod\ M)$.

2). It holds directly by computing the product AA^{-1}.

3-refined Diophantine equations:

Definition.

Let $A = a_0 + a_1I_1 + a_2I_2 + a_3I_3, B = b_0 + b_1I_1 + b_2I_2 + b_3I_3, C = c_0 + c_1I_1 + c_2I_2 + c_3I_3, X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3$, where $a_i, b_i, c_i, x_i, y_i \in Z_3(I)$.

We define the 3-refined neutrosophic linear Diophantine equation with two variables as follows:

$AX + BY = C$.

Example.

Consider the following 3-refined neutrosophic linear Diophantine equation:

$(3 + 2I_1 + I_2 + I_3)X + (2 + 4I_2)Y = 3 + 9I_1 - 7I_3$

Theorem.

Let $AX + BY = C$ be a 3-refined neutrosophic linear Diophantine equation, then it is equivalent to:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \\ (a_0 + a_3)(x_0 + x_3) + (b_0 + b_3)(y_0 + y_3) = c_0 + c_3 \\ (a_0 + a_2 + a_3)(x_0 + x_2 + x_3) + (b_0 + b_2 + b_3)(y_0 + y_2 + y_3) = c_0 + c_2 + c_3 \\ (a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) + (b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) = c_0 + c_1 + c_2 + c_3 \end{cases}$$

Proof.

We compute $AX = a_0x_0 + [(a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) - (a_0 + a_2 + a_3)(x_0 + x_2 + x_3)]I_1 + [(a_0 + a_2 + a_3)(x_0 + x_2 + x_3) - (a_0 + a_3)(x_0 + x_3)]I_2 + [(a_0 + a_3)(x_0 + x_3) - a_0x_0]I_3$

On the other hand, we have:

$$\begin{aligned} BY = b_0y_0 &+ [(b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) - (b_0 + b_2 + b_3)(y_0 + y_2 + y_3)]I_1 \\ &+ [(b_0 + b_2 + b_3)(y_0 + y_2 + y_3) - (b_0 + b_3)(y_0 + y_3)]I_2 \\ &+ [(b_0 + b_3)(y_0 + y_3) - b_0y_0]I_3 \end{aligned}$$

The equation $AX + BY = C$ equivalents:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \\ (a_0 + a_3)(x_0 + x_3) + (b_0 + b_3)(y_0 + y_3) = c_0 + c_3 \\ (a_0 + a_2 + a_3)(x_0 + x_2 + x_3) + (b_0 + b_2 + b_3)(y_0 + y_2 + y_3) = c_0 + c_2 + c_3 \\ (a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) + (b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) = c_0 + c_1 + c_2 + c_3 \end{cases}$$

Example.

Find a solution of the equation:

$(3 + 2I_1 + I_2 + I_3)X + (2 + 4I_2)Y = 3 + 9I_1 - 7I_3$

We have $a_0 = 3, a_1 = 2, a_2 = 1, a_3 = 1, b_0 = 2, b_1 = 0, b_2 = 4, b_3 = 0, c_0 = 3, c_1 = 9, c_2 = 0, c_3 = -7$

The equivalent system is:

$$\begin{cases} 3x_0 + 2y_0 = 3 \dots (1) \\ 4(x_0 + x_3) + 2(y_0 + y_3) = -4 \dots (2) \\ 5(x_0 + x_2 + x_3) + 7(y_0 + y_2 + y_3) = -4 \dots (3) \\ 7(x_0 + x_1 + x_2 + x_3) + 6(y_0 + y_1 + y_2 + y_3) = 5 \dots (4) \end{cases}$$

The equation (1) has a solution $x_0 = 1, y_0 = 0$.

The equation (2) has a solution $x_0 + x_3 = -1, y_0 + y_3 = 0$, thus $x_3 = -2, y_3 = 0$.

The equation (3) has a solution $x_0 + x_2 + x_3 = 9, y_0 + y_2 + y_3 = -7$, thus $x_2 = 10, y_2 = -7$.

The equation (4) has a solution $x_0 + x_1 + x_2 + x_3 = 5, y_0 + y_1 + y_2 + y_3 = 5$, thus $x_1 = -4, y_1 = 12$.

This means that $X = 1 - 4I_1 + 10I_2 - 2I_3, Y = 12I_1 - 7I_2$.

Future research directions and suggestions

3-refined neutrosophic number as generalizations of classical real numbers and integers, may have a great impact on many areas of scientific knowledge.

In the following, we suggest many possible applications of 3-refined neutrosophic real numbers.

1-) How can we build a crypto-system from 3-refined neutrosophic integers which generalize RSA algorithm. [19]

2-) How can we build a crypto-system from 3-refined neutrosophic integers which generalize El-Gamal algorithm. [20-21]

3-) How can we solve 3-refined neutrosophic differential equations, and integral equations.

4-) How can we define Hillbert and Banach 3-refined neutrosophic spaces, and do classical functional inequalities still true in this case.

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On A Novel Security Scheme for The Encryption and Decryption Of 2×2 Fuzzy Matrices with Rational Entries Based on The Algebra of Neutrosophic Integers and El-Gamal Crypto-System

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Abstract:

The main goal behind mathematical cryptography is to keep messages and multimedia messages secret at a time when modern means of communication have spread and become very diverse.

Fuzzy matrices as strong tools which was defined to deal with incomplete and uncertain data and many relationships in real life problems especially those which are related to images and graphs, may considered as important subjects for secret information and communication.

The aim of this research paper is to present a new model and method for encrypting 2×2 fuzzy matrices using the basic concepts in neutrosophic number theory and El Gamal algorithm in cryptography, where we generalize El Gamal algorithm to become applicable to the ring of neutrosophic integer numbers that represents the studied fuzzy matrices.

On the other hand, we study the applications of the novel algorithm to the encryption and decryption of some fuzzy relations represented in terms of fuzzy functions.

In addition, we illustrate many examples to clarify the validity of the new algorithm.

Key words:

Neutrosophic integer, fuzzy matrix, fuzzy relation, fuzzy graph, EL-Gamal crypto-system

Introduction and Preliminaries

The concept of fuzzy logic and fuzzy set was presented by Zadeh [10]. The main point of fuzzy approach is to deal with a degree for truth and a degree for falsity.

Smarandache has generalized fuzzy ideas by introducing neutrosophic logic [16], which deals with a degree of truth (T), a degree of falsity (F), and a degree of indeterminacy (I).

If X is a non-empty set. A fuzzy set (subset) μ of the set X is defined as a function μ: X → [0, 1], and if μ is a fuzzy subset of a set X. For t ∈ [0, 1], the set X_t= $\{x \in X \,; \mu(x) \geq t\}$, then μ is called a t-level subset of the fuzzy subset μ [3].

In the literature, we find many applications and approaches built over the ideas of fuzzy logic especially in probability, algebra, and graph theory [5, 7, 23].

The concept of fuzzy matrix was introduced in [6], and then it was studied widely in [8-9, 13], especially the algebraic properties and applications of these matrices.

A square 2×2 fuzzy matrix is defined as follows:

A=$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{ij} \in [0,1]$.

Mathematical Asymmetric cryptography is a branch of applied mathematics and theoretical computer science that applies mathematical methods and models to encrypt messages and multimedia [4]. Many systems and algorithms were presented such as RSA algorithm and El-Gamal algorithm [4, 15]. In addition, many attacks and applications of some special numbers can be found in [20-21].

In [19], the first suggestion of using the generalizations of integers in cryptology was presented, where authors have suggested the usage of neutrosophic numbers, split-complex numbers, and dual numbers in cryptology.

Neutrosophic cryptography became known recently by using neutrosophic number theory in generalizing classical crypto-systems into more complex and powerful systems. We find a neutrosophic version of RSA and refined El-Gamal crypto-algorithm [18, 22].

In this paper, we continuo the previous efforts for applying neutrosophic number theory in cryptology, where a neutrosophic version of El-Gamal algorithm based on the foundations of neutrosophic number theory will be presented and handled. In addition, we apply this algorithm to encrypt and decrypt fuzzy 2×2 matrices with rational entries.

First, we recall some important concepts and definitions.

The description of El Gamal crypto-scheme:

Assume that we have two sides A and B, the first side A wants to send an encrypted message to B.

The recipient B picks a large prime number p and a generator $1 < g < p - 1$, then B picks x that $0 < x < p - 2$ and computes $X = g^x (mod\, p)$. The number x is kept as the secret key suppose that A wants to send (m) as a message to B.

A should pick $0 < r < p - 2$ and compute $R = g^r (mod\, p)$, the shared key K is computed as follows $K = X^r (mod\, p)$.

A encrypts the message as follows $S = m \times k$, and sends the encrypted message to B as a duplet (R, S).

The second side B decrypts the message by using her/his secret key x as follows $m = R^{-x} \times S$.

Definition: (Neutrosophic integers) [1]

Let R be any ring, I be an indeterminacy with the property $I^2 = I$. Then $R(I) = \{a + bI; a, b \in R\}$ is called a neutrosophic ring.

If $R = Z$ is the ring of integers, then $Z(I) = \{a + bI; a, b \in Z\}$ is called the neutrosophic ring of integers. Elements of Z(I) are called neutrosophic integers.

Theorem: (neutrosophic congruencies) [1]

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in Z(I). Then $x \equiv y(modz)$ if and only if

$a \equiv c(mod\, m), a + b \equiv c + d(mod\, m + n)$.

Theorem: (neutrosophic powers) [2]

$(a + bI)^{c+dI} = a^c + I[(a + b)^{c+d} - a^c]$.

Definition [2]

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers, we say that $a + bI \leq c + dI$ if and only if $a \leq c\ and\ a + b \leq c + d$.

Z(I) is a partially ordered set with the previous relation.

Main Discussion

Neutrosophic Version of EL-Gamal algorithm:

To build a neutrosophic version of EL-Gamal Algorithm, we substitute each integer t by a positive neutrosophic integer $t_1 + t_2I$; $t_1 > 0, t_1 + t_2 > 0$.

The recipient (B) picks a neutrosophic positive integer $p = p_1 + p_2I$, where $p_1, p_1 + p_2$ are large primes.

(B)picks a generator $0 < g = g_1 + g_2I < p = p_1 + p_2I - 1$, i.e $g_1 < p_1 - 1,\ g_1 + g_2 < p_1 + p_2 - 1$.

(B) picks $0 < x = x_1 + x_2I < p = p_1 + p_2I - 2$, i.e $x_1 < p_1 - 2$, $x_1 + x_2 < p_1 + p_2 - 2$ and then computes $X = g^x(mod\, p) = {g_1}^x(mod\, p_1) + I[(g_1 + g_2)^x(mod\, p_1 + p_2) - {g_1}^x(mod\, p_1)]$.

The publish key is (g, X).

Assume that (A) will send $m = m_1 + m_2I$ to (B).

(A) should pick $0 < r = r_1 + r_2I < p = p_1 + p_2I - 2$ and compute:

$R = g^r(mod\ p) = g_1{}^{r_1}(mod\ p_1) + I[(g_1 + g_2)^{r_1+r_2}(mod\ p_1 + p_2) - g_1{}^{r_1}(mod\ p_1)] = t_1 + t_2I.$

The shared key

$$K = X^r(mod\ p)$$
$$= g_1{}^{x_1r_1}(mod\ p_1) + I\left[(g_1 + g_2)^{(x_1+x_2)(r_1+r_2)}(mod\ p_1 + p_2) - g_1{}^{x_1r_1}(mod\ p_1)\right] = k_1 + k_2I$$

(A) encrypts its message as follows:

$S = m \times k = (m_1 + m_2I)(k_1 + k_2I) = m_1k_1 + I(m_1k_2 + m_2k_1 + m_2k_2)$

The other side (B)decrypts the message as follows:

$m = R^{-x}(mod\ p);\ R^{-1} = t_1{}^{-1}(mod\ p_1) + I[(t_1 + t_2)^{-1}(mod\ p_1 + p_2) - t_1{}^{-1}(mod\ p_1)]$

Example.

Consider that (B) has picked $p = p_1 + p_2I = 5 + 6I,$ the generator $g = 3 + 2I = g_1 + g_2I,$ the secret key $x = x_1 + x_2I = 2 + 5I.$

$K = g^x(mod\ p) = 3^2(mod\ 5) + I[5^7(mod\ 11) - 3^2(mod\ 5)] = 4 + I[3 - 4] = 4 - I,$

the publick key is $(g, X) = (3 + 2I, 4 - I)$

Assume that (A)has decided to send $m = 4 + 4J$ to (B).

(A) picks $r = r_1 + r_2I = 2 + I$ and computes:

$R = g^r(mod\ p) = 3^2(mod\ 5) + I[5^3(mod\ 11) - 3^2(mod\ 5)] = 4 + I[5 - 4] = 4.$

The shared key $K \equiv X^r(mod\ p) = 4^2(mod\ 5) + I[3^3(mod\ 11) - 4^2(mod\ 5)] = 1 + I[5 - 1] = 1 + 4I = k_1 + k_2I.$

The encrypted message

$S = m \times k = (4 + 4I)(1 + 4I) = 4 + I(16 + 4 + 16) = 4 + 36I.$

(B) decrypts the message as follows:

$m = R^{-x}.s\ (mod\ p), where:$

$R^{-1} = 4^{-1}(mod\ 5) + I[4^{-1}(mod\ 11) - 4^{-1}(mod\ 5)] = 4 + I(3 - 4) = 4 - I$

$$m \equiv R^{-x}.s\ (mod\ p) = (4 - I)^{2+5I}.(4 + 36I)(modp)$$
$$\equiv [4^2 + I(3^7 - 4^2)](4 + 36I)(modp)$$

$= (16 + 271I)(4 + 36I)(modp) = (64 + 87416I)(modp) \equiv 64(mod\ 5) + I[(87416 + 64)(mod\ 11) - 64(mod\ 5)] = 4 + I(8 - 4) = 4 + 4I.$

Which is the plain text.

Example.

Consider the (B)has picked $p = p_1 + p_2 I = 13 + 6I$, the generator $g = g_1 + g_2 I = 5 + 3I$, the secret key is $x_1 + x_2 I = 6 + 3I$.

$$X \equiv g^x (mod\ p) = 5^6 (mod\ 13) + I[8^9 (mod\ 19) - 5^6 (mod\ 13)] = 12 + I[18 - 12]$$
$$= 12 + 6I$$

The public key is $(g, X) = (5 + 3I, 12 + 6I)$.

Assume that (A) has decided to send $m = 10 + I$ to (B).

(A) picks $r_1 + r_2 I = 3 + 2I$ and computes:

$$R \equiv g^r (mod\ p) = 5^3 (mod\ 13) + I[8^5 (mod\ 19) - 5^3 (mod\ 13)] = 8 + I[12 - 8]$$
$$= 8 + 4I$$

The shared key:

$K \equiv X^r (mod\ p) = 12^3 (mod\ 13) + I[18^5 (mod\ 19) - 12^3 (mod\ 13)] = 12 + I[18 - 12] = 12 + 6I = k_1 + k_2 I.$

The encrypted message is:

$S = m \times k = (10 + I)(12 + 6I) = 120 + I(60 + 12 + 6) = 120 + 78I.$

(B) decrypts the message as follows:

$R^{-1} = 8^{-1} (mod\ 13) + I[12^{-1} (mod\ 19) - 8^{-1} (mod\ 13)] = 5 + I(8 - 5) = 5 + 3I$

$m = (R^{-1})^x S\ (mod\ p), we\ have\ (5 + 3I)^{6+3I}.(modp) \equiv [5^6 + I(8^9 - 5^6)](modp)$

$= 5^6 (mod\ 13) + I[8^9 (mod\ 19) - 5^6 (mod\ 13)] = 12 + (18 - 12) = 12 + 6I$

$(120 + 78I)(modp) = 120(mod\ 13) + I[198(mod\ 19) - 120(mod\ 13)]$
$= 3 + (8 - 3) = 3 + 5I$

$m = (12 + 6I)(3 + 5I) = (36 + 60I + 18I + 30I) = (36 + 108I)(mod\ p) \equiv 36(mod\ 13) + I[144(mod\ 19) - 36(mod\ 13)] = 10 + I[11 - 10] = 10 + I.$

Which is the plain text.

Fuzzy Matrices as Neutrosophic Points:

Definition:

Let A be a fuzzy 2×2 matrix

A=$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Then A can be written in term of a 2-dimensional neutrosophic point as follows:

$A_N = (a_{11} + a_{12}I, a_{21} + a_{22}I)$.

Example:

Consider the following fuzzy matrix:

A=$\begin{pmatrix} 0.3 & 0.2 \\ 1 & 0.9 \end{pmatrix}$, then A can be written in the following form: $A_N = (0.3 + 0.2I, 1 + 0.9I)$.

The encryption/decryption of a fuzzy 2×2 matrix:

Let A be a fuzzy 2×2 matrix with rational entries

A=$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, assume that the sender (X) has decided to send the matrix A to the recipient (Y) as a cipher text.

As a first step, (X) should transform the fuzzy matrix A to a 2-dimensional neutrosophic point

$A_N = (a_{11} + a_{12}I, a_{21} + a_{22}I)$, then (X) picks a weight $w \in Z^+$ with the property $wa_{11}, wa_{22}, wa_{12}, wa_{21} \in Z^+$. This implies that $w(a_{11} + a_{12}I), w(a_{21} + a_{22}I) \in Z(I)$, and (X) should send w to (Y).

The recipient (Y) generates the public key as we explained above in neutrosophic El-Gamal algorithm, and shares his/her key with (X).

(X) decrypts the $w(a_{11} + a_{12}I), w(a_{21} + a_{22}I)$ by using the key, and sends the cipher neutrosophic point to (Y).

(Y) decrypts the message as we have shown previously, and divide it by the weight w. Then (Y) rearranges the values into matrix rows to get the plain text.

Example:

We explain the validity of the novel scheme by the following example.

Consider the following fuzzy matrix:

A=$\begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}$, then A can be written in the following form: $A_N = (0.3 + 0.2I, 0.1 + 0.4I)$. (X) picks w=10 and computes the new point $wA_N = (3 + 2I, 1 + 4I)$, thaen (X) shares w=10 with (Y).

Assume that the recipient (Y) has generated the public key as follows:

Consider the (Y)has picked $p = p_1 + p_2I = 13 + 6I$, the generator $g = g_1 + g_2I = 5 + 3I$, the secret key is $x_1 + x_2I = 6 + 3I$.

$$X \equiv g^x(mod\ p) = 5^6(mod\ 13) + I[8^9(mod\ 19) - 5^6(mod\ 13)] = 12 + I[18 - 12]$$
$$= 12 + 6I$$

The public key is $(g, X) = (5 + 3I, 12 + 6I)$.

(X) will send $wA_N = (3+2I, 1+4I)$ to (Y).

(X) picks $r_1 + r_2 I = 3 + 2I$ and computes:

$R \equiv g^r (mod\ p) = 5^3 (mod\ 13) + I[8^5 (mod\ 19) - 5^3 (mod\ 13)] = 8 + I[12 - 8]$

$= 8 + 4I$

The shared key:

$K \equiv X^r (mod\ p) = 12^3 (mod\ 13) + I[18^5 (mod\ 19) - 12^3 (mod\ 13)] = 12 +$

$I[18 - 12] = 12 + 6I = k_1 + k_2 I.$

The encrypted message is:

$S = wA_N \times k = (3+2I, 1+4I)(12+6I) = (36+54I, 12+78I).$

(Y) decrypts the message as follows:

$R^{-1} = 8^{-1}(mod\ 13) + I[12^{-1}(mod\ 19) - 8^{-1}(mod\ 13)] = 5 + I(8-5) = 5 + 3I$

$m = (R^{-1})^x \times S\ (mod\ p), we\ have\ (5+3I)^{6+3I}(modp) \equiv [5^6 + I(8^9 - 5^6)](modp)$

$= 5^6(mod\ 13) + I[8^9(mod\ 19) - 5^6(mod\ 13)] = 12 + (18 - 12) = 12 + 6I.$

On the other hand, $(36+54I, 12+78I)(mod\ p) = (36(mod\ 13) + I[90(mod\ 19) -$ $36(mod\ 13)], 12\ (mod\ 13) + I[90(mod\ 19) - 12(mod\ 13)])=$

$(10+4I, 12+2I).$

The plain text is $wA_N = (12+6I).(10+4I, 12+2I)(mod\ p) = (120+132I, 144+$ $108I)(mod\ p) \equiv (120(mod\ 13) + I[252(mod\ 19) - 120(mod\ 13)], 144(mod\ 13) +$ $I[252(mod\ 19) - 144(mod\ 13)]) = (3+2I, 1+4I).$

Now, (Y) should divide the plain text by w=10, and rearrange it as rows of a matrix to get:

A=$\begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}$.

A Comparison between El-Gamal algorithm and neutrosophic El-Gamal algorithm:

Since fuzzy matrices may have entries such as 0 or 1, then the encryption by using classical El-Gamal algorithm may be easy to be broken. Meanwhile, transforming them to neutrosophic points keeps the information secret. We explain it through the following example.

Example:

Consider the following fuzzy matrix:

A=$\begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}$, then A can be written in the following form: $A_N = (0.3, 0.4I)$. (X) picks w=10 and computes the new point $wA_N = (3, 4I)$, then (X) shares w=10 with (Y).

Assume that the recipient (Y) has generated the public key as follows:

Consider the (Y)has picked $p = p_1 + p_2 I = 13 + 6I$, the generator $g = g_1 + g_2 I = 5 + 3I$, the secret key is $x_1 + x_2 I = 6 + 3I$.

$$X \equiv g^x (mod\ p) = 5^6 (mod\ 13) + I[8^9 (mod\ 19) - 5^6 (mod\ 13)] = 12 + I[18 - 12]$$

$$= 12 + 6I$$

The public key is $(g, X) = (5 + 3I, 12 + 6I)$.

(X) will send $wA_N = (3 + 2I, 1 + 4I)$ to (Y).

(X) picks $r_1 + r_2 I = 3 + 2I$ and computes:

$$R \equiv g^r (mod\ p) = 5^3 (mod\ 13) + I[8^5 (mod\ 19) - 5^3 (mod\ 13)] = 8 + I[12 - 8]$$

$$= 8 + 4I$$

The shared key:

$$K \equiv X^r (mod\ p) = 12^3 (mod\ 13) + I[18^5 (mod\ 19) - 12^3 (mod\ 13)] = 12 + I[18 - 12] = 12 + 6I = k_1 + k_2 I.$$

The encrypted message is:

$$S = wA_N \times k = (3, 4I)(12 + 6I) = (36 + 18I, 72I).$$

(Y) decrypts the message as follows:

$$R^{-1} = 8^{-1} (mod\ 13) + I[12^{-1} (mod\ 19) - 8^{-1} (mod\ 13)] = 5 + I(8 - 5) = 5 + 3I$$

$$m = (R^{-1})^x \times S\ (mod\ p), we\ have\ (5 + 3I)^{6+3I} (modp) \equiv [5^6 + I(8^9 - 5^6)](modp)$$

$$= 5^6 (mod\ 13) + I[8^9 (mod\ 19) - 5^6 (mod\ 13)] = 12 + (18 - 12) = 12 + 6I.$$

On the other hand, $(36 + 18I, 72I)(mod\ p) = (36(mod\ 13) + I[54(mod\ 19) - 36(mod\ 13)], 0\ (mod\ 13) + I[72(mod\ 19) - 0(mod\ 13)])$= $(10 + 6I, 15I)$.

The plain text is $wA_N = (12+6I).(10+6I,15I)(mod\ p) = (120+168I, 270I)(mod\ p) \equiv (120(mod\ 13) + I[288(mod\ 19) - 120(mod\ 13)], 0(mod\ 13) + I[270(mod\ 19) - 0(mod\ 13)]) = (3, 4I)$.

Now, (Y) should divide the plain text by w=10, and rearrange it as rows of a matrix to get:

A=$\begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}$.

On the other hand, if (X) has ciphered his numbers with classical El-Gamal algorithm, then he gets 0 as a cipher text twice, that is because when he computes $S = (0) \times k = 0$ which is equal to the plain text. Meanwhile, when he uses neutrosophic formulas, he gets $(10+6I, 15I)$ which is different from the original message. From this point of view, we can say that the usage of neutrosophic numbers and neutrosophic El-Gamal algorithm is better that using classical algorithm only, especially in the case of ciphering 0 and 1 entries.

Applications to fuzzy relations

Let $X = \{x_1, x_2\}$,$Y = \{y_1, y_2\}$ be two sets with two elements, with a fuzzy relation R(X,Y) defined on X as follows:

$f_{ij}(x_i, y_j) = a_{ij} \in [0,1]$. Then this relation can be represented as a fuzzy 2×2 matrix

A=$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, so that by using neutrosophic El-Gamal algorithm we can encrypt it as a secret message.

We clarify that by the following example.

Example:

Assume that we have two men m_1, m_2, and two hospitals H_1, H_2. Suppose that the first man goes to the first hospital in 30% of cases of illness, and in 70% of cases, he goes to the second hospital.

As for the second man, he goes to the first hospital in 90% of cases, and he goes to the second hospital in 10% of cases.

Then, we can represent this information as a fuzzy relation, $f(m_1, H_1) = 0.3$, $f(m_1, H_2) = 0.7, f(m_2, H_1) = 0.9, f(m_2, H_2) = 0.1$.

So, it can be described by the following fuzzy matrix with rational entries:

A=$\begin{pmatrix} 0.3 & 0.7 \\ 0.9 & 0.1 \end{pmatrix}$.

Then A can be written in the following form: $A_N = (0.3 + 0.7I, 0.9 + 0.1I)$. (X) picks w=10 and computes the new point $wA_N = (3 + 7I, 9 + I)$, then (X) shares w=10 with (Y).

Assume that the recipient (Y) has generated the public key as follows:
Consider the (Y)has picked $p = p_1 + p_2I = 13 + 6I$, the generator $g = g_1 + g_2I = 5 + 3I$, the secret key is $x_1 + x_2I = 6 + 3I$.

$$X \equiv g^x(mod\ p) = 5^6(mod\ 13) + I[8^9(mod\ 19) - 5^6(mod\ 13)] = 12 + I[18 - 12]$$
$$= 12 + 6I$$

The public key is $(g, X) = (5 + 3I, 12 + 6I)$.

(X) will send $wA_N = (3 + 7I, 9 + I)$ to (Y).

(X) picks $r_1 + r_2I = 3 + 2I$ and computes:

$$R \equiv g^r(mod\ p) = 5^3(mod\ 13) + I[8^5(mod\ 19) - 5^3(mod\ 13)] = 8 + I[12 - 8]$$
$$= 8 + 4I$$

The shared key:

$K \equiv X^r(mod\ p) = 12^3(mod\ 13) + I[18^5(mod\ 19) - 12^3(mod\ 13)] = 12 + I[18 - 12] = 12 + 6I = k_1 + k_2I$.

The encrypted message is:

$S = wA_N \times k = (3 + 7I, 9 + I)(12 + 6I) = (36 + 144I, 108 + 72I)$.

(Y) decrypts the message as follows:

$$R^{-1} = 8^{-1}(mod\ 13) + I[12^{-1}(mod\ 19) - 8^{-1}(mod\ 13)] = 5 + I(8 - 5) = 5 + 3I$$

$m = (R^{-1})^x \times S\ (mod\ p), we\ have\ (5 + 3I)^{6+3I}(modp) \equiv [5^6 + I(8^9 - 5^6)](modp) = 5^6(mod\ 13) + I[8^9(mod\ 19) - 5^6(mod\ 13)] = 12 + (18 - 12) = 12 + 6I$.

On the other hand, $(36 + 144I, 108 + 72I)(mod\ p) = (36(mod\ 13) + I[180(mod\ 19) - 36(mod\ 13)], 108\ (mod\ 13) + I[180(mod\ 19) - 108(mod\ 13)])= (10 - I, 4 + 5I)$.

The plain text is $wA_N = (12+6I).(10-I, 4+5I)(mod\ p) = (120+42I, 48+114I)(mod\ p) \equiv (120(mod\ 13) + I[162(mod\ 19) - 120(mod\ 13)], 48(mod\ 13) + I[162(mod\ 19) - 48(mod\ 13)]) = (3+7I, 9+I)$.

Now, (Y) should divide the plain text by w=10, and rearrange it as rows of a matrix to get:

A=$\begin{pmatrix} 0.3 & 0.7 \\ 0.9 & 0.1 \end{pmatrix}$. This means that (Y) is able to reform the secret fuzzy relation in the original form

$f(m_1, H_1) = 0.3$,

$f(m_1, H_2) = 0.7, f(m_2, H_1) = 0.9, f(m_2, H_2) = 0.1$.

Conclusion

In this paper, we have used the basics of neutrosophic number theory and classical El-Gamal crypto-system to build a new version, which we call neutrosophic EL-Gamal algorithm.

In addition, we use the novel algorithm to encrypt and decrypt messages that contain 2×2 fuzzy matrices with rational entries.

On the other hand, some application of decrypting fuzzy relations and fuzzy functions, which can be represented as 2×2 fuzzy matrices with rational entries were presented and illustrated by examples.

In the future, we aim to find algorithm to encrypt and decrypt $n \times n$ fuzzy matrices with rational entries by using neutrosophic algebraic structures.

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An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings

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Abstract:

The symbolic n-plithogenic sets and algebraic structures are a new branch of pure algebra released as new generalizations of classical algebraic structures.

The main goal of this paper is to define for the first time the concept of symbolic 2-plithogenic module over a symbolic 2-plithogenic ring. Algebraic substructures of symbolic 2-plithogenic modules such as sub-modules, AH-homomorphisms, and algebraic basis.

Keywords: 2-plithogenic symbolic set, 2-plithogenic module, 2-plithogenic ring

Introduction

The concept of symbolic plithogenic sets was defined by Smarandache in [13-17 ,30], and he suggested an algebraic approach of these sets. Laterally, the concept of symbolic 2-plithogenic rings [31], where the concepts such as symbolic AH-ideals, and AH-homomorphisms were presented and discussed.

In general, we can say that symbolic plithogenic structures are very close to neutrosophic algebraic structures with many differences in the definition of multiplication operation [1-10].

Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$2-SP_R=\{a_0+a_1P_1+a_2P_2;\ a_i\in R, {P_j}^2=P_j, P_1\times P_2=P_{max(1,2)}=P_2\}$.

Smarandache has defined algebraic operations on $2-SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2.$

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0{P_1}^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2{P_2}^2 + a_1b_1P_1P_1 = a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2.$

In this paper, we study the symbolic 2-plithogenic modules according to many points of view, where substructures such as AH-submodules, and AH-homomorphisms will be presented in terms of theorems. In addition, many examples will be illustrated to explain the novelty of these ideas.

Main Discussion

Definition.

Let M be a module over the ring R, let $2 - SP_R$ be the corresponding symbolic 2-plithogenic ring.

$2 - SP_R = \{x + yP_1 + zP_2;\ x, y, z \in R, {P_i}^2 = P_i, P_1P_2 = P_2P_1 = P_2\}.$

We define the symbolic 2-plithogenic module as follows:

$2 - SP_M = M + MP_1 + MP_2 = \{a + bP_1 + cP_2;\ a, b, c \in M\}.$

Operations on $2 - SP_M$ can be defined as follows:

Addition: $(+): 2 - SP_M \to 2 - SP_M$, such that:

$[x_0 + x_1P_1 + x_2P_2] + [y_0 + y_1P_1 + ys_2P_2] = (x_0 + y_0) + (x_1 + y_1)P_1 + (x_2 + y_2)P_2$

Multiplication: $(.): 2 - SP_R \times 2 - SP_M \to 2 - SP_M$, such that:

$[a + bP_1 + cP_2].[x_0 + x_1P_1 + x_2P_2] = ax_0 + (ax_1 + bx_0 + bx_1)P_1 + (ax_2 + bx_2 + cx_0 + cx_1 + cx_2)P_2.$

where $x_i, y_i \in M, a, b, c \in R$

Theorem.

Let $(2 - SP_M, +, .)$ Is a module over the ring $2 - SP_R$.

Proof.

Let $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in 2 - SP_M$, $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2 \in 2 - SP_R$ we have:

1. $X = X, (X + Y) + Z = X + (Y + Z), X + (-X) = -X + X = 0, X + 0 = 0 + X = X$

Also

$A(X+Y)=(a_0+a_1P_1+a_2P_2)[(x_0+y_0)+(x_1+y_1)P_1+(x_2+y_2)P_2]$

$=a_0(x_0+y_0)+\big(a_0(x_1+y_1)+a_1(x_0+y_0)+a_1(x_1+y_1)\big)P_1$

$+\big(a_0(x_2+y_2)+a_1(x_2+y_2)+a_2(x_0+y_0)+a_2(x_1+y_1)+a_2(x_2+y_2)\big)P_2$

$=A.X+A.Y$

$(A+B)X=[(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2](x_0+x_1P_1+x_2P_2)$

$=(a_0+b_0)x_0+\big((a_0+b_0)x_1+(a_1+b_1)x_0+(a_1+b_1)x_1\big)P_1$

$+\big((a_0+b_0)x_2+(a_1+b_1)x_2+(a_2+b_2)x_0+(a_2+b_2)x_1+(a_2+b_2)x_2\big)P_2$

$=A.X+B.X$

$(A.B).X=[a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)P_2](x_0+x_1P_1+x_2P_2)=a_0b_0x_0+[a_0b_0x_1+(a_0b_1+a_1b_0+a_1b_1)x_0+(a_0b_1+a_1b_0+a_1b_1)x_1]P_1+[a_0b_0x_2+(a_0b_2+a_2b_0+a_1b_1)x_2+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)x_0+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)x_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)x_2]P_2=A(B.X)$.

Example.

Let $M=Z^3$ be the module over the ring $R=$.

The corresponding symbolic 2-plithogenic vector space over $2-SP_Z$ is:

$2-SP_{Z^3}=\{(x_0,y_0,z_0)+(x_1,y_1,z_1)P_1+(x_2,y_2,z_2)P_2; x_i,y_i,z_i\in Z\}$

Consider $X=(1,1,0)+(2,-1,1)P_1+(0,1,-1)P_2\in 2-SP_{Z^3}, A=2+P_1+P_2\in 2-SP_Z$. We have:

$A.X=(2,2,0)+[(4,-2,2)+(1,1,0)+(2,-1,1)]P_1+[(0,2,2)+(0,1,1)+(1,1,0)+(2,-1,1)+(0,1,1)]P_2=(2,2,0)+(7,-2,3)P_1+(3,4,5)P_2$.

Definition.

Let $2-SP_M$ be a symbolic 2-plithogenic module over $2-SP_R$, let M_0,M_1,M_2 be the three sub-modules of V, we define the AH-submodule as follows:

$W=M_0+M_1P_1+M_2P_2=\{x+yP_1+zP_2;\ x\in M_0,y\in M_1,z\in M_2\}$.

If $M_0=M_1=M_2$, then W is called an AHS-sub-module.

Example.

Consider $2-SP_{Z^3}$, we have $M_0=\{(a,0,0);\ a\in R\},M_1=\{(0,b,0);\ b\in R\},M_2=\{(0,0,c);\ c\in Z\}$ are three sub-modules of $M=Z^3$.

$W=M+M_1P_1+M_2P_2=\{(a,0,0)+(0,b,0)P_1+(0,0,c)P_2;\ a,b,c\in Z\}$ is an AH-submodule of $2-SP_{Z^3}$.

$T = M_1 + MP_1 + M_1P_2 = \{(0,a,0) + (0,b,0)P_1 + (0,c,0)P_2;\ a,b,c \in Z\}$ is an AHS-submodule.

Theorem.

Let $2 - SP_M$ be a symbolic 2-plithogenic module over $2 - SP_R$, let W be an AHS-submodule of $2 - SP_M$, then W is a submodule of $2 - SP_M$.

Proof.

Suppose that W is an AHS-submodule, then there exists a submodule $M_0 \leq M$, such that $W = M_0 + M_0P_1 + M_0P_2 = \{x + yP_1 + zP_2;\ x,y,z \in M_0\}$.

Let $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in W$, then:

$X - Y = (x_0 - y_0) + (x_1 - y_1)P_1 + (x_2 - y_2)P_2 \in W$

$\forall\, A = a_0 + a_1P_1 + a_2P_2 \in 2 - SP_R$, then:

$A.X = a_0x_0 + (a_0x_1 + a_1x_0 + a_1x_1)P_1 + (a_0x_2 + a_1x_2 + a_2x_0 + a_2x_1 + a_2x_2)P_2 \in W$, that is because $a_0x_0 \in M_0, a_0x_1 + a_1x_0 + a_1x_1 \in M_0, a_0x_2 + a_1x_2 + a_2x_0 + a_2x_1 + a_2x_2 \in M_0$, this implies the proof.

Definition.

Let V, W be two modules over the ring R. Let $2 - SP_V$, $2 - SP_W$ be the corresponding symbolic 2-plithogenic modules over $2 - SP_R$.

Let $L_0, L_1, L_2: V \to W$ be three homomorphisms, we define the AH-homomorphism as follows:

$L: 2 - SP_V \to 2 - SP_W, L = L_0 + L_1P_1 + L_2P_2\ ; L(x + yP_1 + zP_2) = L_0(x) + L_1(y)P_1 + L_2(z)P_2$.

If $L_0 = L_1 = L_2$, then L is called AHS-homomorphism.

Definition.

Let $L = L_0 + L_1P_1 + L_2P_2: 2 - SP_V \to 2 - SP_W$ be an AH-homomorphism, we define:

1. $AH - ker(L) = ker(L_0) + ker(L_1)P_1 + ker(L_2)P_2 = \{x + yP_1 + zP_2\}; x \in ker(L_0), y \in ker(L_1), z \in ker(L_2)$.
2. $AH - Im(L) = Im(L_0) + Im(L_1)P_1 + Im(L_2)P_2 = \{a + bP_1 + cP_2\}; a \in Im(L_0), b \in Im(L_1), c \in Im(L_2)$

If L is AHS-linear homomorphism, then we get $AHS - kernel,\ AHS - Image$.

Theorem.

Let $L = L_0 + L_1P_1 + L_2P_2: 2 - SP_V \to 2 - SP_W$ be an AH-homomorphism, then:

1. $AH - ker(L)$ is AH-submodule of $2 - SP_V$.

2. $AH - Im(L)$ is AH-submodule of $2 - SP_W$.

Proof.

1. Since $ker(L_0), ker(L_1), ker(L_2)$ are submodules of V, then $AH - ker(L)$ is an AH-submodule of $2 - SP_V$.
2. It is holds by the same.

Remark.

If L_0, L_1, L_2 are isomorphisms, then $ker(L_0) = ker(L_1) = ker(L_2) = \{0\}, Im(L_0) = Im(L_1) = Im(L_2) = W$, thus $AH - ker(L) = \{0\}, AH - Im(L) = 2 - SP_W$.

Example.

Take $V = Z^3, W = Z,\ L_0, L_1, L_2: V \to W$ such that:

$L_0(x, y, z) = (x), L_1(x, y, z) = (y), L_2(x, y, z) = (z)$

The corresponding AH-homomorphism is:

$L = L_0 + L_1P_1 + L_2P_2: 2 - SP_{Z^3} \to 2 - SP_Z$:

$L[(x_0, y_0, z_0) + (x_1, y_1, z_1)P_1 + (x_2, y_2, z_2)P_2] = L_0(x_0, y_0, z_0) + L_1(x_1, y_1, z_1)P_1 + L_2(x_2, y_2, z_2)P_2 = (x_0) + (y_1)P_1 + (z_2)P_2$.

For example, take $X = (1,9,8) + (9,10,-9)P_1 + (3,2,1)P_2$, then:

$L(X) = 1 + (10)P_1 + P_2$.

$$\begin{cases} ker(L_0) = \{ (0, y_0, z_0);\ y_0, z_0 \in Z\} \\ ker(L_1) = \{(x_1, 0, z_1);\ x_1, z_1 \in Z\} \\ ker(L_2) = \{(x_2, y_2, 0);\ x_2, y_2 \in Z\} \\ AH - ker(L) = \{(0, y_0, z_0) + (x_1, 0, z_1)P_1 + (x_2, y_2, 0)P_2; y_0, z_0, x_1, z_1, x_2, y_2 \in Z\} \end{cases}$$

Also,

$$\begin{cases} Im(L_0) = Z \\ Im(L_1) = Z \\ Im(L_2) = Z \\ AH - Im(L) = Z + ZP_1 + ZP_2 = 2 - SP_W \end{cases}$$

Theorem.

Let $L = f + fP_1 + fP_2: 2 - SP_V \to 2 - SP_W$ be an AHS-homomorphism, then L is a module homomorphism.

Proof.

Let $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in 2 - SP_V$, then:

$L(X+Y)=f(x_0+y_0)+f(x_1+y_1)P_1+f(x_2+y_2)P_2$
$=[f(x_0)+f(x_1)P_1+f(x_2)P_2]+[f(y_0)+f(y_1)P_1+f(y_2)P_2]=L(X)+L(Y)$

Let $A=a_0+a_1P_1+a_2P_2\in 2-SP_F$, then:

$L(A.X)=f(a_0x_0)+f(a_0x_1+a_1x_0+a_1x_1)P_1+f(a_0x_2+a_2x_0+a_2x_2+a_1x_2+a_2x_1)P_2$
$=a_0f(x_0)+\big(a_0f(x_1)+a_1f(x_0)+a_1f(x_1)\big)P_1$
$+\big(a_0f(x_2)+a_2f(x_0)+a_2f(x_2)+a_1f(x_2)+a_2f(x_1)\big)P_2$
$=[a_0+a_1P_1+a_2P_2].[f(x_0)+f(x_1)P_1+f(x_2)P_2]=A.L(X)$

Thus, L is a module homomorphism.

The algebraic relations between symbolic 2-plithogenic modules and neutrosophic modules .

Theorem.

Let M be a module over the ring R, consider $M(I)=M+MI=\{x+yI;x,y\in M\}$ is the corresponding neutrosophic module over the neutrosophic ring $R(I)=\{a+bI;a,b\in R\}$.

$M(I_1,I_2)=M+MI_1+MI_2=\{x+yI_1+zI_2;x,y,z\in M\}$ is the corresponding refined neutrosophic module over the refined neutrosophic ring $R(I_1,I_2)=\{a+bI_1+cI_2;a,b,c\in R\}$.

$2-SP_M=M+MP_1+MP_2=\{x+yP_1+zP_2;x,y,z\in M\}$ is the corresponding symbolic 2-plithogenic module over $2-SP_R$, then:

1. $2-SP_M$ is semi homomorphic to $M(I)$.
2. $2-SP_M$ is semi isomorphic to $M(I_1,I_2)$.

Proof.

1. We define $f:2-SP_M\to M(I),g:2-SP_R\to R(I)$ such that:

$f(x+yP_1+zP_2)=x+yI;x,y,z\in M$

$g(a+bP_1+cP_2)=a+bI;a,b,c\in R$

We have the following:

g is a ring homomorphism, that is because:

$A=a_0+a_1P_1+a_2P_2,B=b_0+b_1P_1+b_2P_2;\ a_i,b_i\in R$, then:

If $A=B$, then $a_i=b_i$ for all i, thus $a_0+a_1I=b_0+b_1I, i.e.\ \ g(A)=g(B)$.

$g(A+B)=g[(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2]=a_0+b_0+(a_1+b_1)I=g(A)+g(B)$.

$g(A.B)=g[a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b+a_2b_1+a_2b_2)P_2]=$
$a_0b_0+(a_0b_1+a_1b_0+a_1b_1)I=(a_0+a_1I)(b_0+b_1I)=g(A).g(B)$.

On the other hand, f is well defined, that is because:

If $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2$, then $x_i = y_i$ for all i, hence $a_0 + a_1I = b_0 + b_1I$, thus $f(X) = f(Y)$.

f preserves addition, that is because:

For $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2$, we have:

$f(X + Y) = f[(x_0 + y_0) + (x_1 + y_1)P_1 + (x_2 + y_2)P_2] = x_0 + y_0 + (x_1 + y_1)I = f(X) + f(Y)$.

f preserves multiplication, that is because:

For $A = a_0 + a_1P_1 + a_2P_2 \in 2 - SP_M$, we have:

$f(A.X) = a_0x_0 + (a_0x_1 + a_1x_0 + a_1x_1)I = (a_0 + a_1I)(x_0 + x_1I) = g(A).f(X)$

Thus f is a semi module homomorphism.

We define $f: 2 - SP_M \to M(I_1, I_2)$, $g: 2 - SP_R \to M(I_1, I_2)$, where $f(x + yP_1 + zP_2) = x + zI_1 + yI_2$, and $g(a + bP_1 + cP_2) = a + cI_1 + bI_2; x, y, z \in M, a, b, c \in R$.

(g) is well defined, that is because:

If $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2$, then:

$a_0 = a_1, b_0 = b_1, c_0 = c_1$, hence: $a_0 + c_0I_1 + b_0I_2 = a_1 + c_1I_1 + b_1I_2$, so that $g(A) = g(B)$.

(f) is well defined by a similar discussion.

(g) is one-to-one mapping, that is because:

$ker(g) = \{a + bP_1 + cP_2; g(a + bP_1 + cP_2) = 0\} = O$

$Im(g) = \{a + cI_1 + bI_2; g(a + bP_1 + cP_2) \in R(I_1, I_2); \exists A \in 2 - SP_R, g(A) = a + cI_1 + bI_2\} = R(I_1, I_2)$.

(f) is one-to-one mapping, it can be proved by the same.

(g) and (f) preserve addition, that is because:

Consider $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2 \in 2 - SP_R$, $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in 2 - SP_M$, then:

$$g(A + B) = g[(a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2] = a_0 + b_0 + (a_1 + b_1)I_1 + (a_2 + b_2)I_2$$
$$= g(A) + g(B)$$

$f(X + Y) = f(X) + f(Y)$ by a similar discussion.

(g) preserves multiplication, that is because:

$g(A.B) = a_0b_0 + (a_0b_2 + a_2b_0 + a_2b_2 + a_1b_2 + a_2b_1)I_1 + (a_0b_1 + a_1b_0 + a_1b_1)I_2 = g(A).g(B)$.

(f) is semi module homomorphism, that is because:

$f(A.X) = a_0x_0 + (a_0x_2 + a_2x_0 + a_2x_2 + a_1x_2 + a_2x_1)I_1 + (a_0x_1 + a_1x_0 + a_1x_1)I_2$
$= (a_0 + a_1I_1 + a_2I_2)(x_0 + x_2I_1 + x_1I_2) = g(A).f(X)$

The basis of a symbolic 2-plithogenic module:

Theorem.

Let $T = \{t_1, \dots, t_n\}$ be a basis of the module V over the ring R, then the set:

$T_P = \{t_i + (t_j - t_i)P_1 + (t_k - t_j)P_2; 1 \le i, j, k \le n\}$ is a basis of $2 - SP_V$.

Proof.

Let $X = x_0 + x_1P_1 + x_2P_2 \in 2 - SP_M, x_0, x_1, x_2 \in M$.

$x_0 = \sum_{i=1}^{n} \alpha_i t_i,\ x_0 + x_1 = \sum_{j=1}^{n} \beta_j t_j,\ x_0 + x_1 + x_2 = \sum_{k=1}^{n} \gamma_k t_k\,; \alpha_i, \beta_j, \gamma_k \in R.$

We put $A_{i,j,k} = \alpha_i + (\beta_j - \alpha_i)P_1 + (\gamma_k - \beta_j)P_2; 1 \le i, j, k \le n$

$T_{i,j,k} = t_i + (t_j - t_i)P_1 + (t_k - t_j)P_2; 1 \le i, j, k \le n$

$$\sum_{i,j,k=1}^{n} A_{i,j,k}\, T_{i,j,k}$$

$$= \sum_{i=1}^{n} [\alpha_i t_i + [\beta_j t_j - \beta_j t_i - \alpha_i t_j + \alpha_i t_i + \beta_j t_i - \alpha_i t_i + \alpha_i t_j - \alpha_i t_i]P_1$$
$$+ [\alpha_i t_k - \alpha_i t_j + \gamma_k t_i - \beta_j t_i - \gamma_k t_j + \gamma_k t_i - \beta_j t_j + \beta_j t_i + \gamma_k t_k - \gamma_k t_j - \beta_j t_k$$
$$+ -\beta_j t_j + \beta_j t_k - \beta_j t_j - \alpha_i t_k + \alpha_i t_j]P_2]$$

$$\sum_{i=1}^{n} \alpha_i t_i + P_1\left[\sum_{j=1}^{n} \beta_j t_j - \sum_{i=1}^{n} \alpha_i t_i\right] + P_2\left[\sum_{k=1}^{n} \gamma_k t_k - \sum_{j=1}^{n} \beta_j t_j\right]$$
$$= x_0 + P_1[x_0 + x_1 - x_0] + P_2[x_0 + x_1 + x_2 - (x_0 + x_1)] = x_0 + x_1P_1 + x_2P_2$$
$$= X$$

Thus T generates $2 - SP_M$.

On the other hand, T is linearly independent, that is because:

If $\sum_{i,j,k=1}^{n} A_{i,j,k}\,.X = 0$, then:

$\sum_{i=1}^{n} \alpha_i t_i = 0, \sum_{j=1}^{n} \beta_j t_j = 0, \sum_{k=1}^{n} \gamma_k t_k = 0$, hence$\alpha_i = \beta_j = \gamma_k = 0$ for all i, j, k, thus $A_{i,j,k} = 0$.

This implies that T is a basis of $2 - SP_M$.

Example.

Find a basis of $2 - SP_{Z^2}$.

Solution.

First of all, we have $\{u_1 = (1,0), u_2 = (0,1)\}$ is a basis of Z^2.

The corresponding basis of $2-SP_{Z^2}$ is:

$T=\{T_1,T_2,T_3,T_4,T_5,T_6,T_7,T_8\}$ such that:

$T_1=(1,0), T_2=(0,1), T_3=u_1+(u_2-u_1)P_1+(u_2-u_2)P_2=(1,0)+(-1,1)P_1$

$T_4=u_1+(u_2-u_1)P_1+(u_1-u_2)P_2=(1,0)+(-1,1)P_1+(1,-1)P_2$

$T_5=u_2+(u_2-u_1)P_1+(u_1-u_1)P_2=(0,1)+(1,-1)P_1$

$T_6=u_2+(u_2-u_1)P_1+(u_2-u_1)P_2=(0,1)+(1,-1)P_1+(-1,1)P_2$

$T_7=u_1+(u_1-u_1)P_1+(u_2-u_1)P_2=(1,0)+(-1,1)P_2$

$T_8=u_2+(u_2-u_2)P_1+(u_1-u_2)P_2=(0,1)+(1,-1)P_2$

Remark.

$$dim\,(2-SP_M)=(dimM)^3$$

Conclusion

In this paper we have defined the concept of symbolic 2-plithogenic modules over a symbolic 2-plithogenic ring, where we have presented some of their elementary properties such as basis, homomorphisms, and AH-submodules. On the other hand, we have suggested many examples to clarify the validity of our work.

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An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces

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Abstract:

The fusion of symbolic plithogenic sets with algebraic structures generates novel algebraic neutrosophic structures that generalize the classical known structures. The objective of this paper is to define the concept of symbolic 2-plithogenic vector space over a symbolic 2-plitogenic field.

Concepts such as AH-subspace and AH-linear transformation will be presented and discussed in terms of theorems.

Keywords: 2-plithogenic symbolic set, 2-plithogenic vector space, 2-plithogenic dimension

Introduction

The concept of symbolic plithogenic sets was defined by Smarandache in [13-17 ,30], and he suggested an algebraic approach of these sets. Laterally, the concept of symbolic 2-plithogenic rings [31], where the concepts such as symbolic AH-ideals, and AH-homomorphisms were presented and discussed.

In general, we can say that symbolic plithogenic structures are very close to neutrosophic algebraic structures with many differences in the definition of multiplication operation [1-10].

Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$2-SP_R = \{a_0 + a_1P_1 + a_2P_2;\ a_i \in R, {P_j}^2 = P_j, P_1 \times P_2 = P_{max(1,2)} = P_2\}$.

Smarandache has defined algebraic operations on $2-SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$.

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0{P_1}^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2{P_2}^2 + a_1b_1P_1P_1 = a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.

It is clear that $(2-SP_R)$ is a ring.

Also, if R is commutative, then $2-SP_R$ is commutative, and if R has a unity (1), than $2-SP_R$ has the same unity (1).

If R is a field, then $2-SP_R$ is called a symbolic 2-plithogenic field.

In this paper, we study the symbolic 2-plithogenic vector spaces according to many points of view, where substructures such as AH-subspaces, and AH-linear transformations will be presented in terms of theorems. In addition, many examples will be illustrated to explain the novelty of these ideas.

Main Discussion

Definition.

Let V be a vector space over the field F, let $2-SP_F$ be the corresponding symbolic 2-plithogenic field.

$2-SP_F = \{x + yP_1 + zP_2;\ x, y, z \in F, {P_i}^2 = P_i, P_1P_2 = P_2P_1 = P_2\}$.

We define the symbolic 2-plithogenic vector space as follows:

$2-SP_V = V + VP_1 + VP_2 = \{a + bP_1 + cP_2;\ a, b, c \in V\}$.

Operations on $2-SP_V$ can be defined as follows:

Addition: $(+): 2-SP_V \to 2-SP_V$, such that:

$[x_0 + x_1P_1 + x_2P_2] + [y_0 + y_1P_1 + ys_2P_2] = (x_0 + y_0) + (x_1 + y_1)P_1 + (x_2 + y_2)P_2$

Multiplication: $(.): 2-SP_F \times 2-SP_V \to 2-SP_V$, such that:

$$[a + bP_1 + cP_2].[x_0 + x_1P_1 + x_2P_2] = ax_0 + (ax_1 + bx_0 + bx_1)P_1 + (ax_2 + bx_2 + cx_0 + cx_1 + cx_2)P_2$$

where $x_i, y_i \in V, a, b, c \in F$

Theorem.

Let $(2-SP_V,+,.)$ Is a module over the ring $2-SP_F$.

Proof.

Let $X=x_0+x_1P_1+x_2P_2, Y=y_0+y_1P_1+y_2P_2 \in 2-SP_V$, $A=a_0+a_1P_1+a_2P_2, B=b_0+b_1P_1+b_2P_2 \in 2-SP_F$ we have:

1. $X=X,(X+Y)+Z=X+(Y+Z),X+(-X)=-X+X=0,X+0=0+X=X$

Also

$$A(X+Y)=(a_0+a_1P_1+a_2P_2)[(x_0+y_0)+(x_1+y_1)P_1+(x_2+y_2)P_2]$$
$$=a_0(x_0+y_0)+\big(a_0(x_1+y_1)+a_1(x_0+y_0)+a_1(x_1+y_1)\big)P_1$$
$$+\big(a_0(x_2+y_2)+a_1(x_2+y_2)+a_2(x_0+y_0)+a_2(x_1+y_1)+a_2(x_2+y_2)\big)P_2$$
$$=A.X+A.Y$$

$$(A+B)X=[(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2](x_0+x_1P_1+x_2P_2)$$
$$=(a_0+b_0)x_0+\big((a_0+b_0)x_1+(a_1+b_1)x_0+(a_1+b_1)x_1\big)P_1$$
$$+\big((a_0+b_0)x_2+(a_1+b_1)x_2+(a_2+b_2)x_0+(a_2+b_2)x_1+(a_2+b_2)x_2\big)P_2$$
$$=A.X+B.X$$

$$(A.B).X=[a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)P_2](x_0$$
$$+x_1P_1+x_2P_2)$$
$$=a_0b_0x_0+[a_0b_0x_1+(a_0b_1+a_1b_0+a_1b_1)x_0+(a_0b_1+a_1b_0+a_1b_1)x_1]P_1$$
$$+[a_0b_0x_2+(a_0b_2+a_2b_0+a_1b_1)x_2+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)x_0$$
$$+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)x_1$$
$$+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)x_2]P_2=A(B.X)$$

Example.

Let $V=R^3$ be the Euclidean space over the field $F=R$.

The corresponding symbolic 2-plithogenic vector space over $2-SP_F$ is:

$2-SP_{R^3}=\{(x_0,y_0,z_0)+(x_1,y_1,z_1)P_1+(x_2,y_2,z_2)P_2; x_i,y_i,z_i \in R\}$

Consider $X=(1,1,0)+(2,-1,1)P_1+(0,1,-1)P_2 \in 2-SP_{R^3}, A=2+P_1+P_2 \in 2-SP_R$. We have:

$$A.X=(2,2,0)+[(4,-2,2)+(1,1,0)+(2,-1,1)]P_1$$
$$+[(0,2,2)+(0,1,1)+(1,1,0)+(2,-1,1)+(0,1,1)]P_2$$
$$=(2,2,0)+(7,-2,3)P_1+(3,4,5)P_2$$

Definition.

Let $2-SP_V$ be a symbolic 2-plithogenic vector space over $2-SP_F$, let V_0,V_1,V_2 be the three subspaces of V, we define the AH-subspace as follows:

$$W = V_0 + V_1P_1 + V_2P_2 = \{x + yP_1 + zP_2;\ x \in V_0, y \in V_1, z \in V_2\}$$

If $V_0 = V_1 = V_2$, then W is called an AHS-subspace.

Example.

Consider $2 - SP_{R^3}$, we have $V_0 = \{(a, 0,0);\ a \in R\}, V_1 = \{(0, b, 0);\ b \in R\}, V_2 = \{(0,0, c);\ c \in R\}$ are three subspaces of $V = R^3$.

$W = V_0 + V_1P_1 + V_2P_2 = \{(a, 0,0) + (0, b, 0)P_1 + (0,0, c)P_2;\ a, b, c \in R\}$ is an AH-subspace of $2 - SP_{R^3}$.

$T = V_1 + V_1P_1 + V_1P_2 = \{(0, a, 0) + (0, b, 0)P_1 + (0, c, 0)P_2;\ a, b, c \in R\}$ is an AHS-subspace.

Theorem.

Let $2 - SP_V$ be a symbolic 2-plithogenic vector space over $2 - SP_F$, let W be an AHS-subspace of $2 - SP_V$, then W is a submodule of $2 - SP_V$.

Proof.

Suppose that W is an AHS-subspace, then there exists a subspace $V_0 \leq V$, such that

$W = V_0 + V_0P_1 + V_0P_2 = \{x + yP_1 + zP_2;\ x, y, z \in V_0\}$.

Let $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in W$, then:

$X - Y = (x_0 - y_0) + (x_1 - y_1)P_1 + (x_2 - y_2)P_2 \in W$

$\forall A = a_0 + a_1P_1 + a_2P_2 \in 2 - SP_F$, then:

$A.X = a_0x_0 + (a_0x_1 + a_1x_0 + a_1x_1)P_1 + (a_0x_2 + a_1x_2 + a_2x_0 + a_2x_1 + a_2x_2)P_2 \in W$, that is because $a_0x_0 \in V_0, a_0x_1 + a_1x_0 + a_1x_1 \in V_0, a_0x_2 + a_1x_2 + a_2x_0 + a_2x_1 + a_2x_2 \in V_0$, this implies the proof.

Definition.

Let V, W be two vector spaces over the field F. Let $2 - SP_V$, $2 - SP_W$ be the corresponding symbolic 2-plithogenic vector spaces over $2 - SP_F$.

Let $L_0, L_1, L_2: V \to W$ be three linear transformations, we define the AH-linear transformation as follows:

$L: 2 - SP_V \to 2 - SP_W, L = L_0 + L_1P_1 + L_2P_2\ ; L(x + yP_1 + zP_2) = L_0(x) + L_1(y)P_1 + L_2(z)P_2$.

If $L_0 = L_1 = L_2$, then L is called AHS-linear transformation.

Definition.

Let $L = L_0 + L_1P_1 + L_2P_2: 2 - SP_V \to 2 - SP_W$ be an AH-linear transformation, we define:

1. $AH - ker(L) = ker(L_0) + ker(L_1)P_1 + ker(L_2)P_2 = \{x + yP_1 + zP_2\}; x \in ker(L_0), y \in ker(L_1), z \in ker(L_2)$.

2. $AH-Im(L)=Im(L_0)+Im(L_1)P_1+Im(L_2)P_2=\{a+bP_1+cP_2\}; a\in Im(L_0), b\in Im(L_1), c\in Im(L_2)$

If L is AHS-linear transformation, then we get $AHS-kernel,\ AHS-Image$.

Theorem.

Let $L=L_0+L_1P_1+L_2P_2: 2-SP_V\to 2-SP_W$ be an AH-linear transformation, then:

1. $AH-ker(L)$ is AH-subspace of $2-SP_V$.
2. $AH-Im(L)$ is AH-subspace of $2-SP_W$.

Proof.

1. Since $ker(L_0), ker(L_1), ker(L_2)$ are subspaces of V, then $AH-ker(L)$ is an AH-subspace of $2-SP_V$.
2. It is holds by the same.

Remark.

If L_0, L_1, L_2 are isomorphism, then $ker(L_0)=ker(L_1)=ker(L_2)=\{0\}, Im(L_0)=Im(L_1)=Im(L_2)=W$, thus $AH-ker(L)=\{0\}, AH-Im(L)=2-SP_W$.

Example.

Take $V=R^3, W=R^3,\ L_0, L_1, L_2: V\to W$ such that:

$L_0(x,y,z)=(x,y), L_1(x,y,z)=(2x,z), L_2(x,y,z)=(x-y,y-z)$

The corresponding AH-linear transformation is:

$L=L_0+L_1P_1+L_2P_2: 2-SP_{R^3}\to 2-SP_{R^2}$:

$$L[(x_0,y_0,z_0)+(x_1,y_1,z_1)P_1+(x_2,y_2,z_2)P_2]$$
$$=L_0(x_0,y_0,z_0)+L_1(x_1,y_1,z_1)P_1+L_2(x_2,y_2,z_2)P_2$$
$$=(x_0,y_0)+(2x_1,z_1)P_1+(x_2-y_2,y_2-z_2)P_2$$

For example, take $X=(1,2,1)+(4,3,-5)P_1+(1,1,1)P_2$, then:

$L(X)=(1,2)+(8,-5)P_1+(0,0)P_2=(1,2)+(8,-5)P_1$.

$$\begin{cases} ker(L_0)=\{(0,0,z_0);\ z_0\in R\} \\ ker(L_1)=\{(0,y_1,0);\ y_1\in R\} \\ ker(L_2)=\{(x_2,x_2,x_2);\ x_2\in R\} \\ AH-ker(L)=\{(0,0,z_0)+(0,y_1,0)P_1+(x_2,x_2,x_2)P_2; z_0,y_1,x_2\in R\} \end{cases}$$

Also,

$$\begin{cases} Im(L_0)=R^2 \\ Im(L_1)=R^2 \\ Im(L_2)=R^2 \\ AH-Im(L)=R^2+R^2P_1+R^2P_2=2-SP_W \end{cases}$$

Example.

Take $W = V = R^2,\ L_0, L_1, L_2: V \to W$ such that:

$L_0(x,y) = (3x,-2x), L_1(x,y) = (x-y,2x), L_2(x,y,z) = (x+2y,y)$

The corresponding AH-linear transformation is $L = L_0 + L_1P_1 + L_2P_2: 2-SP_V \to 2-SP_W$;

$$L[(x_0,y_0) + (x_1,y_1)P_1 + (x_2,y_2)P_2] = L_0(x_0,y_0) + L_1(x_1,y_1)P_1 + L_2(x_2,y_2)P_2$$
$$= (3x_0,-2x_0) + (x_1-y_1,2x_1)P_1 + (x_2+2y_2,y_2)P_2$$

For example $X = (1,4) + (2,8)P_1 + (3,-1)P_2$

$L(X) = (1,4) + (2,8)P_1 + (3,-1)P_2$.

$$\begin{cases} ker(L_0) = \{(0,y_0);\ y_0 \in R\} \\ ker(L_1) = \{0\} \\ ker(L_2) = \{0\} \\ AH-ker(L) = \{(0,y_0) + 0P_1 + 0P_2; y_0 \in R\} \end{cases}$$

Also,

$$\begin{cases} Im(L_0) = \{(a,0);\ a \in R\} \\ Im(L_1) = R^2 \\ Im(L_2) = R^2 \\ AH-Im(L) = \{(a,0) + (a_1,b_1)P_1 + (a_2,b_2)P_2; a,a_1,a_2,b_2,b_1 \in R\} \end{cases}$$

Theorem.

Let $L = f + fP_1 + fP_2: 2-SP_V \to 2-SP_W$ be an AHS-linear transformation, then L is a module homomorphism.

Proof.

Let $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in 2-SP_V$, then:

$$L(X+Y) = f(x_0+y_0) + f(x_1+y_1)P_1 + f(x_2+y_2)P_2$$
$$= [f(x_0) + f(x_1)P_1 + f(x_2)P_2] + [f(y_0) + f(y_1)P_1 + f(y_2)P_2] = L(X) + L(Y)$$

Let $A = a_0 + a_1P_1 + a_2P_2 \in 2-SP_F$, then:

$$L(A.X) = f(a_0x_0) + f(a_0x_1 + a_1x_0 + a_1x_1)P_1 + f(a_0x_2 + a_2x_0 + a_2x_2 + a_1x_2 + a_2x_1)P_2$$
$$= a_0f(x_0) + \big(a_0f(x_1) + a_1f(x_0) + a_1f(x_1)\big)P_1$$
$$+ \big(a_0f(x_2) + a_2f(x_0) + a_2f(x_2) + a_1f(x_2) + a_2f(x_1)\big)P_2$$
$$= [a_0 + a_1P_1 + a_2P_2].[f(x_0) + f(x_1)P_1 + f(x_2)P_2] = A.L(X)$$

Thus, L is a module homomorphism.

The algebraic relations between symbolic 2-plithogenic vector spaces and neutrosophic vector spaces .

Theorem.

Let V be a vector space over the field F, consider $V(I)=V+VI=\{x+yI; x,y\in V\}$ is the corresponding neutrosophic vector space over the neutrosophic field $F(I)=\{a+bI; a,b\in F\}$.

$V(I_1,I_2)=V+VI_1+VI_2=\{x+yI_1+zI_2; x,y,z\in V\}$ is the corresponding refined neutrosophic vector space over the refined neutrosophic field $F(I_1,I_2)=\{a+bI_1+cI_2; a,b,c\in F\}$.

$2-SP_V=V+VP_1+VP_2=\{x+yP_1+zP_2; x,y,z\in V\}$ is the corresponding symbolic 2-plithogenic vector space over $2-SP_F$, then:

1. $2-SP_V$ is semi homomorphic to $V(I)$.
2. $2-SP_V$ is semi isomorphic to $V(I_1,I_2)$.

Proof.

1. We define $f: 2-SP_V\to V(I), g: 2-SP_F\to F(I)$ such that:

$f(x+yP_1+zP_2)=x+yI; x,y,z\in V$

$g(a+bP_1+cP_2)=a+bI; a,b,c\in F$

We have the following:

g is a ring homomorphism, that is because:

$A=a_0+a_1P_1+a_2P_2, B=b_0+b_1P_1+b_2P_2;\ a_i,b_i\in F$, then:

If $A=B$, then $a_i=b_i$ for all i, thus $a_0+a_1I=b_0+b_1I, i.e.\quad g(A)=g(B)$.

$g(A+B)=g[(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2]=a_0+b_0+(a_1+b_1)I=g(A)+g(B)$.

$$g(A.B)=g[a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b+a_2b_1+a_2b_2)P_2]$$
$$=a_0b_0+(a_0b_1+a_1b_0+a_1b_1)I=(a_0+a_1I)(b_0+b_1I)=g(A).g(B)$$

On the other hand, f is well defined, that is because:

If $X=x_0+x_1P_1+x_2P_2, Y=y_0+y_1P_1+y_2P_2$, then $x_i=y_i$ for all i, hence $a_0+a_1I=b_0+b_1I$, thus $f(X)=f(Y)$.

f preserves addition, that is because:

For $X=x_0+x_1P_1+x_2P_2, Y=y_0+y_1P_1+y_2P_2$, we have:

$f(X+Y)=f[(x_0+y_0)+(x_1+y_1)P_1+(x_2+y_2)P_2]=x_0+y_0+(x_1+y_1)I=f(X)+f(Y)$.

f preserves multiplication, that is because:

For $A=a_0+a_1P_1+a_2P_2\in 2-SP_V$, we have:

$f(A.X)=a_0x_0+(a_0x_1+a_1x_0+a_1x_1)I=(a_0+a_1I)(x_0+x_1I)=g(A).f(X)$

Thus f is a semi module homomorphism.

We define $f: 2 - SP_V \to V(I_1, I_2)$, $g: 2 - SP_F \to F(I_1, I_2)$, where $f(x + yP_1 + zP_2) = x + zI_1 + yI_2$, and $g(a + bP_1 + cP_2) = a + cI_1 + bI_2; x, y, z \in V, a, b, c \in F$.

(g) is well defined, that is because:

If $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2$, then:

$a_0 = a_1, b_0 = b_1, c_0 = c_1$, hence: $a_0 + c_0I_1 + b_0I_2 = a_1 + c_1I_1 + b_1I_2$, so that $g(A) = g(B)$.

(f) is well defined by a similar discussion.

(g) is one-to-one mapping, that is because:

$ker(g) = \{a + bP_1 + cP_2; g(a + bP_1 + cP_2) = 0\} = O$

$$Im(g) = \{a + cI_1 + bI_2; g(a + bP_1 + cP_2) \in F(I_1, I_2);\ \exists A \in 2 - SP_F, g(A) = a + cI_1 + bI_2\}$$
$$= F(I_1, I_2)$$

(f) is one-to-one mapping, it can be proved by the same.

(g) and (f) preserve addition, that is because:

Consider $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2 \in 2 - SP_F$, $X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2 \in 2 - SP_V$, then:

$$g(A + B) = g[(a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2] = a_0 + b_0 + (a_1 + b_1)I_1 + (a_2 + b_2)I_2$$
$$= g(A) + g(B)$$

$f(X + Y) = f(X) + f(Y)$ by a similar discussion.

(g) preserves multiplication, that is because:

$g(A.B) = a_0b_0 + (a_0b_2 + a_2b_0 + a_2b_2 + a_1b_2 + a_2b_1)I_1 + (a_0b_1 + a_1b_0 + a_1b_1)I_2 = g(A).g(B)$.

(f) is semi module homomorphism, that is because:

$$f(A.X) = a_0x_0 + (a_0x_2 + a_2x_0 + a_2x_2 + a_1x_2 + a_2x_1)I_1 + (a_0x_1 + a_1x_0 + a_1x_1)I_2$$
$$= (a_0 + a_1I_1 + a_2I_2)(x_0 + x_2I_1 + x_1I_2) = g(A).f(X)$$

The basis of a symbolic 2-plithogenic vector spaces:

Theorem.

Let $T = \{t_1, \dots, t_n\}$ be a basis of the vector space V over the field F, then the set:

$T_P = \{t_i + (t_j - t_i)P_1 + (t_k - t_j)P_2; 1 \le i, j, k \le n\}$ is a basis of $2 - SP_V$.

Proof.

Let $X = x_0 + x_1P_1 + x_2P_2 \in 2 - SP_V, x_0, x_1, x_2 \in V$.

$x_0 = \sum_{i=1}^{n} \alpha_i t_i,\ x_0 + x_1 = \sum_{j=1}^{n} \beta_j t_j,\ x_0 + x_1 + x_2 = \sum_{k=1}^{n} \gamma_k t_k ; \alpha_i, \beta_j, \gamma_k \in F$.

We put $A_{i,j,k} = \alpha_i + (\beta_j - \alpha_i)P_1 + (\gamma_k - \beta_j)P_2; 1 \le i, j, k \le n$

$T_{i,j,k} = t_i + (t_j - t_i)P_1 + (t_k - t_j)P_2; 1 \le i, j, k \le n$

$$\sum_{i,j,k=1}^{n} A_{i,j,k}\, T_{i,j,k}$$

$$= \sum_{i=1}^{n} \left[\alpha_i t_i + \left[\beta_j t_j - \beta_j t_i - \alpha_i t_j + \alpha_i t_i + \beta_j t_i - \alpha_i t_i + \alpha_i t_j - \alpha_i t_i\right] P_1\right.$$

$$+ \left[\alpha_i t_k - \alpha_i t_j + \gamma_k t_i - \beta_j t_i - \gamma_k t_j + \gamma_k t_i - \beta_j t_j + \beta_j t_i + \gamma_k t_k - \gamma_k t_j - \beta_j t_k\right.$$

$$\left.\left. + -\beta_j t_j + \beta_j t_k - \beta_j t_j - \alpha_i t_k + \alpha_i t_j\right] P_2\right]$$

$$\sum_{i=1}^{n} \alpha_i t_i + P_1 \left[\sum_{j=1}^{n} \beta_j t_j - \sum_{i=1}^{n} \alpha_i t_i\right] + P_2 \left[\sum_{k=1}^{n} \gamma_k t_k - \sum_{j=1}^{n} \beta_j t_j\right]$$

$$= x_0 + P_1[x_0 + x_1 - x_0] + P_2[x_0 + x_1 + x_2 - (x_0 + x_1)] = x_0 + x_1 P_1 + x_2 P_2$$

$$= X$$

Thus T generates $2 - SP_V$.

On the other hand, T is linearly independent, that is because:

If $\sum_{i,j,k=1}^{n} A_{i,j,k} \,.\, X = 0$, then:

$\sum_{i=1}^{n} \alpha_i t_i = 0, \sum_{j=1}^{n} \beta_j t_j = 0, \sum_{k=1}^{n} \gamma_k t_k = 0$, hence$\alpha_i = \beta_j = \gamma_k = 0$ for all i, j, k, thus $A_{i,j,k} = 0$.

This implies that T is a basis of $2 - SP_V$.

Example.

Find a basis of $2 - SP_{R^2}$.

Solution.

First of all, we have $\{u_1 = (1,0), u_2 = (0,1)\}$ is a basis of R^2.

The corresponding basis of $2 - SP_{R^2}$ is:

$T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ such that:

$T_1 = (1,0), T_2 = (0,1), T_3 = u_1 + (u_2 - u_1)P_1 + (u_2 - u_2)P_2 = (1,0) + (-1,1)P_1$

$T_4 = u_1 + (u_2 - u_1)P_1 + (u_1 - u_2)P_2 = (1,0) + (-1,1)P_1 + (1,-1)P_2$

$T_5 = u_2 + (u_2 - u_1)P_1 + (u_1 - u_1)P_2 = (0,1) + (1,-1)P_1$

$T_6 = u_2 + (u_2 - u_1)P_1 + (u_2 - u_1)P_2 = (0,1) + (1,-1)P_1 + (-1,1)P_2$

$T_7 = u_1 + (u_1 - u_1)P_1 + (u_2 - u_1)P_2 = (1,0) + (-1,1)P_2$

$T_8 = u_2 + (u_2 - u_2)P_1 + (u_1 - u_2)P_2 = (0,1) + (1,-1)P_2$

Remark.

$dim\,(2 - SP_V) = (dimV)^3$.

Conclusion

In this paper we have defined the concept of symbolic 2-plithogenic vector spaces over a symbolic 2-plithogenic field, where we have presented some of their elementary properties such as basis, linear transformations, and AH-subspaces. On the other hand, we have suggested many examples to clarify the validity of our work.

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On The Symbolic 3-Plithogenic Rings and Their Algebraic Properties

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Abstract:

The aim of this paper is to define and study the concept of symbolic 3-plithogenic rings as a novel extension of classical rings and symbolic 2-plithogenic rings respectively. Also, many related substructures will be presented such as idempotent elements, AH-ideals, AHS-homomorphisms, and kernels.

On the other hand, many examples will be illustrated to show the validity of concepts and theorem.

Keywords: Symbolic 3-plithogenic ring, AH-ideal, AH-homomorphism, symbolic plithogenic set

Introduction

The concept of symbolic neutrosophic algebraic structure has played an important role in the advances of pure algebra and logical algebra. Many interesting structures were defined from this point of view, such as neutrosophic rings, refined neutrosophic rings, neutrosophic spaces, and n-cyclic refined neutrosophic rings [1-5,8-11,13-20].

In [30], Smarandache has presented a novel approach to algebraic structures by using the concept of n-symbolic plithogenic sets, where he defined algebraic operations on these structures and asked many open problems about them.

In [31], the concept of symbolic 2-plithogenic ring was suggested, and concepts such as symbolic 2-plithogenic AH-homomorphisms, ideals, and kernels.

This paper is considered as an additional effort which is dedicated to define a new algebraic structure built over the idea of symbolic n-plithogenic set with algebraic ring in a special case of n=3.

Main Discussion

Definition.

Let R be a ring, the symbolic 3-plithogenic ring is defined as follows:

$3-SP_R=\{a_0+a_1P_1+a_2P_2+a_3P_3;\ a_i\in R, {P_j}^2=P_j, P_i\times P_j=P_{max(i,j)}\}$.

Smarandache has defined algebraic operations on $3-SP_R$ as follows:

Addition:

$[a_0+a_1P_1+a_2P_2+a_3P_3]+[b_0+b_1P_1+b_2P_2+b_3P_3]=(a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2+(a_3+b_3)P_3$.

Multiplication:

$[a_0+a_1P_1+a_2P_2+a_3P_3].[b_0+b_1P_1+b_2P_2+b_3P_3]=a_0b_0+a_0b_1P_1+a_0b_2P_2+a_0b_3P_3+a_1b_0{P_1}^2+a_1b_2P_1P_2+a_2b_0P_2+a_2b_1P_1P_2+a_2b_2{P_2}^2+a_1b_3P_3P_1+a_2b_3P_2P_3+a_3b_3(P_3)^2+a_3b_0P_3+a_3b_1P_3P_1+a_3b_2P_2P_3+a_1b_1P_1P_1=a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)P_2+(a_0b_3+a_1b_3+a_2b_3+a_3b_3+a_3b_0+a_3b_1+a_3b_2)P_3$.

It is clear that $(3-SP_R)$ is a ring.

Also, if R is commutative, then $3-SP_R$ is commutative, and if R has a unity (1), than $3-SP_R$ has the same unity (1).

Example.

Consider the ring $R=Z_5=\{0,1,2,3,4\}$, the corresponding $3-SP_R$ is:

$3-SP_R=\{a+bP_1+cP_2+dP_3; a,b,c,d\in Z_5\}$.

If $X=1+2P_1+3P_2+P_3, Y=P_1+2P_2$, then:

$X+Y=1+3P_1+P_2+P_3, X-Y=1+P_1+P_2+P_3, X.Y=P_1+2P_2+2P_1+4P_2+3P_2+6P_2+P_3+2P_3=3P_1+3P_2+3P_3$.

Invertibility.

Theorem.

Let $3-SP_R$ be a 3-plithogenic symbolic ring, with unity (1).

Let $X=x_0+x_1P_1+x_2P_2+x_3P_3$ be an arbitrary element, then:

1. X is invertible if and only if $x_0, x_0+x_1, x_0+x_1+x_2$, $x_0+x_1+x_2+x_3$ are invertible.
2. $X^{-1} = x_0{}^{-1} + [(x_0+x_1)^{-1} - x_0{}^{-1}]P_1 + [(x_0+x_1+x_2)^{-1} - (x_0+x_1)^{-1}]P_2$ +[$(x_0+x_1+x_2+x_3)^{-1} - (x_0+x_1+x_2)^{-1}]P_3$.

Proof.

1. Assume that X is invertible, than there exists $Y = y_0 + y_1P_1 + y_2P_2 + y_3P_3$ such that $X.Y = 1$, hence:

$$\begin{cases} x_0y_3 + x_1y_3 + x_2y_3 + x_3y_3 + x_3y_1 + x_3y_2 + x_3y_0 = 0 \ (1) \\ x_0y_0 = 1 \dots (2) \\ x_0y_1 + x_1y_0 + x_1y_1 = 0 \dots (3) \\ x_0y_2 + x_2y_0 + x_2y_2 + x_1y_2 + x_2y_1 = 0 \dots (4), \end{cases}$$

Equation (2), means that x_0 is invertible.

By adding (3) to (2), we get $(x_0+x_1)(y_0+y_1) = 1$, thus x_0+x_1 is invertible.

By adding (4) to (3)to (2), we get $(x_0+x_1+x_2)(y_0+y_1+y_2) = 1$, hence $x_0+x_1+x_2$ is invertible.

By adding (1) to (2) to (3) to(4), we get $(x_0+x_1+x_2+x_3)(y_0+y_1+y_2+y_3) = 1$, hence $x_0+x_1+x_2$+x_3 is invertible.

The converse holds by the same.

2. From the previous approach, we can see that:

$y_0 = x_0{}^{-1}, y_0+y_1 = (x_0+x_1)^{-1}, y_0+y_1+y_2 = (x_0+x_1+x_2)^{-1}, (x_0+x_1+x_2+x_3)^{-1} = y_0+y_1+y_2+y_3$, then:

3. $Y = x_0{}^{-1} + [(x_0+x_1)^{-1} - x_0{}^{-1}]P_1 + [(x_0+x_1+x_2)^{-1} - (x_0+x_1)^{-1}]P_2$ +$[(x_0+x_1+x_2+x_3)^{-1} - (x_0+x_1+x_2)^{-1}]P_3$.

$= X^{-1}$.

Example.

Take $R = Z_5 = \{0,1,2,3,4\}$, $3-SP_{Z_5}$ is the corresponding symbolic 3-plithogenic ring, consider $X = 2 + 4P_1 + 2P_2 + P_3 \in 2 - SP_{Z_5}$, then:

$x_0 = 2$ is invertible with $x_0{}^{-1} = 3$, $x_0+x_1 = 1$ is invertible with $(x_0+x_1)^{-1} = 1$, $x_0+x_1+x_2 = 3$ is invertible with $(x_0+x_1+x_2)^{-1} = 2$, $x_0+x_1+x_2+x_3 = 4, (x_0+x_1+x_2+x_3)^{-1} = 4$ hence:

$X^{-1} = 3 + (1-3)P_1 + (2-1)P_2 + (4-2)P_3 = 3 + 3P_1 + P_2 + 2P_3$.

Idempotency.

Definition.

Let $X = a + bP_1 + cP_2 + dP_3 \in 3 - SP_R$, then X is idempotent if and only if $X^2 = X$.

Theorem.

Let $X = a + bP_1 + cP_2 + dP_3 \in 3 - SP_R$, then X is idempotent if and only if $a, a + b, a + b + c,\ a + b + c + d$ are idempotent.

Proof.

$X^2 = X.X = (a + bP_1 + cP_2 + dP_3)(a + bP_1 + cP_2 + dP_3) = a^2 + (ab + ba + b^2)P_1 + (ac + bc + ca + cb + c^2)P_2 + (ad + bd + cd + d.d + da + db + dc)P_3$.

$$X^2 = X.X \text{ equivalents } \begin{cases} ad + bd + cd + d.d + da + db + dc = 0 \ (1) \\ a^2 = a \ldots \ (2) \\ ab + ba + b^2 = b \ldots \ (3) \\ ac + bc + ca + cb + c^2 = c \ldots \ (4) \end{cases}$$

Equation (2) means that a is idempotent.

By adding (3) to (2), we get $(a + b)^2 = a + b$, hence $a + b$ is idempotent.

By adding (3) to (2) to (4), we get $(a + b + c)^2 = a + b + c$, hence $a + b + c$ is idempotent.

By adding (1) to (2) to (3) to (4), we get $(a + b + c + d)^2 = a + b + c + d$, thus $a + b + c + d$ is idempotent.

Thus the proof is complete.

Example.

Take $R = Z_6 = \{0,1,2,3,4,5\}$, $3 - SP_{Z_6}$ is the corresponding symbolic 3-plithogenic ring, consider $X = 3 + P_1 + 5P_2 \in 3 - SP_{Z_5}$, we have:

$X^2 = 9 + 6P_1 + P_1 + 30P_2 + 25P_2 + 10P_2 = 3 + P_1 + 5P_2 = X$.

The following theorem clarifies the natural powers in $2 - SP_R$.

Theorem.

Let $3 - SP_R$ be a commutative symbolic 3-plithogenic ring, hence if $X = a + bP_1 + cP_2 + dP_3$, then $X^n = a^n + [(a + b)^n - a^n]P_1 + [(a + b + c)^n - (a + b)^n]P_2 + [(a + b + c + d)^n - (a + b + c)^n]P_3$ for every $n \in Z^+$.

Proof.

For $n = 1$, it holds easily. Assume that it is true for $n = k$, we prove it for $n = k + 1$.

$X^{k+1} = X.X^k = (a + bP_1 + cP_2 + dP_3)\big(a^k + [(a + b)^k - a^k]P_1 + [(a + b + c)^k - (a + b)^k]P_2 + [(a + b + c + d)^k - (a + b + c)^k]P_3\big) = a^{k+1} + [(a + b)^{k+1} - a^{k+1}]P_1 + [(a + b + c)^{k+1} - (a + b)^{k+1}]P_2 + [(a + b + c + d)^{k+1} - (a + b + c)^{k+1}]P_3$ So, that proof is complete by induction.

Example.

Take $R = Z$, the ring of integers. Let $3 - SP_Z$ be the corresponding symbolic 3-plithogenic ring, hence $X = 1 + 2P_1 + 3P_2 + P_3, X^3 = 1^3 + P_1[(3)^3 - 1^3] + P_2[(6)^3 - 3^3] + (7^3 - 6^3)P_3 = 1 + 26P_1 + 189P_2 + 127P_3$

Definition.

X is called nilpotent if there exists $n \in Z^+$ such that $X^n = 0$.

Theorem.

Let $X \in 3 - SP_R$, where R is a commutative ring, then X is nilpotent if and only if $a, a + b, a + b + c, a + b + c + d$ are nilpotent.

Proof.

$X = a + bP_1 + cP_2 + dP_3$ is nilpotent if and only if there exists $n \in Z^+$ such that $X^n = 0$, hence:

$$\begin{cases} (a+b+c+d)^n - (a+b+c)^n = 0 \\ a^n = 0 \\ (a+b)^n - a^n = 0 \\ (a+b+c)^n - (a+b)^n = 0 \end{cases} \Leftrightarrow \begin{cases} (a+b+c+d)^n = 0 \\ a^n = 0 \\ (a+b)^n = 0 \\ (a+b+c)^n = 0 \end{cases}$$

, thus the proof is complete.

Definition.

Let Q_0, Q_{13}, Q_2, Q_3 be ideals of the ring R, we define the symbolic 3-plithogenic AH-ideal as follows:

$Q = Q_0 + Q_1P_1 + Q_2P_2 + Q_3P_3 = \{x_0 + x_1P_1 + x_2P_2 + x_3P_3;\ x_i \in Q_i\}$.

If $Q_0 = Q_1 = Q_2 = Q_3$, then Q is called an AHS-ideal.

Example.

Let $R = Z$ be the ring of integers, then $Q_0 = 2Z, Q_1 = 3Z, Q_2 = 5Z$ are ideals of R.

$Q = \{2m + 3nP_1 + 5tP_2 + 5sP_3;\ m.n.t, s \in Z\}$ is an AHS-ideal of $3 - SP_Z$.

$M = \{2m + 2nP_1 + 2tP_2 + 2sP_3;\ m.n.t, s \in Z\}$ is an AHS-ideal of $3 - SP_Z$.

Theorem.

Let Q be an AHS- ideal of $3 - SP_R$, then Q is an ideal by the classical meaning.

Proof.

Q can be written as $Q = Q_0 + Q_0P_1 + Q_0P_2 + Q_0P_3$, where Q_0 is an ideal of R. It is clear that $(Q, +)$ is a subgroup of $(3 - SP_R, +)$.

Let $S = s_0 + s_1P_1 + s_2P_2 + s_3P_3 \in 3 - SP_R$, then if $X = a + bP_1 + cP_2 + dP_3 \in Q$, we have:

$S.X = s_0a + (s_0b + s_1a + s_1b)P_1 + (s_0c + s_1c + s_2a + s_2b + s_2c)P_2 + (s_0d + s_1d + s_2d + s_3d + s_3a + s_3b + s_3c)P_3 \in Q$, that is because:

$s_0a \in Q_0, s_0b + s_1a + s_1b \in Q_0, s_0c + s_1c + s_2a + s_2b + s_2c, s_0d + s_1d + s_2d + s_3d + s_3a + s_3b + s_3c \in Q_0$.

Definition.

Let R, T be two rings, $3 - SP_R, 3 - SP_T$ are the corresponding symbolic 3-plithogenic rings, let $f_0, f_1, f_2, f_3: R \to T$ be four homomorphisms, we define the AH-homomorphism as follows:

$f: 3 - SP_R \to 3 - SP_T$ such that:

$f(a + bP_1 + cP_2 + dP_3) = f_0(a) + f_1(b)P_1 + f_2(c)P_2 + f_3(d)P_3$

If $f_0 = f_1 = f_2 = f_3$, then f is called AHS-homomorphism.

Remark.

If f_0, f_1, f_2, f_3 is isomorphisms, then f is called AH-isomorphism.

Example.

Take $R = Z,\ T = Z_6,\ f_0, f_1: R \to T$ such that:

$f_0(x) = x(mod\ 6), f_1(2) = 3x(mod\ 6)$. It is clear that f_0, f_1 are homomorphisms.

We define $f: 3 - SP_R \to 3 - SP_T$, where:

$f(x + yP_1 + zP_2 + sP_3) = f_0(x) + f_1(y)P_1 + f_2(z)P_2 + f_2(s)P_3 = x(mod\ 6) + y(mod\ 6)P_1 + (3z\ mod\ 6)P_2 + (3s\ mod\ 6)P_3$

Which is an AH-homomorphism.

Theorem.

Let $f = f_0 + f_1P_1 + f_2P_2 + f_3P_3: 3 - SP_R \to 3 - SP_T$ be a mapping, then:

1. If f is an AHS-homomorphism, then f is a ring homomorphism by the classical meaning.
2. If f is an AHS-homomorphism, then it is an isomorphism by the classical meaning.

Proof.

1. Assume that f is an AHS-homomorphism, then $f_0 = f_1 = f_2 = f_3$ are homomorphisms.

Let $X = x_0 + x_1P_1 + x_2P_2 + x_3P_3, Y = y_0 + y_1P_1 + y_2P_2 + y_3P_3 \in 3 - SP_R$, we have:

$f(X + Y) = f_0(x_0 + y_0) + f_0(x_1 + y_1)P_1 + f_0(x_2 + y_2)P_2 + f_0(x_3 + y_3)P_3 = f(X) + f(Y)$

$f(X.Y) = f_0(x_0y_0) + f_0(x_0y_1 + x_1y_0 + x_1y_1)P_1 + f_0(x_0y_2 + x_2y_0 + x_2y_2 + x_2y_1 + x_1y_2)P_2 + f_0(x_0y_3 + x_1y_3 + x_2y_3 + x_3y_3 + x_3y_1 + x_3y_0 + x_3y_2)P_3 = f_0(x_0)f_0(y_0) + \big(f_0(x_0)f_0(y_1) + f_0(x_1)f_0(y_0) + f_0(x_1)f_0(y_1)\big)P_1 + \big(f_0(x_0)f_0(y_2) + f_0(x_2)f_0(y_0) + f_0(x_2)f_0(y_2) + f_0(x_2)f_0(y_1) + f_0(x_1)f_0(y_2)\big)P_2 + [\big(f_0(x_0)f_0(y_3) + f_0(x_1)f_0(y_3) + f_0(x_2)f_0(y_3) +$

$f_0(x_3)f_0(y_3) + f_0(x_3)f_0(y_1) + f_0(x_3)f_0(y_2) + f_0(x_3)f_0(y_0))]P_3 = [f_0(x_0) + f_0(x_1)P_1 + f_0(x_2)P_2 + f_0(x_3)P_3][f_0(y_0) + f_0(y_1)P_1 + f_0(y_2)P_2 + f_0(y_3)P_3] = f(X) + f(Y).$

So that, the [roof is complete.

2. By a similar discussion of statement 1, we get the proof.

Definition.

Let $f = f_0 + f_1P_1 + f_2P_2 + f_3P_3: 3 - SP_R \to 3 - SP_T$ be an AH-homomorphism, we define:

1. AH- $ker(f) = ker(f_0) + ker(f_1)P_1 + ker(f_2)P_2 + \ker(f_3)P_3 = \{m_0 + m_1P_1 + m_2P_2 + m_3P_3;\ m_i \in ker(f_i)\}$.
2. AH-factor $3 - SP_R/\text{AH} - ker(f) = R/ker(f_0) + R/ker(f_1)\,P_1 + R/ker(f_2)\,P_2 + + R/ker(f_3)\,P_3$

If $f_0 = f_1 = f_2 = f_3$, then we get an AHS- $ker(f)$ and AHS-factor.

Example.

Take $R = Z_{10},\ f_0: R \to T,\ f_0(x) = (x\ mod\ 10),\ ker(f_0) = 10Z.$

The corresponding AHS-homomorphism is $f = f_0 + f_0P_1 + f_0P_2 + f_0P_3: 3 - SP_R \to 3 - SP_T$, such that:

$$f(x_0 + x_1P_1 + x_2P_2) = f_0(x_0) + f_0(x_1)P_1 + f_0(x_2)P_2 + f_0(x_3)P_3$$
$$= (x_0\ mod\ 10) + (x_1\ mod\ 10)P_1 + (x_2 mod\ 10)P_2 + (x_3 mod\ 10)P_3$$

AHS-$ker(f) = 10Z + 10ZP_1 + 10ZP_2 = \{10x + 10yP_1 + 10zP_2 + 10sP_3;\ x, y, z, s \in Z\}$

AHS-factor= $Z/10Z + Z/10Z\,P_1 + Z/10Z\,P_2 + Z/10Z\,P_3$

Definition.

Let $(F, +, .)$ be a field, then $(3 - SP_F, +, .)$ Is called a symbolic 3-plithogenic field.

$(3 - SP_F, +, .)$ Is not a field in the algebraic meaning, that is because P_i are not invertible, but it is a ring.

Conclusion

In this paper, we have defined the concept of 3-plithogenic rings, and we presented many interesting algebraic properties such as invertibility, nilpotency, and idempotency of their elements.

Also, we have presented many related concepts such as AH-ideals, AH-kernels and homomorphisms with their elementary properties in terms of theorems with many clear examples.

In the future, we look for many symbolic 3-plithogenic structures, especially symbolic 3-pithogenic modules, vector spaces, and matrices.

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On Phi-Euler's Function in Refined Neutrosophic Number Theory and The Solutions of Fermat's Diophantine Equation

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Abstract:

The objective of this paper is to answer the open problem proposed about the validity of phi-Euler's theorem in the refined neutrosophic ring of integers $Z(I_1, I_2)$. This work presents an algorithm to compute the values of Euler's function on refined neutrosophic integers, and it prove that phi-Euler's theorem is still true in $Z(I_1, I_2)$.

On the other hand, we present a solution for another open question about the solutions of Fermat's Diophantine equation in refined neutrosophic ring of integers, where we determine the solutions of Fermat's Diophantine equation $X^n + Y^n = Z^n; n \geq 3$ in $Z(I_1, I_2)$.

Key Words: refined Neutrosophic integer, Neutrosophic Euler's function, Neutrosophic Fermat's equation

1. Introduction

Neutrosophy is a new generalization of fuzzy ideas by considering three membership states (truth, falsity, and indeterminacy) founded by Smarandache in 1995 [1].

In the literature [2], the indeterminacy element I was used to build some interesting extensions of algebraic rings. By adding I (with a logical property $I^2 = I$) to any ring R, we get $R(I)$ the corresponding neutrosophic ring as follows:

$R(I) = \{a + bI; a, b \in R\}$[2].

In [3], Agboola et.al, proposed the structure of refined neutrosophic rings.

As a natural development, neutrosophic number theory was studied in [4,6], where we can find neutrosophic congruencies, Diophantine equations, primes, and neutrosophic Euler's theorem.

In [5], Ibrahim et.al, proposed the basic ideas in refined neutrosophic number theory, where they defined congruencies, Pell's equation, and divisibility in $Z(I_1, I_2)$. On the other hand, an interesting open question has been asked as follows:

Define phi-Euler's function in $Z(I_1, I_2)$? Is Euler's theorem still true ?.

Through this paper, we aim to solve this problem by proving that Euler's theorem is still true in refined neutrosophic number theory.

Also, we find all possible solutions for The non-linear Fermat's Diophantine equation $X^n + Y^n = Z^n; n \geq 3$, which was proposed as an open question in [7].

For more results and findings of neutrosophic number theory and algebraic structures, see [8-15].

For definitions and basic concepts in refined neutrosophic number theory, see [5].

Main discussion :

First of all, we will give an example to explain our idea.

Example :

Let $Z(I_1, I_2)$ be the refined neutrosophic ring of integers, consider $x = (3, I_1, -I_2) \epsilon Z(I_1, I_2)$.

To compute the value of $\varphi(x)$, we have to know the number of refined neutrosophic integers:

$y = (y_0, y_1 I_1, y_2 I_2)$, with the property :

$$\begin{cases} \gcd(x, y) = (1,0,0) \\ 0 < y \leq x \end{cases}$$

According to the definition of (gcd) in refined neutrosophic ring of integers, we get

$\gcd(3, y_0) = 1,\ \gcd(2, y_0 + y_2) = 1,\ \gcd(3, y_0 + y_1 + y_2) = 1$.Also, $y \leq x$ implies that:

$$\begin{cases} 0 \leq y \leq 3 \\ 0 \leq y_0 + y_2 \leq 2 \\ 0 \leq y_0 + y_1 + y_2 \leq 3 \end{cases}$$

The possible values of y_0 are $\{1,2\}$. The possible values of $y_0 + y_2$ are $\{1\}$.The possible values of $y_0 + y_1 + y_2$ are $\{1,2\}$. This implies that we get the following solutions :

$$y = (1,0,0),\ \ y = (1, I_1, 0),\ \ y = (2, 0, I_2),\ \ y = (2, I_1, -I_2)$$

So, $\varphi(x) = 4$ which is equal to $\varphi(3) \times \varphi(2) \times \varphi(3)$. Now, we are able to study the general case.

Definition:

Let $0 < x = (x_0, x_1 I_1, x_2 I_2) \epsilon Z(I_1, I_2)$, we define Euler's function as follows:

$\varphi(x) = |\{y = (y_0, y_1 I_1, y_2 I_2) \epsilon Z(I_1, I_2) : \gcd(x, y) = (1,0,0) and\ 0 < y \leq x\}|$.

Theorem::

Let $x = (x_0, x_1 I_1, x_2 I_2)$ be any positive refined neutrosophic integer, hence: $\varphi(x) = \varphi(x_0) \times \varphi(x_0 + x_2) \times \varphi(x_0 + x_1 + x_2)$.

Proof:

Let $y = (y_0, y_1I_1, y_2I_2)$ be a refined neutrosophic integer with $\begin{cases} 0 \le y \le x \\ \gcd(x,y) = (1,0,0) \end{cases}$

We have, ($y_0 \le x_0$, $y_0 + y_2 \le x_0 + x_2$, $y_0 + y_1 + y_2 \le x_0 + x_1 + x_2$) and ($\gcd(x_0, y_0) = \gcd(x_0 + x_2, y_0 + y_2) = \gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2) = (1,0,0)$. This implies that we have $\varphi(x_0)$ ways to chose y_0, $\varphi(x_0 + x_2)$ ways to chose $y_0 + y_2$ and $\varphi(x_0 + x_1 + x_2)$ ways to chose $y_0 + y_1 + y_2$. By using the essential concept in combinatory, we get $\varphi(x) = \varphi(x_0) \times \varphi(x_0 + x_2) \times \varphi(x_0 + x_1 + x_2)$.

Example:

Let $x = (4,0,2I_2)$, we have :

$\varphi(4) = 2$, $\varphi(4 + 2) = \varphi(6) = 2$, $\varphi(4 + 0 + 2) = \varphi(6) = 2$.

Hence $\varphi(x) = 2 \times 2 \times 2 = 8$.

We shall find the 8 refined neutrosophic integers with the property $\begin{cases} 0 \le y \le x \\ \gcd(x,y) = (1,0,0) \end{cases}$

Let $y = (y_0, y_1I_1, y_2I_2)$, we have

:$\begin{cases} y_0 \le 4, \ \gcd(y_0, 4) = 1 \Longrightarrow y_0 \in \{1,3\} \\ y_0 + y_2 \le 6, \ \gcd(y_0 + y_2, 6) = 1 \Longrightarrow y_0 + y_2 \in \{1,5\} \\ y_0 + y_1 + y_2 \le 6, \ \gcd(y_0 + y_1 + y_2, 6) = 1 \Longrightarrow y_0 + y_1 + y_2 \in \{1,5\} \end{cases}$

The possible solutions are:

1) $y = (1,0,0)$.

2) $y = (1, -4I_1, 4I_2)$.

3) $y = (1,0,4I_2)$.

4) $y = (1,4I_1, 0)$.

5) $y = (3,0,-2I_2)$.

6) $y = (3,4I_1,-2I_2)$.

7) $y = (1,-4I_1,2I_2)$.

8) $y = (1,0,2I_2)$.

The following theorem clarifies how to compute natural powers in $Z(I_1,I_2)$.

Theorem :

Let $x = (x_0,x_1I_1,x_2I_2) \in Z(I_1,I_2)$, let n be any positive integer, hence $x^n = (x_0^n, I_1[(x_0 + x_1 + x_2)^n - (x_0 + x_2)^n], I_2[(x_0 + x_2)^n - x_0^n])$.

Theorem:

Let $Z(I_1,I_2)$ be the refined neutrosophic ring of integers. Let $x = (x_0,x_1I_1,x_2I_2)$, $y = (y_0,y_1I_1,y_2I_2) \in Z(I_1,I_2)$ with $\gcd(x,y)$=1, hence $x^{\varphi(y)} = 1(mod\ y)$.

Proof:

According to the assumption , we have :

$$x^{\varphi(y)} = x^{\varphi(y_0)\times\varphi(y_0+y_2)\times\varphi(y_0+y_1+y_2)} = \left(x_0^{\varphi(y)}, I_1[(x_0 + x_1 + x_2)^{\varphi(y)} - (x_0 + x_2)^{\varphi(y)}], I_2[(x_0 + x_2)^{\varphi(y)} - x_0^{\varphi(y)}]\right).$$

Now, let's compute the following :

$$x_0^{\varphi(y)} = [x_0^{\varphi(y_0)}]^{\varphi(y_0+y_2)\times\varphi(y_0+y_1+y_2)} \equiv 1^{\varphi(y_0+y_2)\times\varphi(y_0+y_1+y_2)}(mod\ y_0) \equiv 1(mod\ y_0)\ .$$

(That is because $\gcd(x_0,y_0) = 1$)

$$(x_0 + x_2)^{\varphi(y)} = [(x_0 + x_2)^{\varphi(y_0+y_2)}]^{\varphi(y_0)\times\varphi(y_0+y_1+y_2)} \equiv 1(mod\ y_0 + y_2).$$

(That is because $\gcd(x_0 + x_2, y_0 + y_2) = 1$)

$$(x_0 + x_1 + x_2)^{\varphi(y)} = [(x_0 + x_1 + x_2)^{\varphi(y_0+y_1+y_2)}]^{\varphi(y_0)\times\varphi(y_0+y_2)} \equiv 1(mod\ y_0 + y_1 + y_2).$$

(That is because $\gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2) = 1$).

We get that:

$$x_0^{\varphi(y)} \equiv 1(mod\ y_0),$$

$$x_0^{\varphi(y)} + \left[(x_0 + x_2)^{\varphi(y)} - x_0^{\varphi(y)}\right] = (x_0 + x_2)^{\varphi(y)} \equiv 1(\text{mod}\, y_0 + y_2),$$

$x_0^{\varphi(y)} + \left[(x_0 + x_2)^{\varphi(y)} - x_0^{\varphi(y)}\right] + [(x_0 + x_1 + x_2)^{\varphi(y)} - (x_0 + x_2)^{\varphi(y)} = (x_0 + x_1 + x_2)^{\varphi(y)} \equiv$

$1(\bmod y_0 + y_1 + y_2)$,

Under the definition of congruencies in refined neutrosophic rings we can write:

$x^{\varphi(y)} \equiv (1,0,0)\ (mod\ y)$.

This implies that Euler's theorem is true in $Z(I_1, I_2)$.

Definition : [7]

Let R be a ring, $F = (X, Y, Z)$ be a triple, where $X, Y, Z \in R$. F is called a general Fermat's triple if and only if $X^n + Y^n = Z^n; for\ all\ integers\ n \geq 3$.
This is equivalent to the condition that (X, Y, Z) is a solution of Fermat's equation.
Theorem :

Let $Z(I_1, I_2)$ be the refined neutrosophic ring of integers. The Equation $X^n + Y^n = Z^n; n \geq 3$ has only 27 solutions.

Proof:

$$X^n + Y^n = Z^n \Leftrightarrow \begin{cases} {x_0}^n + {y_0}^n = {z_0}^n \dots (1) \\ (x_0 + x_2)^n + (y_0 + y_2)^n = (z_0 + z_2)^n \dots (2) \\ (x_0 + x_1 + x_2)^n + (y_0 + y_1 + y_2)^n = (z_0 + z_1 + z_2)^n \dots (3) \end{cases}$$

Now, solutions of (1) is.

$$\begin{cases} x_0 = y_0 = z_0 = 0 \dots (a) \\ x_0 = z_0 = 1, y_0 = 0 \dots (b) \\ y_0 = z_0 = 1, x_0 = 0 \dots (c) \end{cases}$$

solutions of (2) is.

$$\begin{cases} x_0 + x_2 = y_0 + y_2 = z_0 + z_2 = 0 \dots (d) \\ x_0 + x_2 = z_0 + z_2 = 1, y_0 + y_2 = 0 \dots (e) \\ y_0 + y_2 = z_0 + z_2 = 1, x_0 + x_2 = 0 \dots (f) \end{cases}$$

solutions of (3) is.

$$\begin{cases} x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = z_0 + z_1 + z_2 = 0 \dots (g) \\ x_0 + x_1 + x_2 = z_0 + z_1 + z_2 = 1, y_0 + y_1 + y_2 = 0 \dots (h) \\ y_0 + y_1 + y_2 = z_0 + z_1 + z_2 = 1, x_0 + x_1 + x_2 = 0 \dots (i) \end{cases}$$

We discuss possible cases.

Case1. If $(a), (d), (g)$, then $X = Y = Z = (0,0,0)$.

Case2. If $(a), (d), (h)$, then $X = (0,1,0), Z = (0,1,0), Y = (0,0,0)$.

Case3. If $(a), (d), (i)$, then $X = (0,0,0), Z = (0, I_1, 0), Y = (0, I_1, 0)$.

Case4. If $(a), (e), (g)$, then $X = (0, -I_1, I_2), Z = (0, -I_1, I_2), Y = (0,0,0)$.

Case5. If $(a), (e), (g)$, then $X = (0, -I_1, I_2), Z = (0,0, I_2), Y = (0,0,0)$.

Case6. If $(a), (e), (h)$, then $X = (0,0, I_2), Z = (0,0, I_2), Y = (0,0,0)$.

Case7. If $(a), (f), (g)$, then $X = (0,0,0), Z = (0, -I_1, I_2), Y = (0, -I_1, I_2)$.

Case8. If $(a), (f), (h)$, then $X = (0, I_1, 0), Z = (0, -I_1, I_2), Y = (0, -I_1, I_2)$.

Case9. If $(a), (f), (i)$, then $X = (0,0,0), Z = (0,0, I_2), Y = (0,0, I_2)$.

Case10. If $(b), (d), (g)$, then $X = (1,0, -I_2), Z = (1,0, -I_2), Y = (0,0,0)$.

Case11. If $(b), (d), (h)$, then $X = (1, I_1, -I_2), Z = (1, I_1, -I_2), Y = (0,0, I_2)$.

Case12. If $(a), (d), (i)$, then $X = (1,0, -I_2), Z = (1, I_1, I_2), Y = (0, I_1, 0)$.

Case13. If $(b), (e), (g)$, then $X = (1, -I_1, 0), Z = (1, -I_1, 0), Y = (0,0,0)$.

Case14. *If* $(b), (e), (h)$, *then* $X = (1,0,0), Z = (1,0,0), Y = (0, I_1, 0)$.

Case15. If $(b), (e), (i)$, then $X = (1, -I_1, 0), Z = (1,0,0), Y = (0, I_1, 0)$.

Case16. If $(b), (f), (g)$, then $X = (1,0, -I_2), Z = (1, -I_1, 0), Y = (0, -I_1, I_2)$.

Case17. If $(b), (f), (h)$, then $X = (1, I_1, I_2), Z = (1,0,0), Y = (0, -I_1, I_2)$.

Case18. If $(b), (f), (i)$, then $X = (1,0, -I_2), Z = (1,0,0), Y = (0,0, I_2)$.

Case19. If $(c), (d), (h)$, then $X = (0, I_1, 0), Z = (1, I_1, -I_2), Y = (1,0, -I_2)$.

Case20. If $(c), (d), (g)$, then $X = (0,0,0), Z = (1,0, -I_2), Y = (1,0, -I_2)$.

Case21. If $(c), (d), (i)$, then $X = (0,0,0), Z = (1, I_1, -I_2), Y = (1, I_1, -I_2)$.

Case22. If $(c), (e), (g)$, then $X = (0, -I_1, I_2), Z = (1, -I_1, 0), Y = (1,0, -I_2)$.

Case23. If $(a), (e), (h)$, then $X = (0,0, -I_2), Z = (1,0,0), Y = (1,0, -I_2)$.

Case24. If $(c), (e), (i)$, then $X = (0, -I_1, I_2), Z = (1,0,0), Y = (1, I_1, I_2)$.

Case25. If $(c), (f), (g)$, then $X = (0,0,0), Z = (1, -I_1, 0), Y = (1, -I_1, 0)$.

Case26. If $(c), (f), (h)$, then $X = (0, I_1, 0), Z = (1,0,0), Y = (1, -I_1, 0)$.

Case27. If $(c), (f), (i)$, then $X = (0,0,0), Z = (1,0,0), Y = (1,0,0)$.

Conclusion

In this paper, we have defined the Euler's function in the refined neutrosophic ring of integers (I_1, I_2) , as well as, we have presented an algorithm to compute the values of this function.

Also, we have proved that Euler's famous theorem is still true in the case of refined neutrosophic number theory.

In particular, we have determined the possible solutions of Fermat's equation in the refined neutrosophic ring of integers.

As a future research direction, we aim to study the Euler's theorem in n-refined neutrosophic number theory and n-cyclic refined neutrosophic integers, as well as Fermat's equation in these rings.

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On The Algebraic Properties of 2-Cyclic Refined Neutrosophic Matrices and The Diagonalization Problem

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Abstract:

The n-cyclic refined neutrosophic algebraic structures are very diverse and rich materials. In this paper, we study the elementary algebraic properties of 2-cyclic refined neutrosophic square matrices, where we find formulas for computing determinants, eigen values, and inverses. On the other hand, we solve the diagonalization problem of these matrices, where a complete algorithm to diagonlaize every diagonalizable 2-cyclic refined neutrosophic square matrix is obtained and illustrated by many related examples.

Key Words: n-cyclic refined neutrosophic ring, n –cyclic refined neutrosophic matrix, the diagonalization problem.

1.Introduction

Neutrosophic algebraic structures were defined firstly in [1], by adding an algebraic indeterminacy element I to classical algebraic structures to obtain n novel extensions. For example, we can find neutrosophic geometry, neutrosophic functions, neutrosophic rings, and neutrosophic spaces [2-7].

The concept of n-cyclic neutrosophic algebraic structure was supposed in [8], and then it has been studied widely, see [9-12].

As an important algebraic structure, neutrosophic matrices with many types were handled and studied, where we can see many results about inverses, eigen vectors, diagonalizations, and determinants were proven and established [13-24]. In the literature, we have many types of neutrosophic matrices, refined neutrosophic matrices, and n-refined neutrosophic matrices, and n-cyclic refined neutrosophic matrices [17].

The diagonalization algorithm for n-cyclic refined neutrosophic matrix has been asked as an open problem in [12], and it is still open for all values of n.

This motivates us to study the diagonalization problem for n =2, and to present an effective algorithm to diagonlaize a 2-cyclic refined neutrosophic square matrix, as well as many related concepts, especially eigen values computing.

2. Preliminaries

Definition [8]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n}(x_i + y_i) I_i$,$\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n}(x_i \times y_j) I_i I_j = \sum_{i,j=0}^{n}(x_i \times y_j) I_{(i+j \; mod n)}$.

× is the multiplication on the ring R.

In this paper, we study open problem 3, open problem 4, and open problem 5 in [12].

3. Main discussion :

Definition.

Let $M = M_0 + M_1 I_1 + M_2 I_2$ be a 2-cyclic refined neutrosophic matrix, then M is diagonalizable if and only if there exists a 2-cyclic refined neutrosophic diagonal matrix K and invertible matrix U such that $M = UKU^{-1}$.

Theorem.

Let $M=M_0+M_1I_1+M_2I_2$ be a 2-cyclic refined neutrosophic matrix, then M is diagonalizable if and only if: $M_0, M_0+M_1+M_2, M_0-M_1+M_2$ are diagonalizable.

Proof.

Assume that M is diagonalizable, then there exists a diagonal matrix $K=K_0+K_1I_1+K_2I_2$ and an invertible matrix $U=U_0+U_1I_1+U_2I_2$ such that $M=UKU^{-1}$.

The matrix equation $UKU^{-1}=M$ is equivalent to:

$$\begin{aligned}U_0K_0{U_0}^{-1}+\frac{1}{2}I_1[&(U_0+U_1+U_2)(K_0+K_1+K_2)(U_0+U_1+U_2)^{-1}\\ &-(U_0-U_1+U_2)(K_0-K_1+K_2)(U_0-U_1+U_2)^{-1}]\\ &+\frac{1}{2}I_2\big[(U_0+U_1+U_2)(K_0+K_1+K_2)(U_0+U_1+U_2)^{-1}\\ &-(U_0-U_1+U_2)(K_0-K_1+K_2)(U_0-U_1+U_2)^{-1}-2U_0K_0{U_0}^{-1}\big]\\ &=M_0+M_1I_1+M_2I_2\end{aligned}$$

Thus:

$$\begin{cases}U_0K_0{U_0}^{-1}=M_0\\(U_0+U_1+U_2)(K_0+K_1+K_2)(U_0+U_1+U_2)^{-1}=M_0+M_1+M_2\\(U_0-U_1+U_2)(K_0-K_1+K_2)(U_0-U_1+U_2)^{-1}=M_0-M_1+M_2\end{cases}$$

This implies $M_0, M_0+M_1+M_2, M_0-M_1+M_2$ are diagonalizable.

Conversely, assume that $M_0, M_0+M_1+M_2, M_0-M_1+M_2$ are diagonalizable, then there exists diagonal matrices D_0, D_1, D_2 and invertible matrices P_0, P_1, P_2 such that $P_0D_0{P_0}^{-1}=M_0, P_1D_1{P_1}^{-1}=M_0+M_1+M_2,\ P_2D_2{P_2}^{-1}=M_0-M_1+M_2$.

This implies that $M_1=\frac{1}{2}\left(P_1D_1{P_1}^{-1}-P_2D_2{P_2}^{-1}\right), M_2=\frac{1}{2}\left(P_1D_1{P_1}^{-1}+P_2D_2{P_2}^{-1}-2P_0D_0{P_0}^{-1}\right)$

We put $D=D_0+\frac{1}{2}I_1(D_1-D_2)+\frac{1}{2}I_2(D_1+D_2-2D_0)=L_0+\frac{1}{2}I_1L_1+\frac{1}{2}I_2L_2;\begin{cases}L_0=D_0\\L_1=D_1-D_2\\L_2=D_1+D_2-2D_0\end{cases}$.

$$P=P_0+\frac{1}{2}I_1(P_1-P_2)+\frac{1}{2}I_2(P_1+P_2-2P_0)=N_0+\frac{1}{2}I_1N_1+\frac{1}{2}I_2N_2;\begin{cases}N_0=P_0\\N_1=P_1-P_2\\N_2=P_1+P_2-2P_0\end{cases}.$$

We have:

$P^{-1}={N_0}^{-1}+\frac{1}{2}I_1[(N_0+N_1+N_2)^{-1}-(N_0-N_1+N_2)^{-1}]+\frac{1}{2}I_2\big[(N_0+N_1+N_2)^{-1}-(N_0-N_1+N_2)^{-1}-2{N_0}^{-1}\big]={P_0}^{-1}+\frac{1}{2}I_1\big[{P_1}^{-1}-{P_2}^{-1}\big]+\frac{1}{2}I_2\big[{P_1}^{-1}+{P_2}^{-1}-2{P_0}^{-1}\big]$

It is easy to check that $PDP^{-1}=M_0+M_1I_1+M_2I_2=M$, thus M is diagonalizable.

Example.

Consider the following 2×2 2-cyclic refined matrix:

$$X = \begin{pmatrix} 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2 & \frac{1}{2}I_1 + \frac{1}{2}I_2 \\ \frac{1}{2}I_1 - \frac{1}{2}I_2 & 2 - \frac{3}{2}I_1 + \frac{1}{2}I_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix} I_1 + \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} I_2$$

$$= X_0 + X_1 I_1 + X_2 I_2$$

We have:

$$X_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, X_0 + X_1 + X_2 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, X_0 - X_1 + X_2 = \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}$$

X_0 is diagonalizable with $X_0 = {P_0}^{-1} A_0 P_0$, where $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

$X_0 + X_1 + X_2$ is diagonalizable with $X_0 + X_1 + X_2 = {P_1}^{-1} A_1 P_1$, where $P_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$X_0 - X_1 + X_2$ is diagonalizable with $X_0 - X_1 + X_2 = {P_2}^{-1} A_2 P_2$, where $P_2 = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

According the previous theorem, we have.

$X = P^{-1} Y P$, where:

$$Y = A_0 + \frac{1}{2}I_1(A_1 - A_2) + \frac{1}{2}I_2(A_1 + A_2 - 2A_0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-3}{2} \end{pmatrix} I_1 + \begin{pmatrix} \frac{-3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} I_2$$

$$= \begin{pmatrix} 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2 & 0 \\ 0 & 2 - \frac{3}{2}I_1 + \frac{1}{2}I_2 \end{pmatrix}$$

$$P = {P_0}^{-1} + \frac{1}{2}I_1\left[{P_1}^{-1} - {P_2}^{-1}\right] + \frac{1}{2}I_2\left[{P_1}^{-1} + {P_2}^{-1} - 2{P_0}^{-1}\right]$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{-1}{2} & -1 \end{pmatrix} I_1 + \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} I_2 = \begin{pmatrix} 1 - I_1 + I_2 & \frac{1}{2}I_1 + \frac{1}{2}I_2 \\ -\frac{1}{2}I_1 + \frac{1}{2}I_2 & 1 - I_1 - I_2 \end{pmatrix}$$

The Eigen Values.

Definition.

Let $A = A_0 + A_1 I_1 + A_2 I_2$ be an n-cyclic refined neutrosophic matrix, we say that $T = t_0 + t_1 I_1 + t_2 I_2 \in R_2(I)$ is an eigen value if and only if $AX = tX$; $X = X_0 + X_1 I_1 + x_2 I_2$ is an n-cyclic refined neutrosophic vector, where $X_i \in R^n$.

X is called n-cyclic refined neutrosophic vector.

Theorem.

Let $A = A_0 + A_1 I_1 + A_2 I_2$ be an n-cyclic refined neutrosophic matrix, then $T = t_0 + t_1 I_1 + t_2 I_2 \in R_2(I)$ is an eigen value with $X = X_0 + X_1 I_1 + X_2 I_2$ as eigen vector if and only if:

t_0 is an eigen value of A_0 with X_0 as eigen vector, $t_0 + t_1 + t_2$ is an eigen value of $A_0 + A_1 + A_2$ with $X_0 + X_1 + X_2$ as eigen vector, $t_0 - t_1 + t_2$ is an eigen value of $A_0 - A_1 + A_2$ with $X_0 - X_1 + X_2$ as eigen vector.

Proof.

The equation $AX = tX$ is equivalent to:

$$\begin{aligned} A_0X_0 &+ \frac{1}{2} I_1[(A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A_2)(X_0 - X_1 + X_2)] \\ &+ \frac{1}{2} I_2[(A_0 + A_1 + A_2)(X_0 + X_1 + X_2) + (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) \\ &- 2A_0X_0] \\ &= t_0X_0 + \frac{1}{2} I_1[(t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2)] \\ &+ \frac{1}{2} I_2[(t_0 + t_1 + t_2)(X_0 + X_1 + X_2) + (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) - 2t_0X_0] \end{aligned}$$

So that:

$$\begin{cases} t_0X_0 = A_0X_0 \dots (1) \\ (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) = (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A \\ (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) = (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A \end{cases}$$

This equivalents:

$$\begin{cases} A_0X_0 = t_0X_0 \\ (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) = (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) \\ (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) = (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) \end{cases}$$

Thus, the proof is complete.

Example.

Consider the matrix:

$$\begin{aligned} A &= \begin{pmatrix} 3 + \frac{1}{2} I_1 - \frac{3}{2} I_2 & \frac{1}{2} I_1 + \frac{1}{2} I_2 \\ \frac{1}{2} I_1 - \frac{1}{2} I_2 & 2 - \frac{3}{2} I_1 + \frac{1}{2} I_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix} I_1 + \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} I_2 \\ &= A_0 + A_1 I_1 + A_2 I_2 \end{aligned}$$

The eigen values of A_0 are $\{3,2\}$.

The eigen values of $A_0 + A_1 + A_2$ are $\{2,1\}$.

The eigen values of $A_0 - A_1 + A_2$ are $\{1,4\}$.

To find the corresponding 2×2 2-cyclic refined neutrosophic matrix A, we discuss the following cases:

Case(1). If $t_0 = 3,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = \frac{1}{2}\ ,\ t_2 = \frac{-3}{2}$, thus $T_1 = 3 + \frac{1}{2} I_1 - \frac{3}{2} I_2$.

Case(2). If $t_0 = 3,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = -1\ ,\ t_2 = 0$, thus $T_2 = 3 - I_1$.

Case(3). If $t_0 = 3,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = 0\ ,\ t_2 = -2$, thus $T_3 = 3 - 2I_2$.

Case(4). If $t_0 = 3,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = \frac{-3}{2}\ ,\ t_2 = \frac{-1}{2}$, thus $T_4 = 3 - \frac{3}{2} I_1 - \frac{1}{2} I_2$.

Case(5). If $t_0 = 2,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = \frac{1}{2}\ ,\ t_2 = \frac{-1}{4}$, thus $T_5 = 3 + \frac{1}{2} I_1 - \frac{1}{4} I_2$.

Case(6). If $t_0 = 2,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = -1\ ,\ t_2 = 1$, thus $T_6 = 3 - I_1 + I_2$.

Case(7). If $t_0 = 2,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = 0\ ,\ t_2 = -1$, thus $T_7 = 3 - I_2$.

Case(8). If $t_0 = 2,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = \frac{-3}{2}\ ,\ t_2 = \frac{1}{2}$, thus $T_8 = 3 - \frac{3}{2} I_1 + \frac{1}{2} I_2$.

This implies that A has 8 eigen values.

The determinant of an n-cyclic refined neutrosophic matrix.

According to the previous discussion, we have found an algorithm to compute n-cyclic refined neutrosophic matrix.

From the point of view, we are forced to study the computing of eigen values by determinants.

Definition.

Let $A = A_0 + A_1 I_1 + A_2 I_2$ be an n-cyclic refined neutrosophic matrix, we define its determinant as follows:

$\det A = \det A_0 + \frac{1}{2} I_1 [\det(A_0 + A_1 + A_2) - \det(A_0 - A_1 + A_2)] + \frac{1}{2} I_2 [\det(A_0 + A_1 + A_2) + \det(A_0 - A_1 + A_2) - 2 \det A_0]$.

Theorem.

Let $A=A_0+A_1I_1+A_2I_2, B=B_0+B_1I_1+B_2I_2$ be two $n\times n$ n-cyclic refined neutrosophic matrices, then:

1). A is invertible if and only if $\det A$ is invertible.

2). $\det A^T=\det A$.

3). $\det(A.B)=\det A.\det B$.

4). $T=t_0+t_1I_1+t_2I_2$ is an eigen of A if and only if $\det(A-TU_{n\times n})=0$.

Proof.

1). It is clear and easy.

2). $A^T={A_0}^T+{A_1}^TI_1+{A_2}^TI_2$, thus:

$$\det A^T=\det {A_0}^T+\frac{1}{2}I_1[\det(A_0+A_1+A_2)^T-\det(A_0-A_1+A_2)^T]$$
$$+\frac{1}{2}I_2\left[\det(A_0+A_1+A_2)^T=\det(A_0-A_1+A_2)^T-2\det {A_0}^T\right]=\det A$$

3). $A.B=A_0B_0+\frac{1}{2}I_1[(A_0+A_1+A_2)(B_0+B_1+B_2)-(A_0-A_1+A_2)(B_0-B_1+B_2)]+\frac{1}{2}I_2[(A_0+A_1+A_2)(B_0+B_1+B_2)+(A_0-A_1+A_2)(B_0-B_1+B_2)-2A_0B_0]=A_0B_0+\frac{1}{2}I_1(T_1-T_2)+\frac{1}{2}I_2(T_1+T_2-2A_0B_0)$, where:

$T_1=(A_0+A_1+A_2)(B_0+B_1+B_2), T_2=(A_0-A_1+A_2)(B_0-B_1+B_2)$

$$\det(A.B)=\det A_0B_0$$
$$+\frac{1}{2}I_1\left[\det\left(\frac{1}{2}T_1-\frac{1}{2}T_2+\frac{1}{2}T_1+\frac{1}{2}T_2-A_0B_0+A_0B_0\right)\right.$$
$$\left.-\det\left(A_0B_0-\frac{1}{2}T_1+\frac{1}{2}T_2+\frac{1}{2}T_1+\frac{1}{2}T_2-A_0B_0\right)\right]$$
$$+\frac{1}{2}I_2\left[\det\left(\frac{1}{2}T_1-\frac{1}{2}T_2+\frac{1}{2}T_1+\frac{1}{2}T_2-A_0B_0+A_0B_0\right)\right.$$
$$\left.-\det\left(A_0B_0-\frac{1}{2}T_1+\frac{1}{2}T_2+\frac{1}{2}T_1+\frac{1}{2}T_2-A_0B_0\right)-2\det A_0B_0\right]$$
$$=\det A_0\det B_0+\frac{1}{2}I_1[\det T_1-\det T_2]+\frac{1}{2}I_2[\det T_1+\det T_2-2\det A_0\det B_0]$$
$$=\det A_0\det B_0$$
$$+\frac{1}{2}I_1[\det(A_0+A_1+A_2).\det(B_0+B_1+B_2)$$
$$+\det(A_0-A_1+A_2).\det(B_0-B_1+B_2)]$$
$$+\frac{1}{2}I_2[\det(A_0+A_1+A_2).\det(B_0+B_1+B_2)$$
$$-\det(A_0-A_1+A_2).\det(B_0-B_1+B_2)-2\det A_0\det B_0]=\det A\det B$$

4). We have $A - TU_{n\times n} = (A_0 + A_1 I_1 + A_2 I_2) - (t_0 + t_1 I_1 + t_2 I_2)U_{n\times n} = (A_0 - t_0 U_{n\times n}) + (A_1 - t_1 U_{n\times n})I_1 + (A_2 - t_2 U_{n\times n})I_2$.

$$\begin{aligned}\det(A - TU_{n\times n})\\ &= \det(A_0 - t_0 U_{n\times n})\\ &+ \frac{1}{2}I_1[\det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n\times n})\\ &- \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n\times n})]\\ &+ \frac{1}{2}I_2[\det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n\times n})\\ &+ \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n\times n}) - 2\det(A_0 - t_0 U_{n\times n})]\end{aligned}$$

The equation $\det(A - TU_{n\times n}) = 0$ is equivalent to:

$$\begin{cases} \det(A_0 - t_0 U_{n\times n}) = 0 \\ \det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n\times n}) = 0 \\ \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n\times n}) = 0 \end{cases}$$

This is equivalent to:

To is eigen value of A_0, $t_0 + t_1 + t_2$ is eigen value of $A_0 + A_1 + A_2$, $t_0 - t_1 + t_2$ is eigen value of $A_0 - A_1 + A_2$, thus T is an eigen value of A.

Theorem.

Let $A = A_0 + A_1 I_1 + A_2 I_2, B = B_0 + B_1 I_1 + B_2 I_2$ be two $n \times n$ n-cyclic refined neutrosophic matrices, then:

$$\begin{aligned} A.B = A_0 B_0 &+ \frac{1}{2}I_1[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) - (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)]\\ &+ \frac{1}{2}I_2[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) + (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)\\ &- 2A_0 B_0]\end{aligned}$$

The proof is easy and clear.

Example.

Consider the following 2×2 2-cyclic refined neutrosophic matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} I_2 = \begin{pmatrix} 1 + 2I_1 + I_2 & 1 + I_1 \\ I_1 + 3I_2 & 2 + I_1 + I_2 \end{pmatrix} = A_0 + A_1 I_1 + A_2 I_2$$

$A_0^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$, $(A_0 + A_1 + A_2) = \begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}$, $(A_0 + A_1 + A_2)^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -3 \\ -4 & 4 \end{pmatrix}$, $(A_0 - A_1 + A_2) = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$,$(A_0 - A_1 + A_2)^{-1} = -\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix}$

$$A^{-1}=\frac{1}{2}\begin{pmatrix}2 & -2\\0 & 1\end{pmatrix}+\frac{1}{2}I_1\left[\frac{1}{4}\begin{pmatrix}4 & -3\\-4 & 4\end{pmatrix}-\frac{-1}{2}\begin{pmatrix}2 & -1\\-2 & 0\end{pmatrix}\right]$$

$$+\frac{1}{2}I_2\left[\frac{1}{4}\begin{pmatrix}4 & -3\\-4 & 4\end{pmatrix}+\frac{-1}{2}\begin{pmatrix}2 & -1\\-2 & 0\end{pmatrix}-2\left(\frac{1}{2}\right)\begin{pmatrix}2 & -2\\0 & 1\end{pmatrix}\right]$$

$$=\begin{pmatrix}1 & -1\\0 & \frac{1}{2}\end{pmatrix}+\frac{1}{2}I_1\left[\begin{pmatrix}1 & -\frac{3}{4}\\-1 & 1\end{pmatrix}+\begin{pmatrix}1 & -\frac{1}{2}\\-1 & 0\end{pmatrix}\right]$$

$$+\frac{1}{2}I_2\left[\begin{pmatrix}1 & -\frac{3}{4}\\-1 & 1\end{pmatrix}+\begin{pmatrix}-1 & \frac{1}{2}\\1 & 0\end{pmatrix}-\begin{pmatrix}2 & -2\\0 & 1\end{pmatrix}\right]$$

$$=\begin{pmatrix}1 & -1\\0 & \frac{1}{2}\end{pmatrix}+\frac{1}{2}I_1\begin{pmatrix}2 & -\frac{5}{4}\\-2 & 1\end{pmatrix}+\frac{1}{2}I_2\begin{pmatrix}-2 & \frac{7}{4}\\0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}1+I_1-I_2 & -1-\frac{5}{8}I_1+\frac{7}{8}I_2\\-I_1 & \frac{1}{2}+\frac{1}{2}I_1\end{pmatrix}$$

Theorem.

Let $X=X_0+X_1I_1+X_2I_2$ be a 2-cyclic refined neutrosophic matrix, then X is invertible if and only if $X_0, X_0+X_1+X_2, X_0-X_1+X_2$ are invertible, also:

$$X^{-1}={X_0}^{-1}+\frac{1}{2}I_1[(X_0+X_1+X_2)^{-1}-(X_0-X_1+X_2)^{-1}]$$
$$+\frac{1}{2}I_2[(X_0+X_1+X_2)^{-1}+(X_0-X_1+X_2)^{-1}-2X_0]$$

Proof.

Assume that X is invertible, the exists $Y=Y_0+Y_1I_1+Y_2I_2$ such that $X.Y=U_{n\times n}$.

$$X.Y=X_0Y_0+I_1[X_0Y_1+X_1Y_0+X_2Y_1+X_1Y_2]+I_2[X_0Y_2+X_2Y_0+X_1Y_1+X_2Y_2]$$
$$=X_0Y_0+\frac{1}{2}I_1[(X_0+X_1+X_2)(Y_0+Y_1+Y_2)-(X_0-X_1+X_2)(Y_0-Y_1+Y_2)]$$
$$+\frac{1}{2}I_2[(X_0+X_1+X_2)(Y_0+Y_1+Y_2)+(X_0-X_1+X_2)(Y_0-Y_1+Y_2)-2X_0Y_0]$$
$$=U_{n\times n}$$

This implies that:

$$\begin{cases}X_0Y_0=U_{n\times n}\\(X_0+X_1+X_2)(Y_0+Y_1+Y_2)=(X_0-X_1+X_2)(Y_0-Y_1+Y_2)=U_{n\times n}\end{cases}$$

Hence $X_0, X_0+X_1+X_2, X_0-X_1+X_2$ are invertible.

On the other hand, we get $Y_0={X_0}^{-1}$, $Y_0-Y_1+Y_2=(X_0-X_1+X_2)^{-1}$, $Y_0+Y_1+Y_2=(X_0+X_1+X_2)^{-1}$, thus:

$$Y_1=\frac{1}{2}[(X_0+X_1+X_2)^{-1}-(X_0-X_1+X_2)^{-1}]$$

$Y_2 = \frac{1}{2}\left[(X_0 + X_1 + X_2)^{-1} + (X_0 - X_1 + X_2)^{-1} - 2{X_0}^{-1}\right].$

Conclusion

In this paper, we have presented a full solution of the diagonalization problem of 2-cyclic refined neutrosophic matrices, where we have presented a novel algorithm to compute the eigen values and vectors of 2-cyclic refined neutrosophic matrices that helps in representing them as a product $A^{-1}DA$, where A is an invertible matrix, and D is diagonal matrix.

In the future, we suggest researchers to continue our efforts, and to study the possibility of diagonalization problem of 3-cyclic refined neutrosophic matrices.

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On The Classification of The Group of Units of Rational and Real 2-Cyclic Refined Neutrosophic Rings

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Abstract: The objective of this paper is to solve two open problems about the group of units of some 2-cyclic refined neutrosophic rings asked by Sadiq. Where it provides a classification theorem for these rings, and uses this classification property to give a full answer of these open questions.

Also, this work presents a novel algorithm to find all imperfect neutrosophic duplets and triplets in many numerical 2-cyclic refined neutrosophic rings by using the classification isomorphisms.

1. Introduction

Neutrosophic logic as a new generalization of fuzzy logic concerns with indeterminacy in science and real life problems [1]. Neutrosophy was proposed by Smarandache [6] for these logical purposes.

Laterally, neutrosophy was applied to algebra and algebraic structures, were we find many algebraic structures defined by using an indeterminacy element (I) such as neutrosophic rings, neutrosophic spaces, neutrosophic modules, and matrices [2-5].

The concept of n-cyclic refined neutrosophic ring was presented firstly in [7], and studied widely in [8-9].

In [10], Sadiq has studied the group of units problem for 2-CRNR rings, where he proved that it is isomorphic to 3 times direct product of Z_2 . Also, he presented the following open research problems: [10]:

Open problem 1: If the ring R with no zero divisors, then is the group of units of $R_2(I)$ is isomorphic to $U(R) \times U(R) \times U(R)$.

Open problem 2: Find a homomorphism between $R_2(I)$ and the direct product $\times R \times R$.

Open problem 3: Is the group of units of the 2-cyclic refined ring of real numbers isomorphic to $R^* \times R^* \times R^*$.

This motivates us to continuo these efforts to classify the group of units of 2-cyclic refined rings, and to prove the validity of Sadiq's open problems.

On the other hand, we classify all imperfect duplets and triplets in the ring of 2-cyclic refined neutrosophic integers by solving many related Diophantine equations.

We denote the 2-cyclic refined ring by 2-CRNR.

2. Preliminaries

Definition 1.2:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$$\sum_{i=0}^{n} p_i I_i + \sum_{i=0}^{n} q_i I_i$$

$$= \sum_{i=0}^{n} (p_i + q_i) I_i \, , \sum_{i=0}^{n} p_i I_i$$

$$\times \sum_{i=0}^{n} q_i I_i = \sum_{i,j=0}^{n} (p_i \times q_j) I_i I_j = \sum_{i,j=0}^{n} (p_i \times q_j) I_{(i+j \, mod n)}$$

Example 2.2:

(a) The 2-CRNR of integers is defined as follows:

$Z_2(I) = \{t_0 + t_1 I_1 + t_2 I_2;\ t_i \in Z\}$.

(b) Addition on $Z_2(I)$:

$(a + bI_1 + cI_2) + \ (m + nI_1 + tI_2) = (a + m) + I_1(b + n) + I_2(c + t)$.

(c) Multiplication on $Z_2(I)$:

$(a + bI_1 + cI_2)(\, m + nI_1 + tI_2)$= $am + anI_1 + atI_2 + bmI_1 + bnI_2 + btI_1 + cmI_2 + cnI_1 + ctI_2$

$= am + I_1(an + bm + bt + cn) + I_2(at + bn + cm + ct)$.

Where $I_1 I_1 = I_{(1+1 \, mod 2)} = I_2, I_2 I_2 = I_{(2+2 \, mod \, 2)} = I_2, I_1 I_2 = I_{(1+2 \, mod \, 2)} = I_1$.

Definition 3.2:

Let R be a ring, a duplet (x, y) is called an imperfect duplet with x acts as an identity if and only if $xy = yx = y$.

A triple (x, y, z) is called an imperfect triplet with x acts as an identity if and only if $xy = yx = y, xz = zx = z, zy = yz = x$.

3. Main discussion

Theorem 1.3 : Let Z be the ring of integers, and $S = \{(b_0, b_1, b_2);\ b_i \in Z \ and \ b_1 - b_2 \in 2Z\}$, then $(S, +, .)$ Is a subring of $Z \times Z \times Z$.

Proof: It is clear that $S \neq \emptyset$

$\forall \, x, y \in S$, $x = (a_0, a_1, a_2), y = (b_0, b_1, b_2)$, where $b_1 - b_2, a_1 - a_2 \in 2Z$

$x + y = (a_0 + b_0, a_1 + b_1, a_2 + b_2)$, $xy = (a_0 b_0, a_1 b_1, a_2 b_2)$

We have: $(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2) \in 2Z$, thus $x + y \in S$

Also, $a_1 b_1 - a_2 b_2 = a_1 b_1 + a_1 b_2 - a_1 b_2 - a_2 b_2 = a_1(b_1 + b_2) - b_2(a_1 + a_2)$. By the assumption, we have $b_1 - b_2, a_1 - a_2 \in 2Z$, hence

$b_1 + b_2, a_1 + a_2 \in 2Z$, this implies $a_1(b_1 + b_2) - b_2(a_1 + a_2) \in 2Z$ and $x.y \in S$.

Theorem 2.3: Let $Z_2(I)$ be the 2-CRNR of integers, then $Z_2(I) \cong S$.

Proof:

Define $f: Z_2(I) \rightarrow S; f(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

It's clear that f is well defined. On the other hand we have:

(a). f is injective, $\ker f = \{a_0 + a_1 I_1 + a_2 I_2 \in Z_2(I)\ ;\ f(a_0 + a_1 I_1 + a_2 I_2) = (0,0,0)\}$, hence.

$a_0 = 0,\ a_0 + a_1 + a_2 = 0$, $a_0 - a_1 + a_2 = 0,$ thus $a_0 = a_1 = a_2,$ this means that $\ker f = \{0_S\}$.

(b). f is surjective, $\forall \, y = (a_0, a_1, a_2) \in S$, we have: $a_1 - a_2 \in 2Z$, hence $x = a_0 + I_1\left(\frac{a_1 - a_2}{2}\right) + I_2\left(\frac{a_1 + a_2 - 2a_0}{2}\right) \in Z_2(I)$.

This is because $a_1 - a_2,\ a_1 + a_2 - 2a_0 \in 2Z$.

Now, we compute $f(x) = \left(a_0, a_0 + \frac{a_1 - a_2}{2} + \frac{a_1 + a_2 - 2a_0}{2}, a_0 - \frac{a_1 - a_2}{2} + \frac{a_1 + a_2 - 2a_0}{2}\right) = (a_0, a_1, a_2) = y$

(c). $f$is a homomorphism because clearly f preserves addition and multiplication, thus $S \cong Z_2(I)$.

Theorem 3.3: Let R be a of real numbers, $R_2(I)$ be the corresponding 22-CRNR of real numbers, then $R_2(I) \cong R^3$.

Proof. Define $f: R_2(I) \rightarrow R^3; f(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

f is well defined and bijective. (the proof is exactly similar to the previous theorem).

f is a homomorphism. $\forall\, x, y \in R_2(I)\,, x = a_0 + a_1 I_1 + a_2 I_2, y = b_0 + b_1 I_1 + b_2 I_2$.

$x + y = a_0 + b_0 + (a_1 + b_1)I_1 + (a_2 + b_2)I_2$.

$f(x + y) = (a_0 + b_0, a_0 + b_0 + a_1 + b_1 + a_2 + b_2, a_0 + b_0 - (a_1 + b_1) + a_2 + b_2)$

$f(x + y) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2) + (b_0, b_0 + b_1 + b_2, b_0 - b_1 + b_2) = f(x) + f(y)$.

$xy = a_0 b_0 + (a_1 b_0 + a_0 b_1 + a_2 b_1 + a_1 b_2)I_1 + (a_1 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_2)I_2$

$$f(xy) = (a_0 b_0, a_0 b_0 + a_1 b_0 + a_0 b_1 + a_1 b_1 + a_2 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_2, a_0 b_0 - (a_1 b_0 + a_0 b_1 + a_2 b_1 + + a_1 b_2) + a_2 b_0 + a_0 b_2 + a_1 b_1 + a_2 b_2)$$

$f(xy) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2).(b_0, b_0 + b_1 + b_2, b_0 - b_1 + b_2) = f(x).f(y)$, hence $R_2(I) \cong R^3$.

Answers to the open questions

The following theorem answers the open question 3.

Theorem 4.3: Let $\cup\left(R_2(I)\right)$ be the group of units of the 2-CRNR$R_2(I)$, then $\cup\left(R_2(I)\right) \cong R^{*3}$.

Proof.

According to the previous theorem, $R_2(I) \cong R \times R \times R$, hence. $\cup\left(R_2(I)\right) \cong \cup(R) \times \cup(R) \times \cup(R) = R^{*3}$.

The following remark answers the open question 2.

Remark 5.3: If R is a ring, and $R_2(I)$ is the corresponding 2-CRNR, hence the map

$f: R_2(I) \rightarrow R \times R \times R; f(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$, Is a ring homomorphism (the proof is similar to Theorem 3.2). Thus the answer to the open question 2 is yes. Remark that f is not supposed to be an isomorphism, check Theorem 1.3 for example.

The first question is still open, but we can solve the problem in a special case for the ring of integers modulo n, with odd n.

Theorem 6.3: Let R be the ring of integers modulo n, with an odd integer n, then $R_2(I) \cong Z_n \times Z_n \times Z_n$

Proof. . Define $f: R_2(I) \rightarrow Z_n \times Z_n \times Z_n; f(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

It's clear that f is a well defined homomorphism, by a similar argument of the previous theorem, we should prove that f is a bijective map.

$\ker f = \{a_0 + a_1 I_1 + a_2 I_2 \in R_2(I) \,;\, f(a_0 + a_1 I_1 + a_2 I_2) = (0,0,0)\}$, hence.

$$a_0 = 0 \ldots (1)$$

$a_0 + a_1 + a_2 = 0 \ldots \ (2)$

$a_0 - a_1 + a_2 = 0 \ \ldots \ (3)$

From equation (2) and (3), we get $2a_2 = 0$, By the proposition of the theorem, n is odd, this means that gcd(2,n)=1 and 2 cannot be a zero divisor, thus $2a_2 = 0 \Rightarrow a_2 = 0$.

This implies that $a_1 = 0$, and $\ker f = \{(0,0,0)\}$.

f is surjective:

$\forall y = (a_0, a_1, a_2) \in Z_n \times Z_n \times Z_n$, we have $x = a_0 + I_1\big((a_1 + a_2)2^{-1}\big) + I_2\big((a_1 + a_2 - 2a_0)2^{-1}\big) \in R_2(I)$.

That is because 2 is a unit in Z_n and 2^{-1} is existed.

Now, we compute

$$f(x) = (a_0, (a_1 - a_2)2^{-1} + (a_1 + a_2 + 2a_0)2^{-1} + a_0, a_0 + (a_1 - a_2)2^{-1} + (a_1 + a_2 + 2a_0)2^{-1})$$

$$= (a_0, a_1 2^{-1} - a_2 2^{-1} + a_1 2^{-1} + a_2 2^{-1} - 2a_0 2^{-1} + a_0, a_0 - a_1 2^{-1} + a_2 2^{-1} + a_1 2^{-1} + a_2 2^{-1} - 2a_0 2^{-1})$$

$$= (a_0, 2a_1 2^{-1}, 2a_2 2^{-1}) = (a_0, a_1, a_2).$$

So that, $R_2(I) \cong Z_n \times Z_n \times Z_n$.

Theorem 7.3: If $R = Z_n$ the ring of integers modulo n with an odd integer n, we have:

$\cup \left(R_2(I)\right) \cong \cup (Z_n) \times \cup (Z_n) \times \cup (Z_n)$.

The proof holds directly by the previous result.

Theorem 8.3: If $R = Z$ the ring of integers, $Z_2(I)$ be the corresponding 2-CRNR, then $Z_2(I)$ has exactly 8 forms of imperfect duplets.

Proof. We have $Z_2(I) \cong S$; $S = \{(a_0, a_1, a_2);\ a_i \in Z \ and \ a_1 - a_2 \in 2Z\}$.

To find imperfect duplets in $Z_2(I)$, it is sufficient to compute duplets in S:

Let $x = (a_0, a_1, a_2), y = (b_0, b_1, b_2)$ be an imperfect duplet in S, with y acts as an identity, we have.

$$x.y = x \Longrightarrow \begin{cases} a_0 b_0 = a_0 \\ a_1 b_1 = a_1 \\ a_2 b_2 = a_2 \end{cases} \Longrightarrow \begin{cases} a_0 = 0 \ or \ b_0 = 0 \\ a_1 = 0 \ or \ b_1 = 0. \\ a_2 = 0 \ or \ b_2 = 0 \end{cases}$$

The possible imperfect duplets are:

(1).$x = (0, 0,0), y = (b_0, b_1, b_2)$

(With $b_1 - b_2 \in 2Z$)

(2). $x = (0, a_1, a_2), y = (b_0, 1,1)$

(With $a_1 - a_2 \in 2Z$)

(3). $x = (0, 0, a_2), y = (b_0, b_1, 1)$

(With $a_2 \ is \ even \ and \ b_1 \ is \ odd$)

(4). $x = (a_0, 0, a_2), y = (1, b_1, 1)$

(With $a_2 \ is \ even \ and \ b_1 \ is \ odd$)

(5). $x = (a_0, a_1, 0), y = (1, 1, b_2)$

(With $a_1 \ is \ even \ and \ b_2 \ is \ odd$)

(6). $x = (a_0, a_1, a_2), y = (1, 1,1)$

(With $a_1 - a_2 \in 2Z$)

(7). $x = (a_0, 0,0), y = (1, b_1, b_2)$

(With $b_1 - b_2 \in 2Z$)

(8). $x = (0, a_1, 0), y = (b_0, 1, b_2)$

(With a_1 $is\ even\ and\ b_2\ is\ odd$)

Thus, the imperfect duplets in $Z_2(I)$ are the converse image of the duplets in S, according to the isomorphism

$f(a_0 + a_1I_1 + a_2I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

$f^{-1}(a_0 + a_1I_1 + a_2I_2) = \left(a_0 + \frac{a_1-a_2}{2}I_1 + \frac{a_1+a_2-2a_0}{2}I_2\right)$, so that the duplets of $Z_2(I)$are:

(1). $x = 0, y = b_0 + b_1I_1 + b_2I_2$

(With $b_1 - b_2 \in 2Z$)

(2). $x = \frac{a_1-a_2}{2}I_1 + \frac{a_1+a_2}{2}I_2, y = b_0 + \frac{2-2b_0}{2}I_2$

(With $a_1 - a_2 \in 2Z$)

(3). $x = \frac{-a_2}{2}I_1 + \frac{a_2}{2}I_2, y = b_0 + \frac{b_1-1}{2}I_1 + \frac{b_1+1-2b_0}{2}I_2$

(With a_2 $is\ even\ and\ b_1\ is\ odd$)

(4). $x = a_0 + \frac{-a_2}{2}I_1 + \frac{a_2-2a_0}{2}I_2, y = 1 + \frac{b_1-1}{2}I_1 + \frac{b_1+1-2(1)}{2}I_2$

(With a_2 $is\ even\ and\ b_1\ is\ odd$)

(5). $x = a_0 + \frac{a_1}{2}I_1 + \frac{a_1-2a_0}{2}I_2, y = 1 + \frac{1-b_2}{2}I_1 + \frac{1+b_2-2(1)}{2}I_2$

(With a_1 $is\ even\ and\ b_2\ is\ odd$)

(6). $x = a_0 + a_1I_1 + a_2I_2, y = 1$

(With $a_1 - a_2 \in 2Z$)

(7). $x = a_0 - a_0I_2, y = 1 + \frac{b_1-b_2}{2}I_1 + \frac{b_1+b_2-2}{2}I_2$

(With $b_1 - b_2 \in 2Z$)

(8). $x = \frac{a_1}{2}I_1 + \frac{a_1}{2}I_2, y = b_0 + \frac{1-b_2}{2}I_1 + \frac{1++b_2-2b_0}{2}I_2$

(With a_1 $is\ even\ and\ b_2\ is\ odd$)

Theorem 9.3: Let R be the ring of real numbers, $R_2(I)$ be its 2-CRNR, then $R_2(I)$ has exactly 8 forms of imperfect duplets.

Proof. We have $R_2(I) \cong R \times R \times R$ with the isomorphism:

$f: R_2(I) \to R \times R \times R; f(a_0 + a_1I_1 + a_2I_2) = (a_0, a_0 + a_1 + a_2, a_0 - a_1 + a_2)$.

For determining the imperfect duplets in $R_2(I)$, it is sufficient to find duplets in $R \times R \times R$ and go back to $R_2(I)$ by the inverse isomorphism.

The imperfect duplets in $R \times R \times R$ are:

(1). $x = (0,0,0), y = (b_0, b_1, b_2)$

(2). $x = (0, a_1, a_2), y = (b_0, 1,1)$

(3). $x = (0, 0, a_2), y = (b_0, b_1, 1)$

(4). $x = (a_0, 0, a_2), y = (1, b_1, 1)$

(5). $x = (a_0, a_1, 0), y = (1, 1, b_2)$

(6). $x = (a_0, a_1, a_2), y = (1, 1,1)$

(7). $x = (a_0, 0,0), y = (1, b_1, b_2)$

(8). $x = (0, a_1, 0), y = (b_0, 1, b_2)$

Thus$R_2(I)$ has 8 forms of imperfect duplets.

Remark 10.3: To find any imperfect duplets in$R_2(I)$, we should compute the inverse image of the corresponding duplet in $R \times R \times R$as follows:

$$f^{-1}(a_0, a_1, a_2) = a_0 + \frac{a_1 - a_2}{2} I_1 + \frac{a_1 + a_2 - 2a_0}{2} I_2$$

Example 11.3: Let's, take a duplet with form:$x = (2,0,3), y = (1,5,1)$, it is clear $x.y = x$.

The corresponding duplet in $R_2(I)$ is:

$x_1 = f^{-1}(x) = 2 + \frac{-3}{2} I_1 + \frac{-1}{2} I_2, y_1 = f^{-1}(y) = 1 + 2I_1 + 2I_2$.

Remark 12.3: that $x_1.y_1 = 2 + 4I_1 + 4I_2 = \frac{-3}{2} I_1 - 3I_2 - 3I_1 - \frac{1}{2} I_2 - I_1 - I_2 = 2 - \frac{3}{2} I_1 - \frac{1}{2} I_2 = x_1$.

Theorem 13.3: Let $Z_2(I)$ be the 2-CRNR of integers, then it has exactly 14 forms of imperfect triplets.

Proof.

Let x, y, z be a triplet in S, then we have:

$xy = yx = x$, $yz = zy = z$, $xz = zx = y$, so that, (x, y) , (y, z) are imperfect duplets in S.

We discuss the 8 forms of imperfect duplets to find the desired imperfect duplets:

Form 1: $x = (0,0,0), y = (b_0, b_1, b_2), z = (0,0,0)$ it is a triplet if and only if $xy = z,$ thus $, y = (0,0,0)$.

(the first triplet is $x = y = z = (0,0,0)$).

Form 2: $x = (0, a_1, a_2), y = (b_0, 1,1), z = (0, c_1, c_2)$ it is a triplet if and only if $xz = y,$ thus $, b_0 = 0, a_1 c_1 = 1, a_2 c_2 = 1$.

the possible triplets are:

$x = (0,1,1), y = (0,1,1), z = (0,1,1)$.

$x = (0,1,-1), y = (0,1,1), z = (0,1,-1)$

$x = (0,1,-1), y = (0,1,1), z = (0,1,-1)$

$x = (0,-1,1), y = (0,1,1), z = (0,-1,-1)$

Form 3: $x = (0,0,a_2), y = (b_0, b_1, 1), z = (0,0,c_2)$ it is a triplet if and only if $xz = y,$ thus $, b_0 = b_1 = 0, , a_2 = c_2 = 1$.

the possible triplets are:

$x = (0,0,1), y = (0,0,1), z = (0,1,1)$.

$x = (0,0,-1), y = (0,0,1), z = (0,1,-1)$

Form 4: $x = (a_0, 0, a_1), y = (1, b_1, 1), z = (c_0, 0, c_1)$ it is a triplet if and only if $xz = y,$ thus $, a_0 c_0 = a_1 c_1 = 1, b_1 = 0$.

the possible triplets are:

$x = (1,0,1), y = (1,0,1), z = (1,0,1)$

$x = (-1,0,1), y = (1,0,1), z = (-1,0,1)$.

$x = (1,0,-1), y = (1,0,1), z = (1,0,-1)$

$x = (-1,0,-1), y = (1,0,1), z = (-1,0,-1)$

Form 5: $x = (a_0, a_1, 0), y = (1,1, b_2), z = (c_0, c_1, 0)$ it is a triplet if and only if $xz = y,$ thus $, a_0 c_0 = 1, a_1 c_1 = 1, b_2 = 0$.

the possible triplets are:

$x = (1,1,0), y = (1,1,0), z = (1,1,0)$

$x = (-1,-1,0), y = (1,1,0), z = (-1,-1,0)$.

$x = (-1,1,0), y = (1,1,0), z = (-1,1,0)$

$x = (1,1,0), y = (1,1,-1), z = (1,-1,0)$

Form 6: $x = (a_0, a_1, a_2), y = (1,1,1), z = (c_0, c_1, c_2)$ it is a triplet if and only if $xz = y$, thus, $a_0 c_0 = a_1 c_1 = a_2 c_2 = 1$.

the possible triplets are:

$x = (1,1,1), y = (1,1,1), z = (1,1,1)$

$x = (1,1,-1), y = (1,1,1), z = (1,1,-1)$

$x = (1,-1,1), y = (1,1,1), z = (1,-1,1)$

$x = (-1,1,1), y = (1,1,1), z = (-1,1,1)$

$x = (-1,-1,1), y = (1,1,1), z = (-1,-1,1)$

$x = (1,-1,-1), y = (1,1,1), z = (1,-1,-1)$

$x = (-1,1,-1), y = (1,1,1), z = (-1,1,-1)$

$x = (-1,-1,-1), y = (1,1,1), z = (-1,-1,-1)$

Form 7: $x = (a_0, 0,0), y = (1, b_1, b_2), z = (c_0, 0,0)$ it is a triplet if and only if $xz = y$, thus, $a_0 c_0 = 1, b_1 = b_2 = 0$.

the possible triplets are:

$x = (1,0,0), y = (1,0,0), z = (1,0,0)$

$x = (-1,0,0), y = (1,0,0), z = (-1,0,0)$

Form 8: $x = (0, a_1, 0), y = (b_0, 1, b_2), z = (0, c_1, 0)$ it is a triplet if and only if $xz = y$, thus, $a_1 c_1 = 1, b_0 = b_2 = 0$.

the possible triplets are:

$x = (0,1,0), y = (0,1,0), z = (0,1,0)$

$x = (0,-1,0), y = (0,-1,0), z = (0,1,0)$.

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On Some Algorithms for Solving Different Types of Symbolic 2-Plithogenic Algebraic Equations

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Abstract:

The main goal of this paper is to study three different types of algebraic symbolic 2-plithogenic equations. The symbolic 2-plithogenic linear Diophantine equations, symbolic 2-plithogenic quadratic equations, and linear system of symbolic 2-plithgenic equations will be discussed and handled, where algorithms to solve the previous types will be presented and proved by transforming them to classical algebraic systems of equations.

Keywords: symbolic 2-plithogenic Diophantine equation, symbolic 2-plithogenic quadratic equation, linear system, symbolic 2-plithogenic field

Introduction and preliminaries

The process of extending classical algebraic structures by using logical symbols and elements can be considered as a novel approach to generalize algebraic structures, where many algebraic structures were generalized by using neutrosophic elements, fuzzy elements, and refined neutrosophic elements [1-15].

Smarandache has defined the concept of Symbolic 2-plithogenic sets and structures [16-20] as new generalizations of classical structures. Also, he has presented many open research problems [20].

In [21], Smarandache ideas was discussed in a special case of n=2, where the symbolic 2-plithogenic rings were defined and studied with many elementary interesting substructures and properties.

Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$2-SP_R = \{a_0 + a_1P_1 + a_2P_2;\ a_i \in R, {P_j}^2 = P_j, P_1 \times P_2 = P_{max(1,2)} = P_2\}$.

Smarandache has defined algebraic operations on $2-SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$.

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0{P_1}^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2{P_2}^2 + a_1b_1P_1P_1 = a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.

It is clear that $(2-SP_R)$ is a ring.

If R is a field, then $2-SP_R$ is called a symbolic 2-plithogenic field [22].

Also, the following open problems were asked in [22]:

Problem (3):

If F is a field then $2-SP_F$ is called a 2-plithogenic symbolic field. Now, can we find a strong algorithm that explains how can we solve the previous equations by turning it into the classical equations.

Problem (4):

If Z is the ring of integers ring then $2-SP_Z$ is called a 2-plithogenic symbolic ring of integers. Can we find a strong algorithm that explains how can we solve the previous equations by turning it into the classical Diophantine equations.

In this paper, we solve the previous two open problems by suggesting effective algorithms that help us to transform symbolic 2-plithogenic equations to classical algebraic equations.

Main Results

Definition.

Let $2-SP_Z=\{a+bP_1+cP_2;\ a,b,c\in Z\}$ be the symbolic 2-plithogenic ring of integers, the Diophantine equation with two variables is defined as follows:

$AX+BY=C; A=a_0+a_1P_1+a_2P_2, B=b_0+b_1P_1+b_2P_2, C=c_0+c_1P_1+c_2P_2,$

$X=x_0+x_1P_1+x_2P_2, Y=y_0+y_1P_1+y_2P_2, a_i,b_i,c_i,x_i,y_i\in 2-SP_Z.$

The following theorem describes an algorithm to solve the symbolic 2-plithogenic linear Diophantine equation with two variables.

Theorem.

Let $AX+BY=C$ be the symbolic 2-plithogenic linear Diophantine equation with two variables, it is solvable if and only if the following linear Diophantine equations are solvable.

$$\begin{cases} a_0x_0+b_0y_0=c_0 \\ (a_0+a_1)(x_0+x_1)+(b_0+b_1)(y_0+y_1)=c_0+c_1 \\ (a_0+a_1+a_2)(x_0+x_1+x_2)+(b_0+b_1+b_2)(y_0+y_1+y_2)=c_0+c_1+c_2 \end{cases}$$

Proof.

The equation $AX+BY=C$ equivalents:

$$\begin{aligned} a_0x_0+b_0y_0&+(a_0x_1+a_1x_0+a_1x_1+b_0y_1+b_1y_0+b_1y_1)P_1 \\ &+(a_0x_2+a_2x_0+a_2x_2+a_1x_2+a_2x_1+b_0y_2+b_2y_0+b_2y_2+b_1y_2+b_2y_1)P_2 \\ &=c_0+c_1P_1+c_2P_2 \end{aligned}$$

$$\begin{cases} a_0x_0+b_0y_0=c_0 \ldots (1) \\ a_0x_1+a_1x_0+a_1x_1+b_0y_1+b_1y_0+b_1y_1=c_1 \ldots (2) \\ a_0x_2+a_2x_0+a_2x_2+a_1x_2+a_2x_1+b_0y_2+b_2y_0+b_2y_2+b_1y_2+b_2y_1=c_2 \ldots (3) \end{cases}$$

We add (1) to (2), and (1) to (2) to (3), we get:

$$\begin{cases} a_0x_0+b_0y_0=c_0 \\ (a_0+a_1)(x_0+x_1)+(b_0+b_1)(y_0+y_1)=c_0+c_1 \\ (a_0+a_1+a_2)(x_0+x_1+x_2)+(b_0+b_1+b_2)(y_0+y_1+y_2)=c_0+c_1+c_2 \end{cases}$$

And the proof is complete.

The description of the algorithm.

To solve $AX+BY=C$ in $2-SP_Z$, we must follow these steps.

Step1.

We compute $gcd(a_0,b_0), gcd(a_0+a_1,b_0+b_1), gcd(a_0+a_1+a_2,b_0+b_1+b_2)$.

If $gcd(a_0,b_0)/c_0$, $gcd(a_0+a_1,b_0+b_1)/c_0+c_1$, $gcd(a_0+a_1+a_2,b_0+b_1+b_2)/c_0+c_1+c_2$, then it is solvable.

Step2.

We solve the equivalent system and get the values of $x_i, y_i; 0 \leq i \leq 2$.

Example.

Consider the following symbolic 2-plithogenic linear Diophantine equation:

$(2 + P_1 + P_2)X + (3 + 2P_1 - P_2)Y = 8 + 5P_1 + 7P_2$.

$gcd(a_0, b_0) = gcd(2,3) = 1/8$.

$gcd(a_0 + a_1, b_0 + b_1) = gcd(3,5) = 1/c_0 + c_1 = 13$.

$gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) = gcd(4,4) = 4/c_0 + c_1 + c_2 = 20$.

So that, the equation is solvable.

The equivalent system of linear Diophantine equations is:

$$\begin{cases} 2x_0 + 3y_0 = 8 \ldots (1) \\ 3(x_0 + x_1) + 5(y_0 + y_1) = 13 \ldots (2) \\ 4(x_0 + x_1 + x_2) + 4(y_0 + y_1 + y_2) = 20 \ldots (3) \end{cases}$$

The equation (1) has a solution $(x_0 = 1, y_0 = 2)$.

The equation (2) has a solution $(x_0 + x_1 = 1, y_0 + y_1 = 2)$, there for $(x_1 = 0, y_1 = 0)$.

The equation (3) has a solution $(x_0 + x_1 + x_2 = 2, y_0 + y_1 + y_2 = 3)$, there for $(x_2 = 1, y_2 = 1)$.

This implies a solution $X = 1 + P_2, Y = 2 + P_2$.

Example.

Consider the following:

$(3 + P_1 + 5P_2)X + (6 - 2P_1 + 10P_2)Y = 5 + P_1 + P_2$.

$gcd(a_0, b_0) = gcd(3,6) = 3 \ddagger 5$, there for it is not solvable.

2-symbolic plithogenic Quadratic equation.

Let $2 - SP_F$ be a symbolic 2-plithogenic field, the formula

$AX^2 + BY^2 + C = 0; A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2,$

$C = c_0 + c_1P_1 + c_2P_2, X = x_0 + x_1P_1 + x_2P_2, Y = y_0 + y_1P_1 + y_2P_2, a_i, b_i, c_i, x_i, y_i \in 2 - SP_F$.

Is called the symbolic 2-plithogenic quadratic equation.

Theorem.

Let $AX^2 + BY^2 + C = 0$ be a symbolic 2-plithogenic quadratic equation over $2 - SP_F$, then it is solvable if and only if the following system is solvable:

$$\begin{cases} a_0{x_0}^2 + b_0{y_0}^2 + c_0 = 0 \ldots (1) \\ (a_0 + a_1)(x_0 + x_1)^2 + (b_0 + b_1)(y_0 + y_1)^2 + (c_0 + c_1) = 0 \ldots (2) \\ (a_0 + a_1 + a_2)(x_0 + x_1 + x_2)^2 + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2)^2 + (c_0 + c_1 + c_2) = 0 \ldots (3) \end{cases}$$

Proof.

We have $X^2 = {x_0}^2 + P_1[(x_0 + x_1)^2 - {x_0}^2] + P_2[(x_0 + x_1 + x_2)^2 - (x_0 + x_1)^2]$, see [ ].

So that:

$$AX^2 = a_0{x_0}^2 + P_1[a_0(x_0 + x_1)^2 - a_0{x_0}^2 + a_1(x_0 + x_1)^2 - a_1{x_0}^2 + a_1{x_0}^2] + P_2[a_0(x_0 + x_1 + x_2)^2 - a_0(x_0 + x_1)^2 + a_1(x_0 + x_1 + x_2)^2 - a_1(x_0 + x_1)^2 + a_2(x_0 + x_1 + x_2)^2 - a_2(x_0 + x_1)^2 + a_2{x_0}^2 + a_2(x_0 + x_1)^2 - a_2{x_0}^2]$$

$$AX^2 = a_0{x_0}^2 + P_1[(a_0 + a_1)(x_0 + x_1)^2 - a_0{x_0}^2] + P_2[(a_0 + a_1 + a_2)(x_0 + x_1 + x_2)^2 - (a_0 + a_1)(x_0 + x_1)^2]$$

There for, the equation $AX^2 + BY^2 + C = 0$ is equivalent to:

$$\begin{cases} a_0{x_0}^2 + b_0{y_0}^2 + c_0 = 0 \dots (1) \\ (a_0 + a_1)(x_0 + x_1)^2 + (b_0 + b_1)(y_0 + y_1)^2 + (c_0 + c_1) = 0 \dots (2) \\ (a_0 + a_1 + a_2)(x_0 + x_1 + x_2)^2 + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2)^2 + (c_0 + c_1 + c_2) = 0 \dots (3) \end{cases}$$

The description of algorithm.

To solve $AX^2 + BY^2 + C = 0$ in $2 - SP_F$, follow these steps:

Step1.

Solve the equivalent classical system of quadratic equations. If $(1), (2)$, and (3) are solvable in the field F,then the symbolic 2-plithogenic quadratic equation is solvable.

Step2.

Discuss all possible cases of x_0, x_1, x_2.

Remark.

If $AX^2 + BY^2 + C = 0$ is solvable in $2 - SP_F$, then it has at most 8 solutions.

Example.

Consider the following:

$(1 + P_1 + P_2)X^2 + (3 - P_1)X - 4 - 12P_2 = 0$

We have:

$$\begin{cases} a_0 = 1, a_1 = 1, a_2 = 1 \\ b_0 = 3, b_1 = -1, b_2 = 0 \\ c_0 = -4, c_1 = 0, c_2 = -12 \end{cases}$$

The equivalent system is:

$$\begin{cases} {x_0}^2 + 3x_0 - 4 = 0 \dots (1) \\ 2(x_0 + x_1)^2 + 2(x_0 + x_1) - 4 = 0 \dots (2) \\ 3(x_0 + x_1 + x_2)^2 + 2(x_0 + x_1 + x_2) - 16 = 0 \dots (3) \end{cases}$$

The solutions of (1): $x_0 = 1, x_0 = -4$.

The solutions of (2): $x_0 + x_1 = 1, x_0 + x_1 = -2$.

The solutions of (3): $x_0 + x_1 + x_2 = 2, x_0 + x_1 + x_2 = -\frac{8}{3}$.

Case1.

If $x_0 = 1, x_0 + x_1 = 1, x_0 + x_1 + x_2 = 2$, then $x_1 = 0, x_2 = 1$, and $X = 1 + P_2$.

Case2.

If $x_0 = 1, x_0 + x_1 = 1, x_0 + x_1 + x_2 = -\frac{8}{3}$, then $x_1 = 0, x_2 = -\frac{11}{3}$, and $X = 1 - \frac{11}{3}P_2$.

Case3.

If $x_0 = 1, x_0 + x_1 = -2, x_0 + x_1 + x_2 = 2$, then $x_1 = -3, x_2 = 4$, and $X = 1 - 3P_1 + 4P_2$.

Case4.

If $x_0 = 1, x_0 + x_1 = -2, x_0 + x_1 + x_2 = -\frac{8}{3}$, then $x_1 = -3, x_2 = -\frac{2}{3}$, and $X = 1 - 3P_1 - \frac{2}{3}P_2$.

Case5.

If $x_0 = -4, x_0 + x_1 = 1, x_0 + x_1 + x_2 = 2$, then $x_1 = 5, x_2 = 1$, and $X = -4 + 5P_1 + P_2$.

Case6.

If $x_0 = -4, x_0 + x_1 = 1, x_0 + x_1 + x_2 = -\frac{8}{3}$, then $x_1 = 5, x_2 = -\frac{11}{3}$, and $X = -4 + 5P_1 - \frac{11}{3}P_2$.

Case7.

If $x_0 = -4, x_0 + x_1 = -2, x_0 + x_1 + x_2 = 2$, then $x_1 = 2, x_2 = 4$, and $X = -4 + 2P_1 + 4P_2$.

Case8.

If $x_0 = -4, x_0 + x_1 = -2, x_0 + x_1 + x_2 = -\frac{8}{3}$, then $x_1 = 2, x_2 = -\frac{2}{3}$, and $X = -4 + 2P_1 - \frac{2}{3}P_2$.

So that, the solutions of the original symbolic 2-plithogenic quadratic equation are:

$$X \in \left\{-4 + 2P_1 - \frac{2}{3}P_2, -4 + 2P_1 + 4P_2, -4 + 5P_1 - \frac{11}{3}P_2, -4 + 5P_1 + P_2, 1 - 3P_1 - \frac{2}{3}P_2, 1 - 3P_1 + 4P_2, 1 - \frac{11}{3}P_2, 1 + P_2\right\}$$

Example.

Consider the following:

$(2 + 3P_1 - P_2)X^2 + (4 + P_1 + P_2)X - 6 - 4P_1 = 0$

We have:

$$\begin{cases} a_0 = 2, a_1 = 3, a_2 = -1 \\ b_0 = 4, b_1 = 1, b_2 = 1 \\ c_0 = -6, c_1 = -4, c_2 = 0 \end{cases}$$

The equivalent system is:

$$\begin{cases} 2{x_0}^2 + 4x_0 - 6 = 0 \dots (1) \\ 5(x_0 + x_1)^2 + 5(x_0 + x_1) - 10 = 0 \dots (2) \\ 4(x_0 + x_1 + x_2)^2 + 6(x_0 + x_1 + x_2) - 10 = 0 \dots (3) \end{cases}$$

The solutions of (1): $x_0 = 1, x_0 = -3$.

The solutions of (2): $x_0 + x_1 = 1, x_0 + x_1 = -2$.

The solutions of (3): $x_0 + x_1 + x_2 = 1, x_0 + x_1 + x_2 = -\frac{5}{2}$.

Case1.

If $x_0 = 1, x_0 + x_1 = 1, x_0 + x_1 + x_2 = 1$, then $x_1 = x_2 = 0$, and $X = 1$.

Case2.

If $x_0 = 1, x_0 + x_1 = 1, x_0 + x_1 + x_2 = -\frac{5}{2}$, then $x_1 = 0, x_2 = -\frac{7}{2}$, and $X = 1 - \frac{5}{2}P_2$.

Case3.

If $x_0 = 1, x_0 + x_1 = -2, x_0 + x_1 + x_2 = 1$, then $x_1 = -3, x_2 = 3$, and $X = 1 - 3P_1 + 3P_2$.

Case4.

If $x_0 = 1, x_0 + x_1 = -2, x_0 + x_1 + x_2 = -\frac{5}{2}$, then $x_1 = -3, x_2 = -\frac{1}{2}$, and $X = 1 - 3P_1 - \frac{1}{2}P_2$.

Case5.

If $x_0 = -3, x_0 + x_1 = 1, x_0 + x_1 + x_2 = 1$, then $x_1 = 4, x_2 = 0$, and $X = -3 + 4P_1$.

Case6.

If $x_0 = -3, x_0 + x_1 = 1, x_0 + x_1 + x_2 = -\frac{5}{2}$, then $x_1 = 4, x_2 = -\frac{7}{2}$, and $X = -3 + 4P_1 - \frac{7}{2}P_2$.

Case7.

If $x_0 = -3, x_0 + x_1 = -2, x_0 + x_1 + x_2 = 1$, then $x_1 = 1, x_2 = 3$, and $X = -3 + P_1 + 3P_2$.

Case8.

If $x_0 = -3, x_0 + x_1 = -2, x_0 + x_1 + x_2 = -\frac{5}{2}$, then $x_1 = 1, x_2 = -\frac{1}{2}$, and $X = -3 + P_1 - \frac{1}{2}P_2$.

So that, the solutions of the original symbolic 2-plithogenic quadratic equation are:

$$X \in \left\{1, 1 - \frac{5}{2}P_2, 1 - 3P_1 + 3P_2, 1 - 3P_1 - \frac{1}{2}P_2, -3 + 4P_1, -3 + 4P_1 - \frac{7}{2}P_2, -3 + P_1 + 3P_2, -3 + P_1 - \frac{1}{2}P_2\right\}$$

2-plithogenic Linear equations.

We begin the simplest case, a symbolic 2-plithogenic linear equation with one variable $A.X = B$.

This equation is solvable uniquely if and only if A is invertible and $X = A^{-1}B$.

According to [31], $A^{-1} = {a_0}^{-1} + P_1[(a_0 + a_1)^{-1} - {a_0}^{-1}] + P_2[(a_0 + a_1 + a_2)^{-1} - (a_0 + a_1)^{-1}]$.

Example.

Consider the equation $(2 + P_1 + P_2)X = 3 - P_1$ over $2 - SP_R$.

$a_0 = 2, {a_0}^{-1} = \frac{1}{2}, a_0 + a_1 = 3, (a_0 + a_1)^{-1} = \frac{1}{3}, a_0 + a_1 + a_2 = 4, (a_0 + a_1 + a_2)^{-1} = \frac{1}{4}$, thus:

$A^{-1} = \frac{1}{2} - \frac{1}{6}P_1 - \frac{1}{12}P_2$, there for:

$$X = \left(\frac{1}{2} - \frac{1}{6}P_1 - \frac{1}{12}P_2\right)(3 - P_1) = \frac{3}{2} - \frac{1}{2}P_1 - \frac{1}{2}P_1 + \frac{1}{6}P_1 - \frac{1}{4}P_2 + \frac{1}{12}P_2 = \frac{3}{2} - \frac{5}{6}P_1 - \frac{1}{6}P_2$$

The general case is about a linear system of n symbolic 2-plithogenic equations $A_i.X_i = B_i; 1 \leq i \leq n$.

To solve a system like that, we must transform it to an equivalent classical system. We present the following algorithm.

To solve the symbolic 2-plithogenic linear system:

$$\begin{cases} A_{11}.X_1 + A_{12}.X_2 + \cdots + A_{1n}.X_n = B_{1n} \\ A_{21}.X_1 + A_{22}.X_2 + \cdots + A_{2n}.X_n = B_{2n} \\ \vdots \\ A_{n1}.X_1 + A_{n2}.X_2 + \cdots + A_{nn}.X_n = B_{nn} \end{cases}$$

Where: $A_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}P_1 + a_{ij}^{(2)}P_2, X_i = X_i^{(0)} + X_i^{(1)}P_1 + X_i^{(2)}P_2, B_{ij} = b_{ij}^{(0)} + b_{ij}^{(1)}P_1 + b_{ij}^{(2)}P_2 \in 2 - SP_F$.

Follow these steps:

Step1.

Find the classical equivalent system as follows:

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}^{(0)} X_i^{(0)} = \sum_{i,j=1}^{n} b_{ij}^{(0)} \\ \sum_{i,j=1}^{n} \left(a_{ij}^{(0)} + a_{ij}^{(1)}\right)\left(X_i^{(0)} + X_i^{(1)}\right) = \sum_{i,j=1}^{n} \left(b_{ij}^{(0)} + b_{ij}^{(1)}\right) \\ \sum_{i,j=1}^{n} \left(a_{ij}^{(0)} + a_{ij}^{(1)} + a_{ij}^{(2)}\right)\left(X_i^{(0)} + X_i^{(1)} + X_i^{(2)}\right) = \sum_{i,j=1}^{n} \left(b_{ij}^{(0)} + b_{ij}^{(1)} + b_{ij}^{(2)}\right) \end{cases}$$

step2.

Solve each system and remark that:

The first system gives the values of $X_i^{(0)}; 1 \leq i \leq n$.

The second one gives the values of $X_i^{(0)} + X_i^{(1)}; 1 \leq i \leq n$.

The third one gives values of $X_i^{(0)} + X_i^{(1)} + X_i^{(2)}; 1 \leq i \leq n$.

Step3.

If each system is solvable, then the original 2-plithogenic system is solvable, and if the number of solutions of every classical system is k, then the number of solutions for the 2-plithogenic system is k^3.

Example.

Consider the following symbolic 2-plithogenic system of three linear equations with three variables:

$$\begin{cases} (1+P_2)X_1+(3-P_1)X_2+(1+P_1-P_2)X_3=5 \\ P_2X_1+P_1X_2+(P_1-P_2)X_3=2P_1+2P_2 \\ (1+P_1-P_2)X_1+(4+3P_1-P_2)X_2+(5+2P_2)X_3=11+4P_2 \end{cases}$$

the equivalent classical systems are:

$$\begin{cases} X_1^{(0)}+3X_2^{(0)}+X_3^{(0)}=5 \\ 0X_1^{(0)}+0X_2^{(0)}+0X_3^{(0)}=0 \quad \dots \text{system(1)} \\ 2X_1^{(0)}+4X_2^{(0)}+5X_3^{(0)}=11 \end{cases}$$

$$\begin{cases} \left(X_1^{(0)}+X_1^{(1)}\right)+2\left(X_2^{(0)}+X_2^{(1)}\right)+2\left(X_3^{(0)}+X_3^{(1)}\right)=5 \\ 0\left(X_1^{(0)}+X_1^{(1)}\right)+\left(X_2^{(0)}+X_2^{(1)}\right)+\left(X_3^{(0)}+X_3^{(1)}\right)=2 \quad \dots \text{system(2)} \\ 3\left(X_1^{(0)}+X_1^{(1)}\right)+7\left(X_2^{(0)}+X_2^{(1)}\right)+5\left(X_3^{(0)}+X_3^{(1)}\right)=15 \end{cases}$$

$$\begin{cases} 2\left(X_1^{(0)}+X_1^{(1)}++X_1^{(2)}\right)+2\left(X_2^{(0)}+X_2^{(1)}+X_2^{(2)}\right)+\left(X_3^{(0)}+X_3^{(1)}+X_3^{(2)}\right)=5 \\ \left(X_1^{(0)}+X_1^{(1)}++X_1^{(2)}\right)+\left(X_2^{(0)}+X_2^{(1)}+X_2^{(2)}\right)+2\left(X_3^{(0)}+X_3^{(1)}+X_3^{(2)}\right)=4 \quad \dots \text{system(3)} \\ 2\left(X_1^{(0)}+X_1^{(1)}++X_1^{(2)}\right)+6\left(X_2^{(0)}+X_2^{(1)}+X_2^{(2)}\right)+7\left(X_3^{(0)}+X_3^{(1)}+X_3^{(2)}\right)=15 \end{cases}$$

The system(1) has infinite solutions, thus the 2-plithogenic system has infinite solutions.

We will find some solutions to clarify the algorithm.

For example system(1) has a solution $X_1^{(0)}=X_2^{(0)}=X_3^{(0)}=1$.

The system (2) has a solution $X_1^{(0)}+X_1^{(1)}=X_2^{(0)}+X_2^{(1)}=X_3^{(0)}+X_3^{(1)}=1$, thus $X_1^{(1)}=X_2^{(1)}=X_3^{(1)}=0$.

The system (3) has a solution $X_1^{(0)}+X_1^{(1)}+X_1^{(2)}=X_2^{(0)}+X_2^{(1)}+X_2^{(2)}=X_3^{(0)}+X_3^{(1)}+X_3^{(2)}=1$, there for $X_1^{(2)}=X_2^{(2)}=X_3^{(2)}=0$, and $X_1=X_1^{(0)}+X_1^{(1)}P_1+X_1^{(2)}P_2=1$, $X_2=1$,$X_3=1$ is a solution for the 2-plithogenic system.

Also, the system (1) has a solution $X_1^{(0)}=\frac{13}{2},X_2^{(0)}=-\frac{1}{2},X_3^{(0)}=0$.

The system (2) has a solution $X_1^{(0)}+X_1^{(1)}=X_2^{(0)}+X_2^{(1)}=X_3^{(0)}+X_3^{(1)}=1$.

The system (3) has a solution $X_1^{(0)} + X_1^{(1)} + X_1^{(2)} = X_2^{(0)} + X_2^{(1)} + X_2^{(2)} = X_3^{(0)} + X_3^{(1)} + X_3^{(2)} = 1$.

There for $X_1^{(1)} = 1 - \frac{13}{2} = -\frac{11}{2}, X_2^{(1)} = 1 + \frac{1}{2} = \frac{3}{2}, X_3^{(1)} = 1,\ X_1^{(2)} = X_2^{(2)} = X_3^{(2)} = 0$.

This implies that:

$X_1 = \frac{13}{2} - \frac{11}{2} P_1, X_2 = -\frac{1}{2} + \frac{3}{2} P_1, X_3 = P_1$ is a solution of the 2-plithogenic system.

Conclusion

In this paper, we have presented novel algorithms to solve many different types of 2-plithogenic algebraic equations (quadratic, linear, and linear Diophantine equations) by transforming them to classical systems of algebraic equations. Also, many examples were illustrated tp explain the validity of our work.

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On The Symbolic 2-Plithogenic Number Theory and Integers

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Abstract:

The objective of this paper is to study for the first time the foundational concepts of number theory in 2-plithogenic rings of integers, where concepts such as symbolic 2-plithogenic congruencies, division, semi primes, and greatest common divisors.

In addition, many elementary properties will be discussed in details through many theorems and examples.

Keywords: Symbolic 2-plithogenic integer, symbolic 2-plithogenic divison, symbolic 2-plithogenic semi prime.

Introduction and basic concepts

The concept of symbolic plithogenic sets was defined by Smarandache in [13-17 ,30], and he suggested an algebraic approach of these sets. Laterally, the concept of symbolic 2-plithogenic rings [31]. In general, we can say that symbolic plithogenic structures are very close to neutrosophic algebraic structures with many differences in the definition of multiplication operation [1-10].

Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$2-SP_R=\{a_0+a_1P_1+a_2P_2;\ a_i\in R, {P_j}^2=P_j, P_1\times P_2=P_{max(1,2)}=P_2\}$.

Smarandache has defined algebraic operations on $2-SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$.

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0{P_1}^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2{P_2}^2 + a_1b_1P_1P_1 = a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.

It is clear that $(2 - SP_R)$ is a ring.

Also, if R is commutative, then $2 - SP_R$ is commutative, and if R has a unity (1), than $2 - SP_R$ has the same unity (1).

If R is a field, then $2 - SP_R$ is called a symbolic 2-plithogenic field.

In this paper, we study the symbolic 2-plithogenicnumber theoretical concepts according to many points of view, where congruencies, Euclidean division, Euler's function, and gratest common divisors will be presented in terms of theorems. In addition, many examples will be illustrated to explain the novelty of these ideas. In addition, we suggest many future applications of symbolic 2-plithogenic integers in cryptography and public key neutrosophic cryptography.

Main Discussion

Definition.

Let $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2 \in 2 - SP_Z$, we say that $A \setminus B$ if and only if there exists $C \in 2 - SP_Z$ such that $A \times B = C$.

Definition.

Let $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2, C = c_0 + c_1P_1 + c_2P_2$ be three symbolic 2-plithogenic integers, then $A \equiv B(mod\ C)$ if and only if $C \setminus A - B$.

Also, $C = gcd(A, B)$ if and only if $C \setminus A$ and $C \setminus B$ and for any $D \setminus A, D \setminus B$, then $D \setminus C$.

Definition.

We say that $A \leq B$ if $a_0 \leq b_0, a_0 + a_1 \leq b_0 + b_1, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2$.

Theorem.

Let $A = a_0 + a_1P_1 + a_2P_2, B = b_0 + b_1P_1 + b_2P_2, C = c_0 + c_1P_1 + c_2P_2 \in 2 - SP_Z$, then:

1). $(\leq)$ is a partial order relation.

2). $A \setminus B$ if and only if $a_0 \setminus b_0, a_0 + a_1 \setminus b_0 + b_1, a_0 + a_1 + a_2 \setminus b_0 + b_1 + b_2$.

3). $gcd(A, B) = C$ if and only if $gcd(a_0, b_0) = c_0, gcd(a_0 + a_1, b_0 + b_1) = c_0 + c_1, gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) = c_0 + c_1 + c_2$.

4). $A \equiv B(mod\ C)$ if and only if:

$$\begin{cases} a_0 \equiv b_0(mod\ c_0) \\ a_0 + a_1 \equiv b_0 + b_1(mod\ c_0 + c_1) \\ a_0 + a_1 + a_2 \equiv b_0 + b_1 + b_2(mod\ c_0 + c_1 + c_2) \end{cases}$$

Proof.

1). $A \le A$ that is because $a_0 \le a_0, a_0 + a_1 \le a_0 + a_1, a_0 + a_1 + a_2 \le a_0 + a_1 + a_2$.

If $A \le B$ and $B \le A$, then:

$$\begin{cases} a_0 \le b_0, b_0 \le a_0, thus\ a_0 = b_0 \\ a_0 + a_1 \le b_0 + b_1, b_0 + b_1 \le a_0 + a_1, thus\ a_0 + a_1 = b_0 + b_1, hence\ a_1 = b_1 \\ a_0 + a_1 + a_2 \le b_0 + b_1 + b_2, b_0 + b_1 + b_2 \le a_0 + a_1 + a_2, thus\ a_0 + a_1 + a_2 = b_0 + b_1 + b_2, hence\ a_2 = b_2 \end{cases}$$

Hence $A = B$.

If $A \le B$ and $B \le C$, then $a_0 \le b_0 \le c_0$, $a_0 + a_1 \le b_0 + b_1 \le c_0 + c_1$, $a_0 + a_1 + a_2 \le b_0 + b_1 + b_2 \le c_0 + c_1 + c_2$, thus $A \le C$.

2). If $A \setminus B$, then there exists C such that $A.C = B$. This equivalents:

$a_0c_0 + P_1(a_0c_1 + a_1c_0 + a_1c_1) + P_2(a_0c_2 + a_2c_0 + a_2c_2 + a_1c_2 + a_2c_1) = b_0 + b_1P_1 + b_2P_2$,

there for:

$$\begin{cases} a_0c_0 = b_0 \ldots (1) \\ a_0c_1 + a_1c_0 + a_1c_1 = b_1 \ldots (2) \\ a_0c_2 + a_2c_0 + a_2c_2 + a_1c_2 + a_2c_1 = b_2 \ldots (3) \end{cases}$$

We add (1) to (2) and (1) to (2) to (3), to get:

$$\begin{cases} a_0c_0 = b_0 \\ (a_0 + a_1)(c_0 + c_1) = b_0 + b_1 \\ (a_0 + a_1 + a_2)(c_0 + c_1 + c_2) = b_0 + b_1 + b_2 \end{cases}$$

Thus $a_0 \setminus b_0, a_0 + a_1 \setminus b_0 + b_1, a_0 + a_1 + a_2 \setminus b_0 + b_1 + b_2$.

3). Assume that $gcd(A, B) = C$, then for any $D = d_0 + d_1P_1 + d_2P_2 \in 2 - SP_Z$ such that $D \setminus A, D \setminus B$ implies $D \setminus C$.

According to (2), we get $d_0 \setminus c_0, d_0 + d_1 \setminus c_0 + c_1, d_0 + d_1 + d_2 \setminus c_0 + c_1 + c_2$, so that $gcd(a_0, b_0) = c_0, gcd(a_0 + a_1, b_0 + b_1) = c_0 + c_1, gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) = c_0 + c_1 + c_2$.

This implies that $gcd(A, B) = gcd(a_0, b_0) + P_1[gcd(a_0 + a_1, b_0 + b_1) - gcd(a_0, b_0)] + P_2[gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) - gcd(a_0 + a_1, b_0 + b_1)]$.

4). $A \equiv B(mod\ C)$ if and only if $C \setminus A - B$, thus:

$c_0 \setminus a_0 - b_0, c_0 + c_1 \setminus (a_0 + a_1) - (b_0 + b_1), c_0 + c_1 + c_2 \setminus (a_0 + a_1 + a_2) - (b_0 + b_1 + b_2)$

So that:

$$\begin{cases} a_0 \equiv b_0 (mod\ c_0) \\ a_0 + a_1 \equiv b_0 + b_1 (mod\ c_0 + c_1) \\ a_0 + a_1 + a_2 \equiv b_0 + b_1 + b_2 (mod\ c_0 + c_1 + c_2) \end{cases}$$

Theorem.

Let $A = a_0 + a_1 P_1 + a_2 P_2, B = b_0 + b_1 P_1 + b_2 P_2 \in 2 - SP_Z$, then $gcd(A, B) = 1$ if and only if $gcd(a_0, b_0) = 1, gcd(a_0 + a_1, b_0 + b_1) = 1, gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) = 1$.

The proof is clear.

Theorem.

Let $A, B, C, D, E \in 2 - SP_Z$, where:

$A = a_0 + a_1 P_1 + a_2 P_2, B = b_0 + b_1 P_1 + b_2 P_2, C = c_0 + c_1 P_1 + c_2 P_2, D = d_0 + d_1 P_1 + d_2 P_2, E = e_0 + e_1 P_1 + e_2 P_2;\ c_i, a_i, b_i, e_i, d_i \in Z$, then:

1). If $A \equiv B (mod\ C), D \equiv E (mod\ C)$, then $A + D \equiv B + E (mod\ C)\ , A - D \equiv B - E (mod\ C)$.

2). $A.D \equiv B.E (mod\ C)$.

3). If $gcd(A, B) = 1$, then:

$$A^{-1}(mod\ B) = a_0{}^{-1}(mod\ b_0) + P_1[(a_0 + a_1)^{-1}(mod\ b_0 + b_1) - a_0{}^{-1}(mod\ b_0)]$$
$$+ P_2[(a_0 + a_1 + a_2)^{-1}(mod\ b_0 + b_1 + b_2) - (a_0 + a_1)^{-1}(mod\ b_0 + b_1)]$$

Proof.

1). Assume that $A \equiv B (mod\ C), D \equiv E (mod\ C)$, thus:

$$\begin{cases} a_0 \equiv b_0 (mod\ c_0) \\ a_0 + a_1 \equiv b_0 + b_1 (mod\ c_0 + c_1) \\ a_0 + a_1 + a_2 \equiv b_0 + b_1 + b_2 (mod\ c_0 + c_1 + c_2) \end{cases}$$

And

$$\begin{cases} d_0 \equiv e_0 (mod\ c_0) \\ d_0 + d_1 \equiv e_0 + e_1 (mod\ c_0 + c_1) \\ d_0 + d_1 + d_2 \equiv e_0 + e_1 + e_2 (mod\ c_0 + c_1 + c_2) \end{cases}$$

This implies:

$$\begin{cases} a_0 + d_0 \equiv b_0 + e_0 (mod\ c_0) \\ a_0 + a_1 + d_0 + d_1 \equiv b_0 + b_1 + e_0 + e_1 (mod\ c_0 + c_1) \\ a_0 + a_1 + a_2 + d_0 + d_1 + d_2 \equiv b_0 + b_1 + b_2 + e_0 + e_1 + e_2 (mod\ c_0 + c_1 + c_2) \end{cases}$$

So that $A + D \equiv B + E (mod\ C)$.

We can prove that $A - D \equiv B - E (mod\ C)$ by a similar.

2). By using a similar discussion, we can write:

$$\begin{cases} a_0 d_0 \equiv b_0 e_0 (mod\ c_0) \\ (a_0 + a_1)(d_0 + d_1) \equiv (b_0 + b_1)(e_0 + e_1)(mod\ c_0 + c_1) \\ (a_0 + a_1 + a_2)(d_0 + d_1 + d_2) \equiv (b_0 + b_1 + b_2)(e_0 + e_1 + e_2)(mod\ c_0 + c_1 + c_2) \end{cases}$$

Thus $A.D \equiv B.E(mod\ C)$.

3). Suppose that $gcd(A,B)=1$, then $gcd(a_0,b_0)=gcd(a_0+a_1,b_0+b_1)=gcd(a_0+a_1+a_2,b_0+b_1+b_2)=1$.

We put

$$T={a_0}^{-1}(mod\ b_0)+P_1[(a_0+a_1)^{-1}(mod\ b_0+b_1)-{a_0}^{-1}(mod\ b_0)]$$
$$+P_2[(a_0+a_1+a_2)^{-1}(mod\ b_0+b_1+b_2)-(a_0+a_1)^{-1}(mod\ b_0+b_1)]$$

$$A.T=a_0{a_0}^{-1}(mod\ b_0)+P_1[(a_0+a_1)(a_0+a_1)^{-1}(mod\ b_0+b_1)-a_0{a_0}^{-1}(mod\ b_0)]$$
$$+P_2[(a_0+a_1+a_2)(a_0+a_1+a_2)^{-1}(mod\ b_0+b_1+b_2)$$
$$-(a_0+a_1)(a_0+a_1)^{-1}(mod\ b_0+b_1)]=1+P_1(1-1)+P_2(1-1)=1$$

Thus $T=A^{-1}$.

Example:

Consider $A=5+4P_1+2P_2, B=2+P_1+P_2, C=3+4P_2$, we have:

$5\equiv 2(mod\ 3), 5+4=9\equiv(2+1)(mod\ 3+0), 5+4+2=11\equiv(2+1+1)(mod\ 3+0+4)$, thus $A\equiv B(mod\ C)$.

$gcd(A,B)=gcd(5,2)+P_1[gcd(9,3)-gcd(5,2)]+P_2[gcd(11,4)-gcd(9,3)]=1+P_1(3-1)+P_2(1-3)=1+2P_1-2P_2$.

Example.

Consider $A=2+P_1+P_2, B=3+P_1+P_2$, it is clear that $gcd(A,B)=1$.

$A^{-1}(mod\ B)=2^{-1}(mod\ 3)+P_1[3^{-1}(mod\ 4)-2^{-1}(mod\ 3)]+P_2[4^{-1}(mod\ 5)-3^{-1}(mod\ 4)]=2+P_1(3-2)+P_2(4-3)=2+P_1+P_2$.

Definition.

Let $A=a_0+a_1P_1+a_2P_2>0$ be a symbolic 2-plithogenic integer, we define $\varphi_S: 2-SP_Z \to 2-SP_Z$ such that:

$\varphi_S(A)=\varphi(a_0)+P_1[\varphi(a_0+a_1)-\varphi(a_0)]+P_2[\varphi(a_0+a_1+a_2)-\varphi(a_0+a_1)]$.

Where φ is the classical phi-Euler's function.

Example.

Take $A=3+5P_1-P_2,\ a_0=3, a_1=5, a_2=-1$. We have:

$a_0=3>0, a_0+a_1=8>0, a_0+a_1+a_2=7>0$, so that $A>0$.

$\varphi(a_0)=2, \varphi(a_0+a_1)=4, \varphi(a_0+a_1+a_2)=6$, hence:

$\varphi_S(A)=2+P_1[4-2]+P_2[6-4]=2+2P_1+2P_2$.

Theorem.

Let $A=a_0+a_1P_1+a_2P_2, M=m_0+m_1P_1+m_2P_2\in 2-SP_Z$ such that $gcd(A,M)=1$, then

$A^{\varphi_S(M)} \equiv 1(mod\ M)$.

Proof.

According to [ ]:

$$A^{\varphi_S(M)} = a_0{}^{\varphi(m_0)} + P_1\left[(a_0 + a_1)^{\varphi(m_0+m_1)} - a_0{}^{\varphi(m_0)}\right] + P_2\left[(a_0 + a_1 + a_2)^{\varphi(m_0+m_1+m_2)} - (a_0 + a_1)^{\varphi(m_0+m_1)}\right]$$

Since $gcd(A, M) = 1$, then $gcd(a_0, m_0) = gcd(a_0 + a_1, m_0 + m_1) = gcd(a_0 + a_1 + a_2, m_0 + m_1 + m_2) = 1$, so that:

$$\begin{cases} a_0{}^{\varphi(m_0)} \equiv 1(mod\ m_0) \\ (a_0 + a_1)^{\varphi(m_0+m_1)} \equiv 1(mod\ m_0 + m_1) \\ (a_0 + a_1 + a_2)^{\varphi(m_0+m_1+m_2)} \equiv 1(mod\ m_0 + m_1 + m_2) \end{cases}$$

Thus $A^{\varphi_S(M)} \equiv 1 + P_1(1 - 1) + P_2(1 - 1)(\ mod\ M) \equiv 1(mod\ M)$

Example.

Take $A = 2 + 3P_1 - 2P_2, M = 3 + 4P_1 + 4P_2$, we have $gcd(A, M) = 1$.

$\varphi_S(M) = 2 + P_1(6 - 2) + P_2(10 - 6) = 2 + 4P_1 + 4P_2$

$A^{\varphi_S(M)} = 2^2 + P_1[5^6 - 2^2] + P_2[3^{10} - 5^6]$

$2^2 \equiv 1(\ mod\ 3), 5^6 \equiv 1(\ mod\ 7), 3^{10} \equiv 1(\ mod\ 11)$, thus $A^{\varphi_S(M)} \equiv 1(mod\ M)$

Theorem.

Let $C = gcd(A, B) \in 2 - SP_Z$, then there exists $M, N \in 2 - SP_Z$ such that $C = MA + NB$.

Proof.

We assume that $C = gcd(A, B)$, then:

$$\begin{cases} c_0 = gcd(a_0, b_0) \\ c_1 = gcd(a_0 + a_1, b_0 + b_1) - gcd(a_0, b_0) \\ c_2 = gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) - gcd(a_0 + a_1, b_0 + b_1) \end{cases}$$

So there exists $m_0, n_0, m_1, n_1, m_2, n_2 \in Z$ such that:

$$\begin{cases} c_0 = m_0a_0 + n_0b_0 \\ c_0 + c_1 = m_1(a_0 + a_1) + n_1(n_0 + n_1) \\ c_0 + c_1 + c_2 = m_2(a_0 + a_1 + a_2) + n_2(b_0 + b_1 + b_2) \end{cases}$$

We put $M = m_0 + (m_1 - m_0)P_1 + (m_2 - m_1)P_2, N = n_0 + (n_1 - n_0)P_1 + (n_2 - n_1)P_2$, now let us compute:

$M.A = [m_0 + (m_1 - m_0)P_1 + (m_2 - m_1)P_2][a_0 + a_1P_1 + a_2P_2]$

$$M.A = m_0a_0 + P_1(m_0a_1 + m_1a_0 - m_0a_0 + m_1a_1 - m_0a_1) + P_2(m_0a_2 + m_2a_0 - m_1a_0 + m_2a_1 - m_1a_1 + m_1a_2 - m_0a_2 + m_2a_2 - m_1a_2)$$

$$M.A = m_0a_0 + P_1(m_1a_0 + m_1a_1 - m_0a_0) + P_2(m_2a_0 - m_1a_0 + m_2a_1 - m_1a_1 + m_1a_2 + m_2a_2)$$

$N.B = n_0b_0 + P_1(n_1b_0 + n_1b_1 - n_0b_0) + P_2(n_2b_0 - n_1b_0 + n_2b_1 - n_1b + n_1b_2 + n_2b_2)$

$$MA + NB = (m_0a_0 + n_0b_0) + P_1[m_1(a_0 + a_1) + n_1(b_0 + b_1) - n_0b_0 - m_0a_0] + P_2[m_2(a_0 + a_1 + a_2) + n_2(b_0 + b_1 + b_2) - m_1(a_0 + a_1) - n_1(b_0 + b_1)] = c_0 + c_1P_1 + c_2P_2 = C$$

Example.

Consider $A = 3 + 2P_1 + P_2, B = 3 + P_1 + 3P_2$, we have:

$a_0 = 3, a_1 = 2, a_2 = 1, b_0 = 3, b_1 = 1, b_2 = 3.$

$gcd(a_0, b_0) = 3, gcd(a_0 + a_1, b_0 + b_1) = gcd(5,4) = 1, gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2) = gcd(6,7) = 1$

Thus $gcd(A, B) = 3 + (1 - 3)P_1 + (1 - 1)P_2 = 3 - 2P_1.$

On the other hand, we have:

$$\begin{cases} 3 = 1.3 + 0.3 \; hence \; m_0 = 1, n_0 = 0 \\ 1 = 1.5 - 1.4 \; hence \; m_1 = 1, n_1 = -1 \\ 1 = -1.6 + 1.7 \; hence \; m_2 = -1, n_2 = 1 \end{cases}$$

Thus $M = 1 + (1 - 3)P_1 + (-1 - 1)P_2 = 1 - 2P_2$, $N = 0 + (-1 - 0)P_1 + (1 + 1)P_2 = -P_1 + 2P_2$

We can see that:

$$MA + NB = (1 - 2P_2)(3 + 2P_1 + P_2) + (-P_1 + 2P_2)(3 + P_1 + 3P_2) = 3 + 2P_1 + P_2 - 6P_2 - 4P_2 - 2P_2 - 3P_1 - P_1 - 3P_1 + 6P_2 + 2P_2 + 6P_2 = 3 - 2P_1 = C = gcd(A, B)$$

Definition.

Let $S = s_0 + s_1P_1 + s_2P_2 \in 2 - SP_Z$, we say that S is a 2-plithogenic semi prime if $s_0, s_0 + s_1, s_0 + s_1 + s_2$ are primes.

Example.

The 2-plithogenic integer $S = 2 + P_1 + 2P_2$ is a semi prime, that is because $s_0 = 2, s_0 + s_1 = 3, s_0 + s_1 + s_2 = 5$ are primes.

Application In Future Studies

Symbolic 2-plithogenic number theory as a new research direction maybe very useful branches of knowledge.

We suggest the following research points that symbolic 2-plithogenic integers may have a very big effect on it.

1-). How can we use symbolic 2-plithogenic integers in the improvement of crypto-systems [39-41], for example:

a). How can we build a 2-plithogenic version of RSA algorithm.

b). How can we build a 2-plithogenic version of Diffie-Hellman key exchange algorithm.

c). How can we build a 2-plithogenic version of EL-Gamal algorithm for cryptography.

2-). How can we a solve non-linear symbolic 2-plithogenic Diophantine equations and congruencies.

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Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution

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Abstract: In this paper we have successfully constructed the literal neutrosophic Kumaraswamy probability distribution. We mean by literal neutrosophic probability distribution that parameters of the distribution and the values that the random variable describing the distribution all take literal neutrosophic numbers of the form $\theta_N = a + bI\,; I^2 = I$ which differs from interval-valued neutrosophic probability distributions in which parameters of theses distributions take the form $\theta_N \in [L, U]$. We have derived the neutrosophic form of the probability density function, cumulative distribution function, statistical properties and maximum likelihood estimations of the parameters. Finally, a simulation study is performed to show the efficiency of the estimators provided by the neutrosophic MLE method.

Keywords: Literal Neutrosophic Numbers; Probability Distributions Theory; Maximum Likelihood Estimation; Kumaraswamy Distribution; Simulation.

1. Introduction

Neutrosophic probability distributions from one point of view are a generalization of the concept of crisp probability distributions and fuzzy probability distributions that allow for the modeling of indeterminacy and uncertainty. In traditional probability theory, probabilities are assigned to events and are represented as real numbers between 0 and 1. In neutrosophic probability theory, probabilities are assigned as a triplet of values (T, I, F) where T represents the degree of truth, I represents the degree of indeterminacy and F represents the degree of falsity. These values are used to model the degree to which an event is certain, uncertain, or false [1-6,32-38].

From another point of view according to the fact that neutrosophic field of reals $R(I)$ is a generalization of the field of reals R, literal neutrosophic probability theory is another way of generalizing crisp probability theory where each probability can be presented in the form $P = P_1 + P_2 I\ ; P_1, P_2 \in [0,1]\ , I^2 = I$ [7-14].

Neutrosophic probability distributions can be used in a variety of fields such as decision making, artificial intelligence, and data analysis, where traditional probability distributions are inadequate to model the uncertainty and indeterminacy present in real-world systems. [15-29]

The Kumaraswamy distribution [30] is a two-parameter continuous probability distribution that is commonly used in Bayesian statistics, reliability theory and other fields. The probability density function (PDF) of the classical Kumaraswamy distribution is defined as:

$$f(x;\ a,b) = a\ b\ x^{a-1}\ (1-x^{a})^{b-1}; x \in [0,1] \qquad (1)$$

Where a and b are the shape parameters of the distribution, and they are both positive real numbers. The cumulative distribution function (CDF) is given by:

$$F(x;\ a,b)\ =\ 1\ -\ (1-x^{a})^{b} \qquad (2)$$

The Kumaraswamy distribution is a generalization of the beta distribution, in the sense that the beta distribution is a special case of the Kumaraswamy distribution when $a\ =\ b$.

Many generalizations of the Kumaraswamy distribution were made to provide more flexibility in modeling various types of data, and they are widely used in various fields.

It's worth noting that the Kumaraswamy distribution has some desirable properties such as it is closed under convolution, it has increasing failure rate, and it has increasing hazard rate. These properties make it useful for modeling various types of data in different fields.

In this paper, we are going to construct the neutrosophic form of Kumaraswamy distribution and study some properties of it depending on the One-Dimensional AH-Isometry.

2. Preliminaries

Definition 2.1 [7]

Let $R(I) = \{a + bI \,; a, b \in R \,, I^2 = I\}$ be the neutrosophic field of reals. One-dimensional AH-isometry presented by Abobala and Hatip and its inverse are given by:

$$T: R(I) \to R^2 : \ T(a + bI) = (a, a + b) \qquad (5)$$

$$T^{-1}: R^2 \to R(I) : \ T^{-1}(a, b) = a + (b - a)I \qquad (6)$$

Note:

Let $\boldsymbol{x_N}, \boldsymbol{y_N} \in \boldsymbol{R(I)}$ and $\boldsymbol{T}$ be the AH-Isometry, since $\boldsymbol{T}$ is an algebraic isomorphism then it has the following properties:

1. $\boldsymbol{T(x_N + y_N) = T(x_N) + T(y_N)}$
2. $\boldsymbol{T(x_N \cdot y_N) = T(x_N) \cdot T(y_N)}$
3. $\boldsymbol{T}$ is correspondence one-to-one.

Definition 2.2 [8]

Let $f: R(I) \to R(I); f = f(x_N)$ where $x_N = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.3 [9]

Neutrosophic gamma function is a special function is defined by:

$$\Gamma(a_N) = \Gamma(a_1) + I\{\Gamma(a_1 + a_2) - \Gamma(a_1)\} \,; a_N = a_1 + a_2 I \,, I^2 = I$$

Where:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \ ; a > 0$$

Definition 2.4 [9]

Neutrosophic beta function is a special function can be defined in one of the following forms:

$$\beta(a_N, b_N) = \int_0^1 x^{a_N-1}(1-x)^{b_N-1}dx = \beta(a_1, b_1) + \{\beta(a_1 + a_2, b_1 + b_2) - \beta(a_1, b_1)\}I$$

$$= \frac{\Gamma(a_N)\Gamma(b_N)}{\Gamma(a_N + b_N)} \; ; a_N = a_1 + a_2 I \, , b_1 + b_2 I \, , I^2 = I$$

Definition 2.5 [9,11]

A neutrosophic random variable is defined as follows:

$$X_N = X_1 + X_2 I \; ; I^2 = I \, , 0 \cdot I = 0 \qquad (7)$$

Where X, Y are crisp random variables taking values on R.

Definition 2.6 [8]

Neutrosophic power of neutrosophic numbers is defined as follows:

$$(a + bI)^{c+dI} = a^c + I[(a+b)^{c+d} - a^c] \qquad (8)$$

Definition 2.7 [10]

Let $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample of random variables, we call:

$$L_N = L(\mathbb{X}_N; \Theta_N) = f(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} f(X_{iN}; \Theta_N) = L(\mathbb{X}; \Theta_1) + [L(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - L(\mathbb{X}; \Theta_1)]I \qquad (9)$$

The neutrosophic likelihood function.

Definition 2.8 [10]

Let $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample of random variables, we call:

$$\mathcal{L}_N = ln\, L(\mathbb{X}_N; \Theta_N) \qquad (10)$$

The neutrosophic loglikelihood function and we have:

$$\mathcal{L}_N = \mathcal{L}(\mathbb{X}; \Theta_1) + [\mathcal{L}(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - \mathcal{L}(\mathbb{X}; \Theta_1)]I \qquad (11)$$

4. Neutrosophic Kumaraswamy probability distribution

In this section we are going to construct the neutrosophic form of Kumaraswamy probability distribution function, cumulative probability distribution function, statistical properties and MLE estimations. Building this probability distribution and its properties will be in an algebraic approach depending on the one-dimensional AH-Isometry.

3.1 Probability density function and cumulative distribution function

Definition 3.1

Neutrosophic Kumaraswamy probability density function is defined as follows:

$$f(x_N;\ a_N, b_N) = a_N\, b_N\ x_N^{a_N-1}\ \left(1 - x_N^{a_N}\right)^{b_N-1}; x_N \in [0,1] \qquad (12)$$

Where: $x_N = x_1 + x_2 I, a_N = a_1 + a_2 I, b_N = b_1 + b_2 I, I^2 = I$

Theorem 3.1

The neutrosophic formal form of (12) is:

$$f(x_N;\ a_N, b_N) = a_1\, b_1\ x_1^{a_1-1}\ \left(1 - x_1^{a_1}\right)^{b_1-1}$$
$$+ I\left[\ (a_1 + a_2)(b_1 + b_2)\ (x_1 + x_2)^{a_1+a_2-1}\ \left(1 - (x_1 + x_2)^{(a_1+a_2)}\right)^{b_1+b_2-1}\right.$$
$$\left.- a_1\, b_1\ x_1^{a_1-1}\ \left(1 - x_1^{a_1}\right)^{b_1-1}\right]; x_1 \in [0,1] \&\ x_1 + x_2 \in [0,1]$$

Proof

$$T[f(x_N;\ a_N, b_N)] = T\left[a_N\, b_N\ x_N^{a_N-1}\ \left(1 - x_N^{a_N}\right)^{b_N-1}\right]$$
$$= T[a_N]T[b_N\ ]T[\ x_N^{a_N-1}]T\left[\left(1 - x_N^{a_N}\right)^{b_N-1}\right]$$
$$= (a_1, a_1 + a_2)(b_1, b_1$$
$$+ b_2)\left(x_1^{a_1-1}, (x_1 + x_2)^{a_1+a_2-1}\right)\left(\left(1 - x_1^{a_1}\right)^{b_1-1}, (1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$
$$= \left(a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}, (a_1 + a_2)(b_1\right.$$
$$\left.+ b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$
$$= (f(x_1; a_1, b_1), f(x_1 + x_2; a_1 + a_2, b_1 + b_2)\ )$$

Taking T^{-1}:s

$$f(x_N;\ a_N, b_N) = a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}$$
$$+ \left[(a_1 + a_2)(b_1 + b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right.$$
$$\left.- a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}\right] I$$
$$= f(x_1; a_1, b_1) + I[f(x_1 + x_2; a_1 + a_2, b_1 + b_2) - f(x_1; a_1, b_1)]$$

Theorem 3.2

Equation (12) represents probability density function in classical sense.

Proof

We have:

$$T\left[\int_0^1 f(x_N;\ a_N, b_N)\, dx_N\right] = \left(\int_0^1 f(x_1; a_1, b_1) dx_1 , \int_0^1 f(x_1 + x_2; a_1 + a_2, b_1 + b_2) d(x_1 + x_2)\right)$$

$$= \left(\int_0^1 a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1} dx_1 , \int_0^1 (a_1 + a_2)(b_1\right.$$

$$\left. + b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\, d(x_1 + x_2)\right)$$

$$= \left(-\int_0^1 d\left(1 - x_1^{a_1}\right)^{b_1} , -\int_0^1 d(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right) = (1,1)$$

So:

$$\int_0^1 f(x_N;\ a_N, b_N)\, dx_N = T^{-1}(1,1) = 1$$

Also, depending on [7] it is easy to see that $T[f(x_N;\ a_N, b_N)]$ are two continuous functions on $[0,1] \subseteq R$ so $f(x_N;\ a_N, b_N)$ is continuous on $[0,1]$.

Depending on previous results we can prove that given neutrosophic function is a neutrosophic probability density function in classical sense.

Theorem 3.3

Cumulative distribution function of neutrosophic Kumaraswamy distribution is:

$$F(x_N;\ a_N, b_N) = 1 - \left(1 - x_N^{a_N}\right)^{b_N} \qquad (13)$$

Proof

$$F(x_N;\ a_N, b_N) = \int_0^{x_N} f(t_N;\ a_N, b_N)\, dt_N$$

$$T[F(x_N;\ a_N,b_N)] = T\left[\int_0^{x_N} f(t_N;\ a_N,b_N)\,dt_N\right]$$

$$= \left(\int_0^{x_1} f(t_1;a_1,b_1)dt_1\,,\ \int_0^{x_1+x_2} f(t_1+t_2;a_1+a_2,b_1+b_2)d(t_1+t_2)\right)$$

$$= \left(\int_0^{x_1} a_1b_1t_1^{a_1-1}\left(1-t_1^{a_1}\right)^{b_1-1}dt_1\,,\ \int_0^{x_1+x_2} (a_1+a_2)(b_1 + b_2)(t_1+t_2)^{a_1+a_2}(1-(t_1+t_2)^{a_1+a_2-1})^{b_1+b_2-1}\,d(t_1+t_2)\right)$$

$$= \left(-\int_0^{x_1} d\left(1-t_1^{a_1}\right)^{b_1},-\int_0^{x_1+x_2} d(1-(t_1+t_2)^{a_1+a_2-1})^{b_1+b_2-1}\right)$$

$$= \left(1-\left(1-x_1^{a_1}\right)^{b_1}, 1-(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$

So:

$$F(x_N;\ a_N,b_N) = T^{-1}\left[1-\left(1-x_1^{a_1}\right)^{b_1}, 1-(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1}\right]$$

$$= 1-\left(1-x_1^{a_1}\right)^{b_1} + \left[1-(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1} - 1 + \left(1-x_1^{a_1}\right)^{b_1}\right]I$$

Which is the neutrosophic formal form of the function:

$$F(x_N;\ a_N,b_N)\ =\ 1\ -\ \left(1-x_N^{a_N}\right)^{b_N}$$

3.2 Statistical properties of Kumaraswamy distribution

Theorem 3.4

Let X_N be a neutrosophic random variable following Kumaraswamy distribution with parameters a_N, b_N then:

1) $E(X_N^r) = b_1\beta\left(\frac{r}{a_1}+1,b_1\right) + \left[(b_1+b_2)\beta\left(\frac{r}{a_1+a_2}+1,b_1+b_2\right) - b_1\beta\left(\frac{r}{a_1}+1,b_1\right)\right]I$

2) $E(X_N) = b_1\beta\left(\frac{1}{a_1}+1,b_1\right) + \left[(b_1+b_2)\beta\left(\frac{1}{a_1+a_2}+1,b_1+b_2\right) - b_1\beta\left(\frac{1}{a_1}+1,b_1\right)\right]I$

3) $V(X_N) = b_1\beta\left(\frac{2}{a_1}+1,b_1\right) + \left[(b_1+b_2)\beta\left(\frac{2}{a_1+a_2}+1,b_1+b_2\right) - b_1\beta\left(\frac{2}{a_1}+1,b_1\right)\right]I - \left[b_1\beta\left(\frac{1}{a_1}+1,b_1\right) + \left[(b_1+b_2)\beta\left(\frac{1}{a_1+a_2}+1,b_1+b_2\right) - b_1\beta\left(\frac{1}{a_1}+1,b_1\right)\right]I\right]^2$

4) Median= $\left(1-2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}} + \left[\left(1-2^{-\frac{1}{b_1+b_2}}\right)^{\frac{1}{a_1+a_2}} - \left(1-2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}}\right]I$

Proof

1) We have:

$$x_N^r f(x_N;\ a_N,b_N) = a_N\ b_N\ x_N^{a_N+r-1}\ \left(1-x_N^{a_N}\right)^{b_N-1}$$

$$T[x_N^r f(x_N;\ a_N, b_N)] = T[x_N^r]T[f(x_N;\ a_N, b_N)]$$
$$= (x_1^r, (x_1 + x_2)^r)\left(a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}, (a_1 + a_2)(b_1 + b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$
$$= \left(a_1 b_1 x_1^{a_1+r-1}\left(1 - x_1^{a_1}\right)^{b_1-1}, (a_1 + a_2)(b_1 + b_2)(x_1 + x_2)^{a_1+a_2+r-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$

So:

$$T\left[\int_0^1 x_N^r f(x_N;\ a_N, b_N)\, dx_N\right]$$
$$= \left(\int_0^1 a_1 b_1 x_1^{a_1+r-1}\left(1 - x_1^{a_1}\right)^{b_1-1} dx_1, \int_0^1 (a_1 + a_2)(b_1 + b_2)(x_1 + x_2)^{a_1+a_2+r-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1} d(x_1 + x_2)\right) = (L, R)$$

In L let $x_1^{a_1} = t$ then $x_1^r = t^{\frac{r}{a_1}}$ and $a_1 x_1^{a_1-1} dx_1 = dt$ so:

$$L = \int_0^1 b_1 t^{\frac{r}{a}}(1 - t)^{b_1-1} dt = b_1 \beta\left(\frac{r}{a_1} + 1, b_1\right)$$

In R similarly we let $(x_1 + x_2)^{a_1+a_2} = t$ so $(x_1 + x_2)^r = t^{\frac{r}{a_1+a_2}}$ and $(a_1 + a_2)(x_1 + x_2)^{a_1+a_2-1} d(x_1 + x_2) = dt$ that yields:

$$R = \int_0^1 (b_1 + b_2) t^{\frac{r}{a_1+a_2}}(1 - t)^{b_1+b_2-1} dt = (b_1 + b_2)\beta\left(\frac{r}{a_1 + a_2} + 1, b_1 + b_2\right)$$

Then we have:

$$T\left[\int_0^1 x_N^r f(x_N;\ a_N, b_N)\, dx_N\right] = \left(b_1 \beta\left(\frac{r}{a_1} + 1, b_1\right), (b_1 + b_2)\beta\left(\frac{r}{a_1 + a_2} + 1, b_1 + b_2\right)\right)$$

So:

$$E(X_N^r) = \int_0^1 x_N^r f(x_N;\ a_N, b_N)\, dx_N = T^{-1}\left(b_1 \beta\left(\frac{r}{a_1} + 1, b_1\right), (b_1 + b_2)\beta\left(\frac{r}{a_1 + a_2} + 1, b_1 + b_2\right)\right)$$
$$= b_1 \beta\left(\frac{r}{a_1} + 1, b_1\right) + \left[(b_1 + b_2)\beta\left(\frac{r}{a_1 + a_2} + 1, b_1 + b_2\right) - b_1 \beta\left(\frac{r}{a_1} + 1, b_1\right)\right] I$$

2) By substituting $r = 1$ we get the required formula directly.

3) Straightforward from definition of variance (see [9]).

4) Median is the point that 50% of the area under the density curve is preceded by it, so it satisfies the following:

$$\int_{0}^{Median} f(x_N;\ a_N, b_N)\, dx_N = 0.5$$

Or equivalently:

$$F(Median;\ a_N, b_N) = 1 - (1 - Median^{a_N})^{b_N} = 0.5$$

By solving the previous equation with respect to the Median we get:

$$Median = \left(1 - 2^{-\frac{1}{b_N}}\right)^{\frac{1}{a_N}}$$

Following rules of calculating neutrosophic powers presented in equation (8) we get:

$$Median = \left(1 - 2^{-\frac{1}{b_N}}\right)^{\frac{1}{a_N}} = \left(1 - 2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}} + \left[\left(1 - 2^{-\frac{1}{b_1+b_2}}\right)^{\frac{1}{a_1+a_2}} - \left(1 - 2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}}\right] I$$

4.3 Parameters' estimation using neutrosophic MLE method

Let $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from neutrosophic Kumaraswamy distribution presented in equation (12) then the neutrosophic likelihood function will be:

$$L_N = L(\mathbb{X}_N; \Theta_N) = f(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} f(X_{iN}; a_N, b_N) = \prod_{i=1}^{n} a_N\, b_N\, X_{iN}^{a_N-1} \left(1 - X_{iN}^{a_N}\right)^{b_N-1}$$

$$= a_N^n\, b_N^n \prod_{i=1}^{n} X_{iN}^{a_N-1} \prod_{i=1}^{n} \left(1 - X_{iN}^{a_N}\right)^{b_N-1}$$

So, the loglikelihood function will be:

$\mathcal{L}_N = lnL(\mathbb{X}_N; \Theta_N) = n \ln a_N + n \ln\ b_N + (a_N - 1) \sum_{i=1}^{n} \ln X_{iN} + (b_N - 1) \sum_{i=1}^{n} \ln(1 - X_{iN}^{a_N})$ (14)

Taking partial derivatives of equation (14) with respect to a_N, b_N yields to:

$$\frac{\partial}{\partial a_N}\mathcal{L}_N = \frac{n}{a_N} + \sum_{i=1}^{n} \ln X_{iN} + (b_N - 1) \sum_{i=1}^{n} \frac{-X_{iN}^{a_N} \ln X_{iN}^{a_N}}{1 - X_{iN}^{a_N}} \qquad (15)$$

$$\frac{\partial}{\partial b_N}\mathcal{L}_N = \frac{n}{b_N} + \sum_{i=1}^{n} \ln(1 - X_{iN}^{a_N}) \qquad (16)$$

Equations (15-16) are equivalent to the following four equations in R^2 (using the AH-Isometry):

$$\begin{cases} \frac{\partial}{\partial a_1}\mathcal{L}_1 = \frac{n}{a_1} + \sum_{i=1}^{n} \ln X_{i1} + (b_1 - 1)\sum_{i=1}^{n} \frac{-X_{i1}^{a_1} \ln X_{i1}^{a_1}}{1-X_{i1}^{a_1}} \\ \frac{\partial(\mathcal{L}_1+\mathcal{L}_2)}{\partial(a_1+a_2)} = \frac{n}{a_1+a_2} + \sum_{i=1}^{n} \ln(X_{i1}+X_{i2}) + (b_1+b_2-1)\sum_{i=1}^{n} \frac{-(X_{i1}+X_{i2})^{a_1+a_2} \ln(X_{i1}+X_{i2})^{a_1+a_2}}{1-(X_{i1}+X_{i2})^{a_1+a_2}} \end{cases}$$

(17)

$$\begin{cases} \frac{\partial}{\partial b_1}\mathcal{L}_1 = \frac{n}{b_1} + \sum_{i=1}^{n} \ln(1 - X_{i1}^{a_1}) \\ \frac{\partial(\mathcal{L}_1+\mathcal{L}_2)}{\partial(b_1+b_2)} = \frac{n}{b_1+b_2} + \sum_{i=1}^{n} \ln(1-(X_{i1}+X_{i2})^{a_1+a_2}) \end{cases} \quad (18)$$

Solving these sets of equations is not easy analytically, we will provide simulation study to show the efficiency of these neutrosophic MLE estimation.

4.4 Simulation study and random numbers generating

To do a simulation study we first derive a formula for random numbers generating noticing that equation (13) can be written as follows:

$$F(x_N;\, a_N, b_N) = 1 - \left(1 - x_N^{a_N}\right)^{b_N} = p_1 + p_2 I = P_N$$

Where P_N is neutrosophically uniform distributed on $[0,1]$ So:

$$1 - x_N^{a_N} = (1 - P_N)^{\frac{1}{b_N}}$$

$$x_N = \left(1 - (1 - P_N)^{\frac{1}{b_N}}\right)^{\frac{1}{a_N}} \quad (19)$$

Taking AH-isometry to equation (19) yields to the following two equations:

$$x_1 = \left(1 - (1 - p_1)^{\frac{1}{b_1}}\right)^{\frac{1}{a_1}} \quad (20)$$

$$x_1 + x_2 = \left(1 - (1 - p_1 - p_2)^{\frac{1}{b_1+b_2}}\right)^{\frac{1}{a_1+a_2}} \quad (19)$$

We can use equations (20-21) to generate random numbers following classical Kumaraswamy distribution with selected parameters, and takin T^{-1} to the generated numbers yields to neutrosophic Kumaraswamy distribution.

Now, performance of MLE estimators will be evaluated based on Monte Carlo simulation to the Kumaraswamy neutrosophic probability distribution with total replication of $N = 10000$ times and with sample sizes of 5,15,30,50 and 100 and with fixed parameters $a_N = 3 + 2I, b_N = 2 + 4I$.

Goodness of estimation is assessed depending on average bias and root mean square error defined below: [31]

$$AB = \frac{\sum_{i=1}^{N}\left(\hat{\theta}_{Ni} - \theta_N\right)}{N}$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\theta}_{Ni} - \theta_N\right)^2}{N}}$$

Where $\hat{\theta}_{Ni}$ is the ith estimator of θ_N.

Table 1. Simulation results of neutrosophic Kumaraswamy distribution parameters' estimation

N	Average $\hat{a}_N$	$RMSE(\hat{a}_N)$	$AB(\hat{a}_N)$	Average $\hat{b}_N$	$RMSE(\hat{b}_N)$	$AB(\hat{b}_N)$
5	$4.287 + 1.642I$	$2.380 - 0.237I$	$1.287 - 0.358I$	$3.311 + 2.109I$	$2.522 - 0.279I$	$1.311 - 1.891I$
15	$3.434 + 2.149I$	$1.109 + 0.390I$	$0.434 + 0.149I$	$2.586 + 3.365I$	$1.344 + 0.615I$	$0.586 - 0.635I$
30	$3.209 + 2.075I$	$0.714 + 0.270I$	$0.209 + 0.075I$	$2.297 + 3.953I$	$0.807 + 0.923I$	$0.297 - 0.047I$
50	$3.093 + 2.090I$	$0.503 + 0.233I$	$0.093 + 0.090I$	$2.139 + 4.123I$	$0.522 + 0.977I$	$0.139 + 0.123I$
100	$3.043 + 2.017I$	$0.339 + 0.169I$	$0.043 + 0.017I$	$2.063 + 4.140I$	$0.330 + 0.869I$	$0.063 + 0.140I$

Table (1) shows results of simulation analysis for neutrosophic Kumaraswamy distribution where we notice that average bias of estimators is when sample size increases, which proves by simulation that proposed estimators are asymptotically unbiased.

5. Conclusions and future research directions

We have derived the neutrosophic Kumaraswamy probability distribution function, cumulative distribution function and statistical properties of the distribution, such as the mean, median, variance, and general moments. Additionally, we have derived the maximum likelihood estimations of the distributions' parameters.

The simulation study demonstrated the efficiency of the derived estimators and have shown that the estimators are unbiased. These results indicate that the neutrosophic Kumaraswamy distribution and its associated estimators can be useful in a variety of applications, including those involving uncertain or incomplete information.

Overall, this work has contributed to the development of neutrosophic probability theory and has practical implications for data analysis in various fields. Further research can be done to explore the potential of the neutrosophic Kumaraswamy distribution in other statistical applications.

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A Review Study on Some Properties of The Structure of Neutrosophic Ring

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Abstract: determinacy is a philosophical concept which introduced by Smarandache and used in structure of mathematical systems. In this article we use this concept to introduced Particular Structure of neutrosophic ring and studied some theorem and properties according to classical axiomatic ring theory.

Keywords: Neutrosophic rings; Neutrosophic rings of Integers; Neutrosophic rings of Complex; Neutrosophic rings of Integers of modulo n.

1. Introduction

Neutrosophic ring established first time by Kandasamy and Smarandache in 2006 see [19], in this paper we introduced particular neutrosophic ring depend on classical axioms of ring theory and studied some theorems and properties of neutrosophic ring theory.

2. Neutrosophic Rings and Their Examples

In this section we introduced the concept of neutrosophic ring was introduced in 2006 by Kandasamy and Smarandache see [19] with examples, but by applying the axioms of classical ring theory with concept of indeterminate. The neutrosophic element as I where I is an indeterminate and I is such that $I^2 = I$. If $I^2 = I \nRightarrow I(I-1) = 0$ or any relation is just saying $I^2 = I$

Definition 2.1. [19] Let R be any ring. The neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and I under the operations of R.

Theorem 2.2. [19] Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is a ring.

Note. In sated of notation $\langle R \cup I \rangle$ and $\langle R\backslash\{0\} \cup I \rangle$, we use notation $R[I]$ and $R^*[I]$ respectively.

Definition 2.2. Let R be a nonempty set and the triple $(R, +, \bullet)$ be a ring, and consider the neutrosophic (NS):$R[\mathrm{I}] = \{a + bI: a, b \in R\}$, then the neutrosophic algebra structure (NAS):

$N(R) = \langle R[I], +, \bullet \rangle$ is called the neutrosophic associative ring which is a generated by I and R under operations $+$ " addition "and $\bullet$ " multiplications" respectively if satisfies the axiomatic conditions of ring:

NR1: For all x, y and $z \in N(R)$, $N(R) = \langle R[I], + \rangle$ is a neutrosophic an abelian group under addition;

NR2: For all x, y and $z \in N(R^*)$, $N(R^*) = \langle R^*[I], \bullet \rangle$ is a mathematical associative neutrosophic system under multiplications, that is, $N(R^*) = \langle R^*[I], \bullet \rangle$ is neutrosophic semi group and

NR3: $x \bullet (y + \mathrm{z}) = (x \bullet y) + (x \bullet z)$ and $(y + \mathrm{z}) \bullet x = (y \bullet x) + (z \bullet x)$ " left and right distribution laws".

Observations.

- If $N(R^*) = \langle R^*[I], \bullet \rangle$ has neutrosophic identity (or unit), then $N(R) = \langle R[I], +, \bullet \rangle$ is called a neutrosophic ring with a neutrosophic identity (or neutrosophic unit).
- If $N(R^*) = \langle R^*[I], \bullet \rangle$ has neutrosophic inverse, then $N(R) = \langle R[I], +, \bullet \rangle$ is called a neutrosophic ring with a neutrosophic inverse and the neutrosophic structure $N(R^*) = \langle R^*[I], \bullet \rangle$ is called neutrosophic group.
- If $N(R^*) = \langle R^*[I], \bullet \rangle$ is a neutrosophic abelian, that is, all $x, y \in N(R)$, we have

$x.y = y.x$, in addition,$N(R^*) = \langle R^*[I],\bullet\rangle$ is called a neutrosophic abelian group, consequently, the $N(R) = \langle R[I],+,\bullet\rangle$ is called a filed and denoted by $N(F) = \langle F[I],+,\bullet\rangle$.

Definition 2.3. Let R be a finite set and the triple $(R,+,\bullet)$ be a finite ring, then $N(R) = \langle R[I],+,\bullet\rangle$ is called a finite neutrosophic ring, otherwise, $N(R) = \langle R[I],+,\bullet\rangle$ is called is an infinite neutrosophic ring.

Definition 2.4. Let $N(R) = \langle R[I],+,\bullet\rangle$be a neutrosophic ring. Define the neutrosophic set: $N(C(R) = \{x \in N(R): xy = yx, \forall\, y \in N(R)\}$ which is called the neutrosophic center of $N(R)$. Also, $N(R)$ is abelian iff $N(R) = N(C(R)$.

Definition 2.5.[19] Let $\mathbb{Z}$ be a set of integer numbers and $\mathbb{Z}[I] = \{a + bI: a, b \in \mathbb{Z}\}$ be a neutrosophic- integer set, where $a + bI$ is a neutrosophic integer number.

Preposition 2.1. Let $(\mathbb{Z},+,\bullet)$ be a ring of integers under usual addition and multiplication, then the neutrosophic algebra structure (NAS): $N(\mathbb{Z}) = \langle \mathbb{Z}[I],+,\bullet\rangle$ is called the neutrosophic integer ring which is generated by I and $\mathbb{Z}$.

Proof. Let $(\mathbb{Z},+,\bullet)$ be a ring of integers under usual addition and multiplication and $\langle \mathbb{Z} \cup I\rangle = \{a + bI: a, b \in \mathbb{Z}\}$ be a neutrosophic- integer set, where $a + bI$ is a neutrosophic integer number. Then by proposition 2.1 in [5] , $N(\mathbb{Z}) = \langle \mathbb{Z}[I],+\rangle$ is a neutrosophic abelian group, so NR1 axioms is hold. Now Let , $N(\mathbb{Z}) = \langle \mathbb{Z}[I],\bullet\rangle$ such that all $x , y \in N(\mathbb{Z})$, then:

$$x \cdot y = \big((x_1 + x_2I) + (y_1 + y_2I)\big)$$

$$= \Big((x_1.y_1) + \Big(\big((x_1.y_2) + (x_2.y_1)\big) + (x_2.y_2)\Big)I\Big) \in, N(\mathbb{Z}) = \langle \mathbb{Z}[I],\cdot\rangle,$$ it's a closure,

moreover,

$$(x \cdot y) \cdot z = \big((x_1 + x_2I) + (y_1 + y_2I)\big) \cdot (z_1 + z_2I)$$

$$= \Big((x_1.y_1) + \Big(\big((x_1.y_2) + (x_2.y_1)\big) + (x_2.y_2)\Big)I\Big) \cdot (z_1 + z_2I)$$

$$= \left(\begin{array}{c}(x_1.y_1).z_1 \\ + \\ \Big((x_1.y_1).z_2 + \big((x_1.y_2) + (x_2.y_1) + (x_2.y_2)\big).z_1 + \big((x_1.y_2) + (x_2.y_1) + (x_2.y_2)\big).z_2\Big)I\end{array}\right)$$

$$=\begin{pmatrix} x_1.(y_1.z_1) \\ + \\ \big((x_1.y_1).z_2 + \big((x_1.y_2)+(x_2.y_1)+(x_2.y_2)\big).z_1 + \big((x_1.y_2)+(x_2.y_1)+(x_2.y_2)\big).z_2\big)I \end{pmatrix}$$

$$=\begin{pmatrix} x_1.(y_1.z_1) \\ + \\ \big((x_1.y_1).z_2 + \big((x_1.y_2).z_1+(x_2.y_1).z_1+(x_2.y_2).z_1\big) + \big((x_1.y_2).z_2+(x_2.y_1).z_2+(x_2.y_2).z_2\big)\big)I \end{pmatrix}$$

$$=\begin{pmatrix} x_1.(y_1.z_1) \\ + \\ \big(x_1.(y_1.z_2) + \big(x_1.(y_2.z_1)+x_2.(y_1.z_1)+x_2.(y_2.z_1)\big) + \big(x_1.(y_2.z_2)+x_2.(y_1.z_2)+x_2.(y_2.z_2)\big)\big)I \end{pmatrix}$$

$$= (x_1 + x_2I)\Big((y_1.z_1) + \big(\big((y_1.z_2)+(y_2.z_1)\big)+(y_2.z_2)\big)I\Big) = x.(y.z).$$

Hence the associative law is hold.

Finally, $x.(y + z) = (x_1 + x_2I).\big((y_1 + y_2I) + (z_1 + z_2I)\big)$

$$= (x_1 + x_2I).\big((y_1 + z_1) + (y_2 + z_2)I\big)$$

$$= \big(x_1.(y_1 + z_1) + \big(x_1.(y_2 + z_2) + x_2.(y_1 + z_1) + x_2.(y_2 + z_2)\big)I\big)$$

$$= \Big(\big((x_1.y_1) + (x_1.+z_1)\big) + \Big(\big((x_1.y_2) + (x_1.z_2)\big) + \big((x_2.y_1) + (x_2.z_1)\big) + \big((x_2.y_2) + (x_2.z_2)\big)\Big)I\Big)$$

$$= \Big((x_1.y_1) + \big((x_1.y_2) + (x_2.y_1) + (x_2.y_2)\big)I\Big) + \Big((x_1.+z_1) + \big((x_1.z_2) + (x_2.z_1) + (x_2.z_2)\big)I\Big)$$

$= \big((x_1 + x_2I).(y_1 + y_2I)\big) + \big((y_1 + y_2I).(z_1 + z_2I)\big) = (x.y) + (x.z)$. By similar procedure, we can deduce that: $(y + z).x = (y.x) + (z.x)$. Moreover there exists $1 \in, N(\mathbb{Z}) = \langle \mathbb{Z}[I], \bullet \rangle$such that $1.x = x.1 = x$. Hence $N(\mathbb{Z}) = \langle \mathbb{Z}[I], +, \bullet \rangle$ is neutrosophic integer ring with identity■. The neutrosophic integer ring will plays an important role in the study of neutrosophic ring theory.

Example 2.1. Let $(\mathbb{Z}^+ \cup \{0\}, +, \bullet)$ be a unit ring of positive integers under neutrosophic addition and multiplication, then the neutrosophic algebra structure (NAS): $N(\mathbb{Z}^+) = \{\langle \mathbb{Z}^+ \cup \{0\} \cup I \rangle, +, \cdot\}$ is called the neutrosophic unit integer ring which is a generated by I and $\mathbb{Z}^+ \cup \{0\}$.

Defintione2.6. (Number theory) Let $\mathbb{Z}$ be the set of integers and $x \in \mathbb{Z}$, then x is called even number if there exists $k \in \mathbb{Z}$ such that $x = 2k$. If $x, y \in \mathbb{Z}$ and both are even numbers, then $x+y$ and $x.y$ are even numbers. Because, $x+y=2k_1+2k_2=2(k_1+k_2)=2k_3$, where,

$k_3=(k_1+k_2)\in\mathbb{Z}$. Also, $x.y=(2k_1).(2k_2)=2\big(k_1.(2k_2)\big)=2k_3$, where, $k_3=\big(k_1.(2k_2)\big)\in\mathbb{Z}$.

Defintione2.7. (Neutrosophic Number Theory) Let $\mathbb{Z}[I]=\{a+bI: a,b\in\mathbb{Z}\}$ be the set of neutrosophic integers and $x\in\mathbb{Z}[I]$, then $x=x_1+x_2I$ is called the neutrosophic even number if, x_1 and x_2 are even number. So, $0, 2I, 4I, \dots, 2+2I, 2+4I, \dots$ etc, are neutrosophic even integers.

Example 2.2. Let $(\mathbb{Z}_{even},+,\bullet)$ be a ring of even integers without unit under neutrosophic addition and multiplication, then the neutrosophic algebra structure (NAS): $N(\mathbb{Z}_{even})=\{\langle\mathbb{Z}_{even}\cup I\rangle,+,\bullet\}$ is called the neutrosophic integer ring which is a generated by I and $\mathbb{Z}_{even}$. This is a neutrosophic integer ring without neutrosophic unit elements.

Definition 2.8.[19] Let $\mathbb{R}$ be a set of real numbers and $\langle\mathbb{R}\cup I\rangle=\{a+bI: a,b\in\mathbb{R}\}$ be a neutrosophic- real set, where $a+bI$ is a neutrosophic real number.

Preposition 2.2. Let $(\mathbb{R},+,\cdot)$ be a ring of real numbers under usual addition, then the neutrosophic algebra structure (NAS): $N(\mathbb{R})=\langle\mathbb{R}[I],+,\cdot\rangle$ is called the neutrosophic real ring with identity which is a generated by I and $\mathbb{R}$. In addition, $N(\mathbb{R})=\langle\mathbb{R}[I],+,\cdot\rangle$ is a neutrosophic real field**.**

Proof. By the same argument of preceding preposition 2.1. In addition, $N(\mathbb{R}^*)=\langle\mathbb{R}^*[I],\cdot\rangle$ is a commutative group. consider $a=a_1+a_2I\in\mathbb{R}(I)$. Suppose that $x=x_1+x_2I\in\mathbb{R}(I)$ is the neutrosophic inverse of a, that is,

$a.x=1\Leftrightarrow(a_1+a_2I).(x_1+x_2I)=1+0I$

$$\Leftrightarrow\Big((a_1.x_1)+\big((a_1.x_2)+(a_2.x_1)+(a_2.x_2)\big)I\Big)=1+0I.$$

$$\Rightarrow a_1.x_1=1 \text{ and } (a_1.x_2)+(a_2.x_1)+(a_2.x_2)=0.$$

$$\Rightarrow x_1=\frac{1}{a_1} \text{ and } (a_1+a_2)x_2+a_2.\frac{1}{a_1}=0\Rightarrow x_1=\frac{1}{a_1} \text{ and } x_2=-\frac{a_2}{a_{1(a_1+a_2)}}.$$ To check the axiom of inverse, $a.x=(a_1+a_2I).\left(\frac{1}{a_1}-\frac{a_2}{a_{1(a_1+a_2)}}I\right)$

$$=\left(\left(a_1.\frac{1}{a_1}\right)+\left(-\frac{a_1.a_2}{a_{1(a_1+a_2)}}\right)+\left(a_2.\frac{1}{a_1}\right)-\left(\frac{{a_2}^2}{a_{1(a_1+a_2)}}\right)I\right)$$

$= 1 + \left(\frac{-a_1.a_2+a_2.(a_1+a_2)-a_2{}^2}{a_{1(a_1+a_2)}}\right) I.$

$= 1 + \left(\frac{-a_1.a_2+a_2.a_1+a_2{}^2-a_2{}^2}{a_{1(a_1+a_2)}}\right) I.$

$= 1 + 0I = 1$.By similar way we have $x.a = 1$. Also for all $a, b \in \mathbb{R}(I)$,we have

$ab = ba$. Hence $N(\mathbb{R}) = \langle \mathbb{R}[I], +,\cdot \rangle$ is the neutrosophic field of real ■.

Definition 2.9.[19] Let $\mathbb{C}$ be a set of complex numbers and $\mathbb{C}[I] = \{a + bI : a, b \in \mathbb{C}\}$ be a neutrosophic- complex set, where $a + bI$ is a neutrosophic complex number.

Preposition 2.3. Let $(\mathbb{C}, +.\cdot)$ be a ring of complex numbers under usual addition, then the neutrosophic algebra structure (NAS): $N(\mathbb{C}) = \langle \mathbb{C}[I], +,\cdot \rangle$ is called the neutrosophic complex ring with identity which is a generated by I and $\mathbb{C}$. Moreover, $N(\mathbb{C}) = \langle \mathbb{C}[I], +,\cdot \rangle$ is the neutrosophic field of complex numbers.

Proof. Let $N(\mathbb{C}) = \langle \mathbb{C}[I], +,\cdot \rangle$ be the neutrosophic algebra structure and let

$a = a_1 + a_2 I$, $b = b_1 + b_2 I$ and $c = c_1 + c_2 I$ be three elements in $\mathbb{C}[I]$ Then $N(\mathbb{C}) = \langle \mathbb{C}[I], + \rangle$ is a neutrosophic complex abelian group by prop2.3 in [5]. Also, $N(\mathbb{C}^*) = \langle \mathbb{C}^*[I],\cdot \rangle$ is a neutrosophic commutative complex group, 1 is the neutrosophic identity element, now if we consider

$a = a_1 + a_2 I \in \mathbb{C}(I)$, $a_1, a_2 \in \mathbb{C}$, then suppose that $a^{-1} = \frac{1}{a_1} - \left(\frac{a_2}{a_{1(a_1+a_2)}}\right) I$ is the neutrosophic inverse element of a by the same argument in pervious proposition 2.2.Hence $N(\mathbb{C}^*) = \langle \mathbb{C}^*[I],\cdot \rangle$is a commutative neutrosophic complex group and consequently , $N(\mathbb{C}) = \langle \mathbb{C}[I], +,\cdot \rangle$ is neutrosophic field of complex■.

Theorem 2.2. Condiser $N(\mathbb{Z}_n) = \{\mathbb{Z}_n \cup I, \oplus_n, \otimes_n\}$ is a finite neutrosophic ring under addition and multiplication with modulo n. Moreover $N(\mathbb{Z}_n) = \{\mathbb{Z}_n \cup I, \oplus_n, \otimes_n\}$ is a finite neutrosophic ring under addition and multiplication with modulo n.In addition it is a field.

Proof. See theorems 2.4 and 2.5. in [5].

Example2.3. $N(\mathbb{Z}_3) = \langle \mathbb{Z}_3 [I], \oplus_3, \otimes_3 \rangle$ is a finite neutrosophic ring under addition and multiplication with modulo 3. Moreover, it's a finite neutrosophic field. As we know, $\mathbb{Z}_3 = \{0,1,2\}$ and,

$Z_3[I] = \{a + bI : a, b \in Z_3\} = \{0,1,2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$, to construct then the neutrosophic algebra structure (NAS):$N(Z_3) = \langle Z_3[I], \oplus_3 \rangle$ by the visualizing table as shown in table.2.1.

Table.2.1, of (NAS of $N(Z_3) = \langle Z_3[I], \oplus_3 \rangle$.

$\oplus_3$	0	1	2	I	$2I$	$1+I$	$1+2I$	$2+I$	$2+2I$
0	0	1	2	I	$2I$	$1+I$	$1+2I$	$2+I$	$2+2I$
1	1	2	0	$1+I$	$1+2I$	$2+I$	$2+2I$	I	$2I$
2	2	0	1	$2+I$	$2+2I$	I	$2I$	$1+I$	$1+2I$
I	I	$1+I$	$2+I$	$2I$	0	$1+2I$	1	$2+2I$	2
$2I$	$2I$	$1+2I$	$2+2I$	0	I	1	$2+I$	2	$2+I$
$1+I$	$1+I$	$2+I$	I	$1+2I$	1	$2+2I$	2	$2I$	0
$1+2I$	$1+2I$	$2+2I$	$2I$	1	$1+I$	2	$2+I$	0	I
$2+I$	$2+I$	I	$1+I$	$2+2I$	2	$2I$	0	$1+2I$	1
$2+2I$	$2+2I$	$2I$	$1+2I$	2	$2+I$	0	I	1	$1+I$

The (NAS) is a closure under operation $\oplus_3$ modulo 3 and associative, there exists identity element is zero and for any elements in x has inverse as shown in the table 2.2.

Table 2.2, of inverse element.

x	0	1	2	I	$2I$	$1+I$	$1+2I$	$2+I$	$2+2I$
x^{-1}	0	2	1	$2I$	I	$2+2I$	$2+I$	$1+2I$	$1+I$

The (NAS) $N(Z_3) = \langle Z_3[I], \oplus_3 \rangle$ is represents a neutrosophic commutative group (NS). In addition,

$\mathbb{Z}_3{}^* = \{1,2\}$ and $\mathbb{Z}_3{}^*[\mathrm{I}] = \{a + bI: a, b \in Z_3\} = \{1+I, 1+2I, 2+I, 2+2I\}$, to construct the neutrosophic algebra structure (NAS):$N(\mathbb{Z}_3{}^*) = \langle \mathbb{Z}_3{}^*[I], \otimes_3 \rangle$ by the visualizing table as shown in table.2.3.

Table.2.3, of (NAS of $N(\mathbb{Z}_3{}^*) = \langle \mathbb{Z}_3{}^*[I], \otimes_3 \rangle$.

$\otimes_3$	$1+I$	$1+2I$	$2+I$	$2+2I$
$1+I$	1	$1+2I$	$2+I$	2
$1+2I$	$1+2I$	$1+2I$	$2+I$	2
$2+I$	$2+I$	$2+I$	$1+2I$	$1+2I$
$2+2I$	2	$2+I$	$1+2I$	1

We see that $N(\mathbb{Z}_3{}^*) = \langle \mathbb{Z}_3{}^*[I], \otimes_3 \rangle$ is a neutrosophic semigroup, but in classical ring theory $\langle \mathbb{Z}_3{}^*, \otimes_3 \rangle$ is a group. Also, this table is a correction of table 2.1 in [5]. Moreover, NR3 is hold, for instance, $(1+I).\left((2+I)+(2+2I)\right) = (1+I).(4+3I) = 4+10I$and,

$(1+I).(2+I)+(1+I).(2+2I) = (2+4I)+(2+6I) = 4+10I.$ Hence $N(\mathbb{Z}_3) = \langle \mathbb{Z}_3\,[I], \oplus_3, \otimes_3 \rangle$ is a neutrosophic ring.

Theorem2.3. [6] Let $A, B,$ and C be three neutrosophic matrices of the same capacity, and consider x and y are two neutrosophic scalars, then:

i. $A + B = B + A$;

ii. $(A+B)+C = A+(B+C)$" " associative law";

iii. $A + 0 = A$;

iv. $x(A+B) = xA + xB$;

v. $(x+y)A = xA + yA$;

vi. $x(yA) = (xy)A$,and

vii. $1.A = A$

Theorem2.4.[6]. Let A, B, and C be three neutrosophic matrices which are defined under multiplication, with x is a neutrosophic scalars, then:

i. $(AB)C = A(BC)$ " associative law";

ii. $A(B+C) = AB + AC$ " left distributive law";

iii. $(B+C)A = BA + CA$ "right distributive law" and

iv. $x(AB) = (xA)B = A(xB)$.

v. $0A. = 0$, $B.0 = 0$. Where 0 is a neutrosophic zero matrix.

Theorem 2.5. Consider the $n - square$ neutrosophic matrix set $M_{n\times n} = \{a_{ij} + b_{ij}I : a_{ij}, b_{ij} \in \mathbb{R}, 0I = 0 \;\&\; I^2 = I\}$, such that $M_{n\times n}$ has inverse, that is $det\left(\left[a_{ij} + b_{ij}I\right]\right) \neq 0$, then $N(M_{n\times n}) = \left\{\left[a_{ij} + b_{ij}I\right], +, \times\right\}$,where " + " defined as definition 2.11 and " × " defined as definition 2.13 respectively in [4,6]. Then $N(M_{n\times n}) = \left\{\left[a_{ij} + b_{ij}I\right], +, \times\right\}$ is non- commutative neutrosophic ring with unit.

Proof.

NR1: $N(M) = \left\{\left[a_{ij} + b_{ij}I\right], +\right\}$ is a commutative group under +. By theorem2.2.[6]. From (i) to(iii) the neutrosophic inverse element:

$$A + (-A) = \left[a_{ij} + b_{ij}I\right] + \left[\left(-a_{ij}\right) + \left(-b_{ij}\right)I\right]$$

$$= \left[a_{ij} + \left(-a_{ij}\right) + \left(b_{ij} + \left(-b_{ij}\right)\right)I\right], \text{ for } i, j = 1,2,3, \dots, n$$

$= [0 + 0I] = 0$. By the same argument we have $-A + A = 0$. Hence,

$N(M_{n\times n}) = \left\{\left[a_{ij} + b_{ij}I\right], +\right\}$ is a neutrosophic abelian group.

NR2: $N(M) = \left\{\left[a_{ij} + b_{ij}I\right], \times\right\}$ is monoid according to theorems 2.2.and 2.3.[6].

NR3: From part (ii) and (iii) in theorem 2.4, the neutrosophic distributive law is hold. Hence

$N(M_{n\times n}) = \left\{\left[a_{ij} + b_{ij}I\right], +, \times\right\}$ is non- commutative neutrosophic ring with unit■.

Example 2.4. Consider the following two matrices: $A = \begin{bmatrix} 1+0I & 1+0I \\ 0+0I & 1+0I \end{bmatrix}$ and $B = \begin{bmatrix} 1+0I & 0+0I \\ 1+0I & 1+0I \end{bmatrix}$.

Then:$AB = \begin{bmatrix} 1+0I & 1+0I \\ 0+0I & 1+0I \end{bmatrix}\begin{bmatrix} 1+0I & 0+0I \\ 1+0I & 1+0I \end{bmatrix} = \begin{bmatrix} 2+0I & 1+0I \\ 1+0I & 1+0I \end{bmatrix}$, and,

$$BA = \begin{bmatrix} 1+0I & 0+0I \\ 1+0I & 1+0I \end{bmatrix}\begin{bmatrix} 1+0I & 1+0I \\ 0+0I & 1+0I \end{bmatrix} = \begin{bmatrix} 1+0I & 1+0I \\ 1+0I & 2+0I \end{bmatrix}, \text{ we see that } AB \neq BA.$$

Definition 2.10. Let $N(R) = \langle \mathrm{R}[I], , +,\cdot\rangle$be a neutrosophic ring contains a neutrosophic unit element and $x = (x_1 + x_2 I) \neq 0 \in N(R)$ (not necessarily to be a commutative neutrosophic ring), then x is called a neutrosophic unit in $N(\mathrm{R})$ if there exists a multiplication inverse y such that

$xy = \mathrm{y}x = 1$ and y denoted by x^{-1}.

Theorem 2.6. Consider $N(R) = \langle \mathrm{R}[I], , +,\cdot\rangle$ is a neutrosophic ring contains a neutrosophic unit Let $U(N(R)) = \{x \in N(R) : \exists\, y \in N(R) \ni xy = yx = 1\}$ be the set of all units. Then: $\langle U(N(R)),\cdot\rangle$ is a neutrosophic group under multiplication.

Proof. Since $1 \in N(\mathrm{R})$, then $1 \in U(N(R))$ and $U(N(R)) \neq \emptyset$. Suppose that $x, y \in U(N(R))$,then there exists $x^{-1}, y^{-1} \in N(\mathrm{R})$ such that $xx^{-1} = x^{-1}x = 1$ and $yy^{-1} = y^{-1}y = 1$. Now,

$(y^{-1}x^{-1})(xy) = 1$ and $(xy)(y^{-1}x^{-1}) = 1$ by theorem 3.2. part.2 in [5], hence $x.y \in U(N(R))$. Also, if $x \in U(N(R))$, then $x^{-1} \in U(N(R))$, therefore for all if $x \in U(N(R))$, there is a multiplication neutrosophic inverse $x^{-1} \in U(N(R))$. Moreover, $N(R) = \langle \mathrm{R}[I], , +,\cdot\rangle$ is a neutrosophic ring , then the multiplication is associative in particular of elements of $U(N(R))$ and consequently, $\langle U(N(R)),\cdot\rangle$ is a neutrosophic group.

Definition 2.9.[19]: Let $\langle R \cup I\rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I\rangle$ is said to be a neutrosophic subring if P itself is a neutrosophic ring under the operations of $\langle R \cup I\rangle$. It is essential that $P = \langle S \cup nI\rangle, n$ a positive integer where S is a subring of R. i.e. $\{P$ is generated by the subring S together with $n\,I. (n \in Z+)\}$. Note: Even if P is a ring and cannot be represented as $\langle S \cup nI\rangle$ where S is a subring of R then we do not call P a neutrosophic subring of $\langle R \cup I\rangle$.

Theorem 2.7. Consider $N(R) = \langle R[I], +, \bullet \rangle$ is neutrosophic ring and $N(S) \neq \emptyset \subseteq N(R)$, then $N(S)$ is called a neutrosophic subring of $N(R)$ **iff** :

1. $\forall\, a, b \in N(S) \Rightarrow a - b \in N(S)$, and,
2. $\forall\, a, b \in N(S) \Rightarrow ab \in N(S)$.

Note. If $N(S)$ is a neutrosophic subring of $N(R)$, then denoted by: $N(S) \preccurlyeq N(R)$.

Proof. Frist direction, consider $N(R) = \langle R[I], +, \bullet \rangle$ is neutrosophic ring and $N(S) \neq \emptyset \subseteq N(R)$**.** Assume that $a, b \in N(S) = \{a + bI: a, b \in S\}$

$\Rightarrow a - b = (a_1 + a_2 I) - (b_1 + b_2 I) = \big((a_1 - b_1) + (a_2 - b_2)I\big) \in N(S)$. Also,

$\Rightarrow a.b = (a_1 + a_2 I).(b_1 + b_2 I) = (a_1.b_1) + \big((a_1.b_2) + (a_2.b_1) + (a_2.b_2)\big)I \in N(S)$.

Conversely,

Suppose that $a + b$ and $ab \in N(S)$ for all $a, b \in N(S)$,then $N(S)$ its closure under addition, since $N(R) = \langle R[I], + \rangle$ is a commutative neutrosophic group,, then $N(S) = \langle R[I], + \rangle$ in particular elements is commutative neutrosophic group. Also, $N(R) = \langle R[I], \bullet \rangle$ is a neutrosophic semigroup, so

$N(S) = \langle S[I], \bullet \rangle$ is a neutrosophic semigroup in particular elements of $N(S)$. Finally, $N(R) = \langle R[I], +, \bullet \rangle$ has the property of NR3, so NR3 is hold in $N(S) = \langle S[I], +, \bullet \rangle$ for particular elements, therefore $N(S) = \langle S[I], +, \bullet \rangle$ is a neutrosophic ring■.

Example 2.5. Consider $N(\mathbb{Z}_6) = \langle \mathbb{Z}_6 [I], \oplus_6, \otimes_6 \rangle$is a finite neutrosophic ring under addition and multiplication with modulo 6,where $\mathbb{Z}_6 [I] = \{a + bI: a, b \in \mathbb{Z}_6\}$, that is,

$\mathbb{Z}_6 [I] = \{0,1,2,3,4,5,1 + I, \ldots ,1 + 5I, 2 + I, \ldots ,2 + 5I, 3 + I, \ldots ,3 + 5I, 4 + I, \ldots ,4 + 5I, 5 + I, \ldots ,5 + 5I\}$,and

Take $S[I] = \{0, 2I, 4I\} \subseteq \mathbb{Z}_6 [I]$. Then $S[I] \preccurlyeq \mathbb{Z}_6 [I]$.

Table.2.4, of (NAS of $N(S[I]) = \langle S[I], \oplus_6 \rangle$.

$\oplus_6$	0	$2I$	$4I$
0	0	$2I$	$4I$
$2I$	$2I$	$4I$	0
$4I$	$4I$	0	$2I$

We see that $S[I]$is closed under addition modulo 6.

Table.2.5, of (NAS of $NS[I]) = \langle S[I], \otimes_6 \rangle$.

$\oplus_6$	0	$2I$	$4I$
0	0	0	0
$2I$	0	$4I$	2I
$4I$	0	2I	$4I$

Since $S[I]$ is closed under multiplication modulo 6.

If $S[I] = \{0,2,4,2I,4I,2+2I,2+4I,4+2I,4+4I\}$, then $S[I] \preccurlyeq \mathbb{Z}_6\,[I]$, because $S[I]$ is closed under addition and multiplication of modulo 6.

Defintion2.11. Let $N(R) = \langle R[I],+,\bullet\rangle$ is neutrosophic ring**,** then the center of neutrosophic ring is denoted by $C(N(R))$ and defined by: $C(N(R)) = \{x \in R[I]: xy = yx, \forall\, y \in R[I]\}$.

Proposition 2.5. If $N(R) = \langle R[I],+,\bullet\rangle$ is neutrosophic ring contains a neutrosophic unit element, then $C(N(R)) \preccurlyeq R[I]$.

Proof. Since $1 = 1 + 0I \in C(N(R))$, then $C(N(R)) \neq \emptyset$. Suppose that $a, b \in C(N(R))$, now, since

$a \in C\big(N(R)\big) \Longrightarrow ax = xa, \forall\, x \in R[I]$

$\Leftrightarrow (a_1 + a_2I)(x_1 + x_2I) = (x_1 + x_2I)(a_1 + a_2I), \forall\, x \in R[I]$

$\Leftrightarrow (a_1.x_1) + \big((a_1.x_2) + (a_2.x_1) + (a_2.x_2)I\big) = (\,x_1.a_1) + \big((x_2.a_1) + (x_1.a_2) + (x_2.a_2)I\big), \forall\, x \in R[I]$.

Also, $b \in C\big(N(R)\big) \Longrightarrow bx = xb, \forall\, x \in R[I]$

$\Leftrightarrow (b_1 + b_2I)(x_1 + x_2I) = (x_1 + x_2I)(b_1 + b_2I), \forall\, x \in R[I]$

$\Leftrightarrow (b_1.x_1) + \big((b_1.x_2) + (b_2.x_1) + (b_2.x_2)I\big) = (\,x_1.b_1) + \big((x_2.b_1) + (x_1.b_2) + (x_2.b_2)I\big), \forall\, x \in R[I]$.

Hence $(a - b)x = \big((a_1 + a_2I) - (b_1 + b_2I)\big).(x_1 + x_2I)$

$= \big((a_1 + a_2I).(x_1 + x_2I) - (b_1 + b_2I)\big).(x_1 + x_2I)$

$= (a_1.x_1) + \big((a_1.x_2) + (a_2.x_1) + (a_2.x_2)I\big) - (\,x_1.b_1) + \big((x_2.b_1) + (x_1.b_2) + (x_2.b_2)I\big)$

$= (x_1.a_1) + \big((x_2.a_1) + (x_1.a_2) + (x_2.a_2)I\big) - (x_1.b_1) + \big((x_2.b_1) + (x_1.b_2) + (x_2.b_2)I\big)$

$= (x_1 + x_2I)(a_1 + a_2I) - (x_1 + x_2I)(b_1 + b_2I)$

$= (x_1 + x_2I)\big((a_1 + a_2I) - (b_1 + b_2I)\big)$

$= x(a - b)$. Hence $(a - b) \in C\big(N(R)\big)$. Moreover,

$(ab)x = (a_1 + a_2I)\big((b_1 + b_2I).(x_1 + x_2I)\big)$.

$= (a_1 + a_2I)\big((x_1 + x_2I).(b_1 + b_2I)\big)$.

$= \big((a_1 + a_2I).(x_1 + x_2I)\big).(b_1 + b_2I)$.

$= \big((x_1 + x_2I).(a_1 + a_2I)\big).(b_1 + b_2I)$.

$= (x_1 + x_2I).\big((a_1 + a_2I).(b_1 + b_2I).\big)$

$= x(ab)$. Therefore $ab \in C\big(N(R)\big)$. By theorem 2.6. $C\big(N(R)\big) \preccurlyeq R[I]$■.

Example 2.6. Consider $N(\mathrm{Z}_3) = \langle \mathrm{Z}_3[I], \oplus_3, \otimes_3 \rangle$is a finite neutrosophic ring under addition and multiplication with modulo 3, where $\mathrm{Z}_3[I] = \{a + bI : a, b \in \mathrm{Z}_3\}$, then $C(N(\mathrm{Z}_3) = \mathrm{Z}_3[I]$, because $\mathrm{Z}_3[I]$ is a commutative neutrosophic ring. Also $U\big(N(\mathrm{Z}_3)\big) = \{1 + I, 2 + 2I\}$.

3. Properties of Neutrosophic Elements in Neutrosophic Ring

Definition3.1. Let $N(\mathrm{R}) = \langle \mathrm{R}[I], +, \bullet \rangle$ be a neutrosophic commutative ring and $x \neq 0 \in N(\mathrm{R})$, then x is said to be a zero-divisor, if there exists $y \neq 0 \in N(R)$ such that $x.y = 0$.

Example 3.1. $N(\mathbb{Z}) = \langle \mathbb{Z}[I], +, \bullet \rangle, N(\mathbb{Q}) = \langle \mathbb{Q}[I], +, \bullet \rangle, N(\mathbb{R}) = \langle \mathbb{R}[I], +, \bullet \rangle$ and $N(\mathbb{C}) = \langle \mathbb{C}[I], +, \bullet \rangle$ has no zero divisor. Also $N(\mathrm{Z}_4) = \langle \mathrm{Z}_4[I], \oplus_4, \otimes_4 \rangle$ and $N(\mathrm{Z}_6) = \langle \mathrm{Z}_6[I], \oplus_6, \otimes_6 \rangle$ has no zero divisor, but

$\langle \mathrm{Z}_4, \oplus_4, \otimes_4 \rangle$ and $\langle \mathrm{Z}_6, \oplus_6, \otimes_6 \rangle$ in classical ring theory has zero divisor.

Definition3.2. Let $N(\mathrm{R}) = \langle \mathrm{R}[I], +, \bullet \rangle$ be a neutrosophic commutative ring, then $N(\mathrm{R})$ is called a neutrosophic integral domain, if $N(\mathrm{R})$ it has no zero divisor.

Example 3.2. All neutrosophic ring structure in pervious example are neutrosophic integral domain.

Theorem3.1. Consider $N(\mathrm{Z}_p) = \langle \mathrm{Z}_p\,[I], \oplus_p, \otimes_\mathrm{p} \rangle$ is a neutrosophic ring, then $N(\mathrm{Z}_p) = \langle \mathrm{Z}_p\,[I], \oplus_p, \otimes_\mathrm{p} \rangle$

Is not a neutrosophic field.

Proof. By pervious example 2.3.

Example 3.3. Consider the $N(M_{n\times n}) = \{[a_{ij} + b_{ij}I], +, \times\}$ is non- commutative neutrosophic ring with unit. Take $A = \begin{bmatrix} 0 & 2I \\ 0 & 4I \end{bmatrix} \neq 0$ and $B = \begin{bmatrix} 0 & 2+2I \\ 0 & 0 \end{bmatrix} \neq 0$, then: $AB = \begin{bmatrix} 0 & 2I \\ 0 & 4I \end{bmatrix}\begin{bmatrix} 0 & 2+2I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

Hence A and B are zero dvisors.

Definition3.3.[19] Let $N(\mathrm{R}) = \langle \mathrm{R}[I], +, \bullet \rangle$ be a neutrosophic ring. A characteristic of $N(\mathrm{R})$ is the smallest positive integer n (if there is one) such that $nx = 0, \forall\, x \in \mathrm{R}[I]$. If there is no such integer, we say that neutrosophic ring $\mathrm{R}[I]$ has characteristic zero, otherwise $\mathrm{R}[I]$ has characteristic n and denoted by $N(chR[I]) = n$.

Example 3.4. $N(\mathbb{Z}) = \langle \mathbb{Z}[I], +, \bullet \rangle, N(\mathbb{Q}) = \langle \mathbb{Q}[I], +, \bullet \rangle, N(\mathbb{R}) = \langle \mathbb{R}[I], +, \bullet \rangle$ and $N(\mathbb{C}) = \langle \mathbb{C}[I], +, \bullet \rangle$ have characteristic zero.

Proposition 3.1. Let $N(\mathrm{Z}_n) = \langle \mathrm{Z}_n[I], , \oplus_n, \otimes_\mathrm{n} \rangle$ be a neutrosophic ring. Then $N(ch\,\mathrm{Z}_n[I]) = n$.

Proof. By Principle of Mathematical Induction.

First, If $n = 1$, then $\mathrm{Z}_1[I] = \{a + bI : a, b \in \mathrm{Z}_1\} = \{0 + 0I\}$ and $1.(0 + 0I) = 0$, hence $N(ch\,\mathrm{Z}_1[I]) = 1$. Hence is true statement when $n = 1$.If $n = 2$, then $\mathrm{Z}_2[I] = \{a + bI : a, b \in \mathrm{Z}_2\} = \{0,1, I + 1 + I\}$ and $2.0 = 0, 2.1 = 0(\,mod 2), 2.I = 2I = 0(mod 2), 2.(1 + I) = 2 + 2I = 0\ (\,mod\ 2)$.herefore $N(ch\,\mathrm{Z}_2[I]) = 2$. Hence is true statement when $n = 2$.

Second. Suppose that, $n = k$, then $\mathrm{Z}_k[I] = \{a + bI : a, b \in \mathrm{Z}_k\}$

$\mathrm{Z}_k[I] = \{0,1,2, \dots, k-1, I, 2I, \dots, (k-1)I, 1+I, 1+2I, \dots, 1+(k-1)I, 2+I, 2+2I, \dots, 2+(k-1)I, \dots, (k-1)+I, (k-1)+2I, \dots, (k-1)+(k-1)I\}$. Such that $k.x = 0, \forall\, x \in \mathrm{Z}_k[I]$ is true stamen.

Third, to show that the statement $n = k + 1$ is also true, that is $(k+1).x = 0, \forall\, x \in \mathrm{Z}_{k+1}[I]$.Now

$(k+1).x = k.x + 1.x = 0 + x = x\ (\,mod\ (k+1)) \Longrightarrow (k+1).x = 0\ (mod\ (k+1)$. Hence,

$N(ch\, \mathbb{Z}_{k+1}[I]) = k + 1$. Is also true, we deduced that $N(ch\, \mathbb{Z}_n[I]) = n, \forall n \in \mathbb{N}$■.

Theorem3.2. Let $N(R)$ be a neutrosophic ring and x, y and $z \in N(R)$.Then:

1. $x.0 = 0.x = 0$;
2. $x.(-y) = (-x).y = -(xy)$;
3. $(-x).(-y) = xy$, and,

Proof.

1. $x.0 = (x_1 + x_2I).(0 + 0I) = (x_1 + x_2I).(0 + 0I)$

$$= (x_1.0 + (x_1.0 + x_2.0 + x_2.0)I)$$

$$= 0 + 0I = 0.$$ By similar way $0.x = 0$.

2. We have from (1) $0 = x.0 = (x_1 + x_2I)\big((-y_1 - y_2I) + (y_1 + y_2I)\big)$

$$= (x_1 + x_2I).(-y_1 - y_2I) + (x_1 + x_2I).(y_1 + y_2I)$$

(1).

Also, $0 = -(xy) + (xy) = -\big((x_1 + x_2I).(y_1 + y_2I)\big) + \big((x_1 + x_2I).(y_1 + y_2I)\big)$

(2).

From (1) and (2), we get:

$(x_1 + x_2I).(-y_1 - y_2I) + (x_1 + x_2I).(y_1 + y_2I)$

$= -\big((x_1 + x_2I).(y_1 + y_2I)\big) + \big((x_1 + x_2I).(y_1 + y_2I)\big)$. By theorem 3.2, part 3in [5], this is

implying that, $(x_1 + x_2I).(-y_1 - y_2I) = -\big((x_1 + x_2I).(y_1 + y_2I)\big) \Leftrightarrow x.(-y) = -(xy)$.

3. By the same procedure we can deduced that $(-x).y = -(xy)$. From (2), we have

$(-x).y = -(xy) \Rightarrow (-x).(-y) = -\big(x(-y)\big) = -\big((-x).y\big) = xy$■.

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On The Refined Neutrosophic Real Analysis Based on Refined Neutrosophic Algebraic AH-Isometry

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Abstract:

The objective of this paper is to study the basic concepts of real refined neutrosophic analysis by using the refined neutrosophic AH-isometry, where refined neutrosophic real famous functions such as polynomials, exponents, Gamma functions and logarithmic refined neutrosophic real functions will be presented and discussed in terms of formulas and theorems. Also, many related examples will be illustrated.

Keywords: refined neutrosophic function, refined neutrosophic AH-isometry, refined neutrosophic Gamma function

Introduction and Preliminaries

The concept of refined neutrosophic algebraic structure was released in 2020 by neutrosophic rings, groups, spaces, modules and matrices [1-10].

The main idea behind the refined neutrosophic algebraic structures is that they are considered as a new generalization of classical and neutrosophic structures and other similar structures respectively [11-15]. Also, the refined neutrosophic functions were suggested and discussed.

The Element I can be split into I_1, I_2 satisfying the following:

$I_1^2 = I_1, I_2^2 = I_2, I_1 \cdot I_2 = I_2 \cdot I_1 = I_1$.

The structure $R(I_1, I_2) = \{a + bI_1 + cI_2 \,; a, b, c \in R\}$ is called the refined neutrosophic field of reals. Let $f: R(I_1, I_2) \to R(I_1, I_2)$ be a function with one variable, i.e.,$f = f(X) \,; X \in R(I_1, I_2)$ then f is called a refined neutrosophic real function with one refined neutrosophic real variable.

To study the analytical properties of this type of functions we must use the refined AH-Isometry defined in [7] as follows:

$$T: R(I_1, I_2) \to R \times R \times R$$

$$T(a + bI_1 + cI_2) = (a, a + b + c, a + c)$$

And its inverse is defined as follows:

$$T^{-1}: R \times R \times R \to R(I_1, I_2)$$

$$T^{-1}(a, b, c) = a + (b - c)I_1 + (c - a)I_2$$

Example:

Let $f: R(I_1, I_2) \to R(I_1, I_2)$ be a function defined as follows:

$$f(X) = X^2 + I_1X - I_2 \,; X = x_0 + x_1I_1 + x_2I_2 \in R(I_1, I_2)$$

By using the refined AH-Isometry we can turn f into three classical real functions:

$$T[f(X)] = T(X^2) + T(I_1)T(X) - T(I_2)$$

$$= (x_0^2, (x_0 + x_1 + x_2)^2, (x_0 + x_2)^2) + (0,1,0)(x_0, x_0 + x_1 + x_2, x_0 + x_2)$$

$$- (0,1,1) = (x_0^2, (x_0 + x_1 + x_2)^2 + x_0 + x_1 + x_2 - 1, (x_0 + x_2)^2 - 1)$$

So that, the refined neutrosophic real function f has been splat into three classical real functions:

$$g: R \to R \,; g(x_0) = x_0^2$$

$$h: R \to R \,; h(x_0 + x_1 + x_2) = (x_0 + x_1 + x_2)^2 + x_0 + x_1 + x_2 - 1$$

$$l: R \to R \,; l(x_0 + x_2) = (x_0 + x_2)^2 - 1$$

In this work, we use the previous algebraic AH-isometry to define and study the real refined neutrosophic real analysis and functions as a continuing of efforts released to study neutrosophic analysis [16-18].

Main Discussion

Definition:

The neutrosophic real function $f: R(I_1, I_2) \to R(I_1, I_2)$ is called:

(a) Continuous if and only if corresponding functions g, h, l are continuous on R.

(b) Differentiable if and only if g, h, l are differentiable.

(c) Integrable if and only if g, h, l are integrable.

Example:

Take $f: R(I_1, I_2) \to R(I_1, I_2)\ ; f(X) = X^2 - I_1 + 2I_2$

$$T\big(F(X)\big) = (x_0^2, (x_0 + x_1 + x_2)^2, (x_0 + x_2)^2) - (0,1,0) + 2(0,1,1)$$
$$= (x_0^2, (x_0 + x_1 + x_2)^2 + 1, (x_0 + x_2)^2 + 2)$$

We have:

$g: R \to R\ ; g(x_0) = x_0^2$ is continuous, differentiable and integrable on R.

$h: R \to R\ ; h(x_0 + x_1 + x_2) = (x_0 + x_1 + x_2)^2 + 1$ is continuous, differentiable and integrable on R.

$l: R \to R\ ; l(x_0 + x_2) = (x_0 + x_2)^2 + 2$ is continuous, differentiable and integrable on R.

Thus f is continuous, differentiable and integrable on $R(I_1, I_2)$.

Now let's compute the derived function of f by using the refined AH-Isometry:

$g'(x_0) = 2x_0, h'(x_0 + x_1 + x_2) = 2(x_0 + x_1 + x_2), l'(x_0 + x_2) = 2(x_0 + x_2)$, thus:

$$f'(X) = T^{-1}\big(2x_0, 2(x_0 + x_1 + x_2), 2(x_0 + x_2)\big)$$
$$= 2x_0 + I_1[2(x_0 + x_1 + x_2) - 2(x_0 + x_2)] + I_2[2(x_0 + x_2) - 2x_0]$$
$$= 2x_0 + 2x_1 I_1 + 2x_2 I_2 = 2X$$

Same result can be found by direct computing where:

$f'(X) = 2X$.

Now let's integrate f directly:

$$\int f(X)dX = \frac{1}{3}X^3 + (-I_1 + 2I_2)X$$

The second is to integrate f by using refined AH-Isometry as follows:

$$\int g(x_0)dx_0 = \frac{x_0^3}{3}, \int h(x_0+x_1+x_2)d(x_0+x_1+x_2)$$
$$= \frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2), \int l(x_0+x_2)d(x_0+x_2)$$
$$= \frac{(x_0+x_2)^3}{3} + 2(x_0+x_2)$$

So:

$$T\left(\int f(X)d(X)\right) = \left(\frac{x_0^3}{3}, \frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2), \frac{(x_0+x_2)^3}{3} + 2(x_0+x_2)\right)$$

Thus:

$$\int f(X)d(X) = T^{-1}\left(\frac{x_0^3}{3}, \frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2), \frac{(x_0+x_2)^3}{3} + 2(x_0+x_2)\right)$$
$$= \frac{x_0^3}{3} + I_1\left[\frac{(x_0+x_1+x_2)^3}{3} + (x_0+x_1+x_2) - \frac{(x_0+x_2)^3}{3} - 2(x_0+x_2)\right]$$
$$+ I_2\left[\frac{(x_0+x_2)^3}{3} + 2(x_0+x_2) - \frac{x_0^3}{3}\right].$$

It is easy to catch that:

$$T\left(\int f\right) = T\left(\frac{X^3}{3} - I_1X + I_2X\right) = \left(\int g, \int h, \int l\right)$$

Definition:

Let $R(I_1, I_2) = \{a + bI_1 + cI_2\ ; a, b, c \in R\}$ be the refined neutrosophic field of reals, we say that $a_0 + a_1I_1 + a_2I_2 \leq_N b_0 + b_1I_1 + b_2I_2$ if and only if $a_0 \leq b_0, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2, a_0 + a_2 \leq b_0 + b_2$.

Theorem 1:

The previous relation is a partial order relation.

Proof:

Let $x = a_0 + a_1I_1 + a_2I_2, y = b_0 + b_1I_1 + b_2I_2, z = c_0 + c_1I_1 + c_2I_2 \in R(I_1, I_2)$, we have:

$x \leq x$ because $a_0 \leq a_0, a_0 + a_1 + a_2 \leq a_0 + a_1 + a_2, a_0 + a_2 \leq a_0 + a_2$

Assume that $x \leq y$ and $y \leq x$ so: $a_0 \leq b_0, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2, a_0 + a_2 \leq b_0 + b_2$ and $b_0 \leq a_0, b_0 + b_1 + b_2 \leq a_0 + a_1 + a_2, b_0 + b_2 \leq a_0 + a_2$ which means that $a_0 = b_0, a_0 + a_1 + a_2 = b_0 + b_1 + b_2, a_0 + a_2 = b_0 + b_2,$ we conclude that $a_0 = b_0, a_1 = b_1, a_2 = b_2$ so $x = y$

Suppose that $x \leq y$ and $y \leq z$ so: $a_0 \leq b_0, a_0 + a_1 + a_2 \leq b_0 + b_1 + b_2, a_0 + a_2 \leq b_0 + b_2$and $b_0 \leq c_0, b_0 + b_1 + b_2 \leq c_0 + c_1 + c_2, b_0 + b_2 \leq c_0 + c_2$ which yields $a_0 \leq c_0, a_0 + a_1 + a_2 \leq c_0 + c_1 + c_2, a_0 + a_2 \leq c_0 + c_2$ which means that $x \leq z$

Finally, we conclude that $\leq_N$ is a partial order relation.

Computing Refined Neutrosophic Powers in $R(I_1, I_2)$

we call $(a_0 + a_1 I_1 + a_2 I_2)^{n_0 + n_1 I_1 + n_2 I_2}; a_0, a_1, a_2, n_0, n_1, n_2 \in R$a refined neutrosophic power. Here will present a theorem helps in finding such powers:

Theorem:

$$(a_0 + a_1 I_1 + a_2 I_2)^{n_0 + n_1 I_1 + n_2 I_2}$$
$$= a_0^{n_0} + [(a_0 + a_1 + a_2)^{n_0 + n_1 + n_2} - (a_0 + a_2)^{n_0 + n_2}] I_1$$
$$+ \left[(a_0 + a_2)^{n_0 + n_2} - a_0^{n_0}\right] I_2$$

Proof:

Taking refined AH-Isometry to the left side yields:

$$T[(a_0 + a_1 I_1 + a_2 I_2)^{n_0 + n_1 I_1 + n_2 I_2}] = (a_0^{n_0}, (a_0 + a_1 + a_2)^{n_0 + n_1 + n_2}, (a_0 + a_2)^{n_0 + n_2})$$

Now taking inverse isometry T^{-1} we get:

$$(a_0 + a_1 I_1 + a_2 I_2)^{n_0 + n_1 I_1 + n_2 I_2} = T^{-1}\left(a_0^{n_0}, (a_0 + a_1 + a_2)^{n_0 + n_1 + n_2}, (a_0 + a_2)^{n_0 + n_2}\right)$$
$$= a_0^{n_0} + [(a_0 + a_1 + a_2)^{n_0 + n_1 + n_2} - (a_0 + a_2)^{n_0 + n_2}] I_1$$
$$+ \left[(a_0 + a_2)^{n_0 + n_2} - a_0^{n_0}\right] I_2$$

Example:

let $x_N = (3 + 2I_1 - 2I_2)^{1 + I_1 + 2I_2}$, we have: $T(x_N) = T[(3 + 2I_1 - 2I_2)^{1 + I_1 + 2I_2}] = (3,3,1)^{(1,4,3)} = (1,0,1)$, which yields that:

$$x_N = T^{-1}(3,81,1) = 3 + 80I_1 - 2I_2$$

If our result is right, then $(3 + 80I_1 - 2I_2)^{\frac{1}{1 + I_1 + 2I_2}}$ should be equal to $3 + 2I_1 - 2I_2$.

Let $y_N = (3 + 80I_1 - 2I_2)^{\frac{1}{1 + I_1 + 2I_2}}$ then $T(y_N) = T\left[(3 + 80I_1 - 2I_2)^{\frac{1}{1 + I_1 + 2I_2}}\right] = (3,81,1)^{\frac{(1,1,1)}{(1,4,3)}} = (3,81,1)^{\left(1, \frac{1}{4}, \frac{1}{3}\right)} = (3,3,1)$

So: $y_N = T^{-1}(3,3,1) = 3 + 2I_1 - 2I_2$

Refined Neutrosophic Trigonometric Functions:

Here we are going to present some definitions and theorems related to refined neutrosophic trigonometric functions which are functions in $\theta_N = \theta_0 + \theta_1 I_1 + \theta_2 I_2\ ; \theta_0, \theta_1, \theta_2 \in R$

Theorems:

Let $R(I_1, I_2)$ be refined neutrosophic field of reals then:

1. $\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \sin\theta_0 + [\sin(\theta_0 + \theta_1 + \theta_2) - \sin(\theta_0 + \theta_2)]I_1 + [\sin(\theta_0 + \theta_2) - \sin\theta_0]I_2$
2. $\cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \cos\theta_0 + [\cos(\theta_0 + \theta_1 + \theta_2) - \cos(\theta_0 + \theta_2)]I_1 + [\cos(\theta_0 + \theta_2) - \cos\theta_0]I_2$
3. $\tan(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \tan\theta_0 + [\tan(\theta_0 + \theta_1 + \theta_2) - \tan(\theta_0 + \theta_2)]I_1 + [\tan(\theta_0 + \theta_2) - \tan\theta_0]I_2$
4. $\sin^2(\theta_0 + \theta_1 I_1 + \theta_2 I_2) + \cos^2(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = 1$
5. $-1 \le \sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) \le 1$
6. $-1 \le \cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2) \le 1$

Proof:

1. $\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} - e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2i}\ ; i^2 = -1$

$$T[\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2)] = T\left[\frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} - e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2i}\right]$$

$$= \frac{1}{2i}\left(e^{\theta_0} - e^{-\theta_0}, e^{\theta_0 + \theta_1 + \theta_2} - e^{-(\theta_0 + \theta_1 + \theta_2)}, e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}\right)$$

So:

$$\sin(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \frac{1}{2i}T^{-1}\left(e^{\theta_0} - e^{-\theta_0}, e^{\theta_0 + \theta_1 + \theta_2} - e^{-(\theta_0 + \theta_1 + \theta_2)}, e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}\right)$$

$$= \frac{e^{\theta_0} - e^{-\theta_0}}{2i} + \left[\frac{e^{\theta_0 + \theta_1 + \theta_2} - e^{-(\theta_0 + \theta_1 + \theta_2)}}{2i} - \frac{e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}}{2i}\right]I_1 + \left[\frac{e^{\theta_0 + \theta_2} - e^{-(\theta_0 + \theta_2)}}{2i} - \frac{e^{\theta_0} - e^{-\theta_0}}{2i}\right]I_2$$

$$= \sin\theta_0 + [\sin(\theta_0 + \theta_1 + \theta_2) - \sin(\theta_0 + \theta_2)]I_1 + [\sin(\theta_0 + \theta_2) - \sin\theta_0]I_2$$

2. $\cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2) = \frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} + e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2}$

$$T[\cos(\theta_0 + \theta_1 I_1 + \theta_2 I_2)] = T\left[\frac{e^{\theta_0 + \theta_1 I_1 + \theta_2 I_2} + e^{-(\theta_0 + \theta_1 I_1 + \theta_2 I_2)}}{2}\right]$$

$$= \frac{1}{2}\left(e^{\theta_0} + e^{-\theta_0}, e^{\theta_0 + \theta_1 + \theta_2} + e^{-(\theta_0 + \theta_1 + \theta_2)}, e^{\theta_0 + \theta_2} + e^{-(\theta_0 + \theta_2)}\right)$$

So:

$$\cos(\theta_0+\theta_1I_1+\theta_2I_2)=\frac{1}{2}T^{-1}\left(e^{\theta_0}+e^{-\theta_0},e^{\theta_0+\theta_1+\theta_2}+e^{-(\theta_0+\theta_1+\theta_2)},e^{\theta_0+\theta_2}+e^{-(\theta_0+\theta_2)}\right)$$

$$=\frac{e^{\theta_0}+e^{-\theta_0}}{2}+\left[\frac{e^{\theta_0+\theta_1+\theta_2}+e^{-(\theta_0+\theta_1+\theta_2)}}{2}-\frac{e^{\theta_0+\theta_2}+e^{-(\theta_0+\theta_2)}}{2}\right]I_1$$

$$+\left[\frac{e^{\theta_0+\theta_2}+e^{-(\theta_0+\theta_2)}}{2}-\frac{e^{\theta_0}+e^{-\theta_0}}{2}\right]I_2$$

$$=\cos\theta_0+[\cos(\theta_0+\theta_1+\theta_2)-\cos(\theta_0+\theta_2)]I_1$$

$$+[\cos(\theta_0+\theta_2)-\cos\theta_0]I_2$$

3. Similar to 1 and 2.

4. Using refined neutrosophic powers theorem we get:

$$\sin^2(\theta_0+\theta_1I_1+\theta_2I_2)=\sin^2\theta_0+[\sin^2(\theta_0+\theta_1+\theta_2)-\sin^2(\theta_0+\theta_2)]I_1$$

$$+[\sin^2(\theta_0+\theta_2)-\sin^2\theta_0]I_2$$

Also:

$$\cos^2(\theta_0+\theta_1I_1+\theta_2I_2)$$

$$=\cos^2\theta_0+[\cos^2(\theta_0+\theta_1+\theta_2)-\cos^2(\theta_0+\theta_2)]I_1$$

$$+[\cos^2(\theta_0+\theta_2)-\cos^2\theta_0]I_2$$

So:

$$\sin^2(\theta_0+\theta_1I_1+\theta_2I_2)+\cos^2(\theta_0+\theta_1I_1+\theta_2I_2)$$

$$=\sin^2\theta_0+[\sin^2(\theta_0+\theta_1+\theta_2)-\sin^2(\theta_0+\theta_2)]I_1$$

$$+[\sin^2(\theta_0+\theta_2)-\sin^2\theta_0]I_2+\cos^2\theta_0$$

$$+[\cos^2(\theta_0+\theta_1+\theta_2)-\cos^2(\theta_0+\theta_2)]I_1+[\cos^2(\theta_0+\theta_2)-\cos^2\theta_0]I_2$$

$$=\sin^2\theta_0+\cos^2\theta_0+[(\sin^2(\theta_0+\theta_1+\theta_2)+\cos^2(\theta_0+\theta_1+\theta_2)$$

$$-\sin^2(\theta_0+\theta_2)-\cos^2(\theta_0+\theta_2)]I_1$$

$$+[\sin^2(\theta_0+\theta_2)+\cos^2(\theta_0+\theta_2)-\sin^2\theta_0-\cos^2\theta_0]I_2$$

$$=1+[1-1]I_1+[1-1]I_2=1$$

5. Since $T[\sin(\theta_0+\theta_1I_1+\theta_2I_2)]=(\sin\theta_0,\sin(\theta_0+\theta_1+\theta_2),\sin(\theta_0+\theta_2))$

And it is known that $(-1,-1,-1)\le(\sin\theta_0,\sin(\theta_0+\theta_1+\theta_2),\sin(\theta_0+\theta_2))\le(1,1,1)$

Also $T^{-1}(-1,-1,-1)=-1+(-1+1)I_1+(-1+1)I_2=-1$

$$T^{-1}(1,1,1)=1+(1-1)I_1+(1-1)I_2=1$$

So the theorem holds.

6. Similar to 5.

Refined Neutrosophic Exponential and Logarithmic Functions:

Theorem:

The refined neutrosophic exponential function form is

$$e^{x_0+x_1I_1+x_2I_2} = e^{x_0} + (e^{x_0+x_1+x_2} - e^{x_0+x_2})I_1 + (e^{x_0+x_2} - e^{x_0})I_2$$

Proof:

$$T[e^{x_0+x_1I_1+x_2I_2}] = e^{(x_0, x_0+x_1+x_2, x_0+x_2)} = (e^{x_0}, e^{x_0+x_1+x_2}, e^{x_0+x_2})$$

Thus:

$$e^{x_0+x_1I_1+x_2I_2} = T^{-1}(e^{x_0}, e^{x_0+x_1+x_2}, e^{x_0+x_2}) = e^{x_0} + (e^{x_0+x_1+x_2} - e^{x_0+x_2})I_1 + (e^{x_0+x_2} - e^{x_0})I_2$$

Theorem:

The refined neutrosophic logarithmic function form is

$$\ln(x_0 + x_1I_1 + x_2I_2) = \ln x_0 + (\ln(x_0 + x_1 + x_2) - \ln(x_0 + x_2))I_1 + (\ln(x_0 + x_2) - \ln x_0)I_2$$

Proof:

We will search for $a_0, a_1, a_2 \in R$ where:

$$\ln(x_0 + x_1I_1 + x_2I_2) = a_0 + a_1I_1 + a_2I_2$$

Taking inverse function, we have:

$$x_0 + x_1I_1 + x_2I_2 = e^{a_0+a_1I_1+a_2I_2} = e^{a_0} + (e^{a_0+a_1+a_2} - e^{a_0+a_2})I_1 + (e^{a_0+a_2} - e^{a_0})I_2$$

Corresponding to the last equality we get:

$$x_0 = e^{a_0} \Rightarrow a_0 = \ln x_0$$

$$x_2 = e^{a_0+a_2} - e^{a_0} = x_0e^{a_2} - x_0 \Rightarrow e^{a_2} = \frac{x_2 + x_0}{x_0} \Rightarrow a_2 = \ln(x_2 + x_0) - \ln x_0$$

$$x_1 = e^{a_0+a_1+a_2} - e^{a_0+a_2} = x_0\frac{x_2 + x_0}{x_0}(e^{a_1} - 1) = (x_2 + x_0)(e^{a_1} - 1) \Rightarrow e^{a_1} = \frac{x_0 + x_1 + x_2}{x_0 + x_2}$$

$$\Rightarrow a_1 = \ln(x_0 + x_1 + x_2) - \ln(x_0 + x_2)$$

So:

$$\ln(x_0 + x_1I_1 + x_2I_2) = \ln x_0 + [\ln(x_0 + x_1 + x_2) - \ln(x_0 + x_2)]I_1 + [\ln(x_2 + x_0) - \ln x_0]I_2$$

Some Refined Neutrosophic Special Functions:

Refined Neutrosophic Gamma Function:

We can find the value of refined neutrosophic gamma function at refined neutrosophic point $a_N = a_0 + a_1I_1 + a_2I_2$ using the formula:

$$\Gamma(a_N) = \Gamma(a_0) + [\Gamma(a_0 + a_1 + a_2) - \Gamma(a_0 + a_2)]I_1 + [\Gamma(a_0 + a_2) - \Gamma(a_0)]I_2$$

Where:

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx \ ; a > 0$$

Proof:

Let $f(x) = x^{a_N - 1}e^{-x}$, so:

$$T\big(f(x)\big) = (x^{a_0-1}e^{-x}, x^{a_0+a_1+a_2-1}e^{-x}, x^{a_0+a_2-1}e^{-x})$$

Then:

$$T\left(\int_0^\infty f(x)dx\right) = \left(\int_0^\infty x^{a_0-1}e^{-x}\,dx, \int_0^\infty x^{a_0+a_1+a_2-1}e^{-x}\,dx, \int_0^\infty x^{a_0+a_2-1}e^{-x}\,dx\right)$$

$$= \big(\Gamma(a_0), \Gamma(a_0+a_1+a_2), \Gamma(a_0+a_2)\big)$$

Taking T^{-1} yields to:

$$\Gamma(a_N) = \Gamma(a_0) + [\Gamma(a_0+a_1+a_2) - \Gamma(a_0+a_2)]I_1 + [\Gamma(a_0+a_2) - \Gamma(a_0)]I_2$$

Remark:

Neutrosophic gamma function $\Gamma(a_N)$ is defined when $a_N >_N 0_N$ i.e., $a_0 > 0, a_0 + a_1 + a_2 > 0, a_0 + a_2 > 0$.

Examples:

$\Gamma(I_1 + I_2)$ is undefined because $I_1 + I_2 = 0 + 1 \cdot I_1 + 1 \cdot I_2$ and $0 \ngtr 0$.

$$\Gamma(a_N) = \Gamma(a_0) + [\Gamma(a_0+a_1+a_2) - \Gamma(a_0+a_2)]I_1 + [\Gamma(a_0+a_2) - \Gamma(a_0)]I_2$$

$$\Gamma(0.5 + 2I_1 + I_2) = \Gamma(0.5) + [\Gamma(3.5) - \Gamma(1.5)]\mathrm{I}_1 + [\Gamma(1.5) - \Gamma(0.5)]\mathrm{I}_2$$

$$= \sqrt{\pi} + \big(2.5 * 1.5 * 0.5 * \sqrt{\pi} - 0.5 * \sqrt{\pi}\big)I_1 + \big(0.5 * \sqrt{\pi} - \sqrt{\pi}\big)I_2$$

Remark:

Since:

$$T[\Gamma(n_0 + n_1I_1 + \mathrm{n}_2\mathrm{I}_2 + 1)] = \big(\Gamma(n_0+1), \Gamma(n_0+n_1+n_2+1), \Gamma(n_0+n_2+1)\big)$$

$$= (n_0!, (n_0+n_1+n_2)!, (n_0+n_2)!)\ ; n_0, \mathrm{n}_1, \mathrm{n}_2 \in \mathbb{N}.$$

So $\Gamma(n_0 + n_1I_1 + \mathrm{n}_2\mathrm{I}_2 + 1) = (n_0 + n_1I_1 + \mathrm{n}_2\mathrm{I}_2)! = \mathrm{T}^{-1}(n_0!, (n_0+n_1+n_2)!, (n_0 + n_2)!) = \mathrm{n}_0! + [(n_0+n_1+n_2)! - (n_0+n_2)!]\mathrm{I}_1 + [(n_0+n_2)! - n_0!]\mathrm{I}_2$

And it's the formal form of refined neutrosophic factorial function.

Refined Neutrosophic Beta Function:

We can find the value of refined neutrosophic beta function at refined neutrosophic points $a_N = a_0 + a_1I_1 + a_2I_2$, $b_N = b_0 + b_1I_1 + b_2I_2$ using the formula:

$$\beta(a_N, b_N) = \int_0^1 x^{a_N-1}(1-x)^{b_N-1}dx =$$

Where:

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \ ; a,b > 0$$

Proof:

Let $f(x) = x^{a_N-1}(1-x)^{b_N-1}$, so:

$$T[f(x)] = (x^{a_0-1}(1-x)^{b_0-1}, x^{a_0+a_1+a_2-1}(1-x)^{b_0+b_1+b_2-1}, x^{a_0+a_2-1}(1-x)^{b_0+b_2-1})$$

Then:

$$\left(\int_0^1 x^{a_0-1}(1-x)^{b_0-1}dx, \int_0^1 x^{a_0+a_1+a_2-1}(1-x)^{b_0+b_1+b_2-1}dx, \int_0^1 x^{a_0+a_2-1}(1-x)^{b_0+b_2-1}dx\right)$$

$$= \left(\beta(a_0,b_0), \beta(a_0+a_1+a_2, b_0+b_1+b_2), \beta(a_0+a_2, b_0+b_2)\right)$$

So:

$$\beta(a_N,b_N) = \beta(a_0,b_0) + [\beta(a_0+a_1+a_2, b_0+b_1+b_2) - \beta(a_0+a_2, b_0+b_2)]I_1 + [\beta(a_0+a_2, b_0+b_2) - \beta(a_0,b_0)]I_2$$

We let it an exercise to the reader to prove that:

$\beta(a_N,b_N) = \frac{\Gamma(a_N)\Gamma(b_N)}{\Gamma(a_N+b_N)}$.

Conclusion

In this paper, we have used the refined neutrosophic algebraic AH-isometry to study the functions defined on the real refined neutrosophic field, where refined neutrosophic Beta functions, Gamma functions, Logarithmic functions, and trigonometric functions were presented and formulated.

As a future research direction, we aim to study the refined neutrosophic probability continuous distributions based on this approach.

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Introduction to Neutrosophic Stochastic Processes

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Abstract: In this article, the definition of literal neutrosophic stochastic processes is presented for the first time in the form $\mathcal{N}_t = \xi_t + \eta_t I \,; I^2 = I$ where both $\{\xi(t), t \in T\}$ and $\{\eta(t), t \in T\}$ are classical real valued stochastic processes. Characteristics of the literal neutrosophic stochastic process are defined and its formulas are driven including neutrosophic ensemble mean, neutrosophic covariance function and neutrosophic autocorrelation function. Concept of literal neutrosophic stationary stochastic processes is well defined and many theorems are presented and proved using classical neutrosophic operations then using the one-dimensional AH-Isometry. Some solved examples are presented and solved successfully. We have proved that studying the literal neutrosophic stochastic process $\{\mathcal{N}(t), t \in T\}$ is equivalent to studying two classical stochastic processes which are $\{\xi(t), t \in T\}$ and $\{\xi_t + \eta_t, t \in T\}$.

Keywords: AH-Isometry; Neutrosophic Field of Reals; Neutrosophic Random Variables; Stationary Stochastic Processes; Characteristics of Stochastic Processes; Ensemble Mean; Covariance Function; Autocorrelation Function.

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1. Introduction

In probability theory, a family of random variables is called a stochastic process usually noted by $\{\xi(t)\,, t \in T\}$. Stochastic processes have many applications in many fields of science like biology, physics, ecology, information theory, chemistry, telecommunications, finance, etc. [1]

Classical stochastic process depends on parameters which are determined and known with high precision and confidence, but sometimes those parameters may have some uncertainty and it may be imprecise which led to define what is known by fuzzy stochastic processes **[2]**, **[3]**, **[4]**.

In the recent years Prof. Smarandache introduced an extension of fuzzy and intuitionistic fuzzy sets called neutrosophic sets where elements are described using three independent functions; membership, indeterminacy and non-membership. Also, Smarandache extended the field of reals adding the indeterminacy component $\boldsymbol{I}$ which satisfies $\boldsymbol{I}^2 = \boldsymbol{I}$ and introduced the literal neutrosophic reals field $\boldsymbol{R}(\boldsymbol{I}) = \boldsymbol{R} \cup \{\boldsymbol{I}\}$.

These extensions have been applied in many fields of sciences like probability theory, statistics, game theory, geometry, decision making, artificial intelligence, machine learning, abstract algebra, linear algebra, operations research, etc.[5-34]

Zeina and Hatip defined literal neutrosophic random variable in the form $\xi_N = \xi + I$ and studied its properties including literal neutrosophic expected value, literal neutrosophic variance, literal neutrosophic moments, literal neutrosophic characteristic function, literal neutrosophic moments generating function, literal neutrosophic probability density function and literal neutrosophic cumulative distribution function, and this study has been extended by Carlos Granados et al in [5-8].

Abobala and Hatip defined an isometry mapping between $R(I)$ and $R \times R$ called One-Dimensional AH-Isometry [9]. Based on this isometry, strong theorems and definitions of Euclidian geometry was written. This isometry is a powerful tool to build mathematical concepts strongly and with logical steps.

In this paper we generalize the definition of literal neutrosophic random variables to literal neutrosophic stochastic processes which are families of literal neutrosophic random variables depending on the one-dimensional AH-isometry and depending on direct computation based on neutrosophic rules.

This paper opens new research fields in probability theory like queueing theory, dynamic systems, reliability theory, stochastic differential equations, etc.

2. Preliminaries

Definition 2.1

Literal neutrosophic real number $\boldsymbol{N}$ is defined by:

$$\boldsymbol{N} = \boldsymbol{n_1} + \boldsymbol{n_2 I}\,;\ \boldsymbol{I^2} = \boldsymbol{I}\ \&\ \boldsymbol{n_1}, \boldsymbol{n_2} \in \boldsymbol{R}$$

And we call $\boldsymbol{R(I)} = \{\boldsymbol{n_1} + \boldsymbol{n_2 I}\,; \boldsymbol{n_1}, \boldsymbol{n_2} \in \boldsymbol{R}\ \boldsymbol{and}\ \boldsymbol{I^2} = \boldsymbol{I}\}$ the literal neutrosophic real set.

Definition 2.2

Let $\boldsymbol{R(I)}$ be the literal neutrosophic real set, we say $\boldsymbol{n_1} + \boldsymbol{n_2 I} \le \boldsymbol{n_3} + \boldsymbol{n_4 I}$ iff $\boldsymbol{n_1} \le \boldsymbol{n_3}$ & $\boldsymbol{n_1} + \boldsymbol{n_2} \le \boldsymbol{n_3} + \boldsymbol{n_4}$.

Definition 2.3

AH-Isometry is an isomorphism preserves distances between $\boldsymbol{R(I)}$ and $\boldsymbol{R} \times \boldsymbol{R}$ and defined as in the following equation:

$$\boldsymbol{g}: \boldsymbol{R(I)} \to \boldsymbol{R} \times \boldsymbol{R}\,; \boldsymbol{g}(\boldsymbol{n_1} + \boldsymbol{n_2 I}) = (\boldsymbol{n_1}, \boldsymbol{n_1} + \boldsymbol{n_2}) \quad (1)$$

and its inverse is defined as follows:

$$\boldsymbol{g^{-1}}: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R(I)}\,; \boldsymbol{g}(\boldsymbol{n_1}, \boldsymbol{n_2}) = \boldsymbol{n_1} + (\boldsymbol{n_2} - \boldsymbol{n_1})\boldsymbol{I} \qquad (2)$$

Definition 2.4

Let $\vec{\boldsymbol{n}} = (\boldsymbol{n_1} + \boldsymbol{n_2 I}, \boldsymbol{n_3} + \boldsymbol{n_4 I})$ be a vector, then its norm is defined as:

$$\|\vec{\boldsymbol{n}}\| = \sqrt{(\boldsymbol{n_1} + \boldsymbol{n_2 I})^2 + (\boldsymbol{n_3} + \boldsymbol{n_4 I})^2}$$

Remark 2.1

Since the one-dimensional AH-Isometry is an algebraic isomorphism and preserves distances then it has the following properties:

1. $g(n_1 + n_2 I + \mathrm{n}_3 + n_4 I) = g(n_1 + n_2 I) + g(n_3 + n_4 I)$

2. $g[(n_1 + n_2 I) \cdot (n_3 + n_4 I\,)] = g(n_1 + n_2 I) \cdot g(n_3 + n_4 I)$

3. g is correspondence one-to-one

4. $g(\|\overrightarrow{AB}\|) = \|g(\overrightarrow{AB})\|$

Definition 2.5 [11]

Let $\boldsymbol{\xi}, \boldsymbol{\eta}$ be two classical random variables, then literal neutrosophic random variable (LNRV) is defined by:

$$\xi_N = \xi + \eta I \; ; I^2 = I$$

Remark 2.2

Let $\boldsymbol{\xi_N}$ be a LNRV then:

1. $E(\xi_N) = E(\xi) + I\, E(\eta)$

2. $V(\xi_N) = V(\xi) + I\, [V(\xi + \eta) - V(\xi)]$

3. Literal Neutrosophic Stochastic Processes

Definition 3.1

Let $\{\boldsymbol{\xi}\ (\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{T}\}$ and $\{\boldsymbol{\xi}(\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{T}\}$ be two crisp (classic) stochastic processes, we define the literal neutrosophic stochastic process $\{\boldsymbol{\mathcal{N}}(\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{T}\}$ as follows:

$$\boldsymbol{\mathcal{N}}: (\boldsymbol{\Omega} \times \boldsymbol{T}) \to \boldsymbol{R}(\boldsymbol{I}) \; ; \; \boldsymbol{\mathcal{N}}(\boldsymbol{t}) = \boldsymbol{\xi}(\boldsymbol{t}) + \boldsymbol{\eta}(\boldsymbol{t})\boldsymbol{I}; \boldsymbol{I}^{\boldsymbol{2}} = \boldsymbol{I}$$

We call $\boldsymbol{\xi}(\boldsymbol{t})$ the determinant part of $\boldsymbol{\mathcal{N}}(\boldsymbol{t})$ and we call $\boldsymbol{\eta}(\boldsymbol{t})$ the indeterminant part of $\boldsymbol{\mathcal{N}}(\boldsymbol{t})$.

Theorem 1

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process then the ensemble average function of $\{\mathcal{N}(t), t \in T\}$ is:

$$\boldsymbol{\mu_{\mathcal{N}}}(\boldsymbol{t}) = \boldsymbol{\mu_{\xi}}(\boldsymbol{t}) + \boldsymbol{I}\boldsymbol{\mu_{\eta}}(\boldsymbol{t}) \quad (3)$$

Proof

For a fixed $t \in T$ both $\{\xi(t), t \in T\}$ and $\{\eta(t), t \in T\}$ become random variables (not stochastic processes), the $\{\mathcal{N}(t), t \in T\}$ becomes a literal neutrosophic random variable, so based on properties of literal neutrosophic random variables we can write:

$$\boldsymbol{\mu_{\mathcal{N}}(t) = E[\mathcal{N}(t)] = E[\xi(t) + I\,\eta(t)] = E[\xi(t)] + IE[\eta(t)] = \mu_{\xi}(t) + I\mu_{\eta}(t)}$$

Theorem 2

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process then autocorrelation function is:

$$R_{\mathcal{N}}(s,t) = R_{\xi}(s,t) + I\{R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_{\eta}(s,t)\} \quad (4)$$

Proof

$$R_{\mathcal{N}}(s,t) = E[\mathcal{N}(s) \cdot \mathcal{N}(t)] = E\{[\xi(s) + I\,\eta(s)] \cdot [\xi(t) + I\,\eta(t)]\}$$

$$= E\{\xi(s)\xi(t) + I\xi(s)\eta(t) + I\eta(s)\xi(t) + I^2\eta(s)\eta(t)\}$$

$$= R_{\xi}(s,t) + I\{R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_{\eta}(s,t)\}$$

Remark 3.1

Notice that $R_{\mathcal{N}}(t,t) = R_{\xi}(t,t) + I\{2R_{\xi\eta}(t,t) + R_{\eta}(t,t)\} = E[\xi^2(t)] + I\{2R_{\xi\eta}(t,t) + E[\eta^2(t)]\}$

Theorem 3

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process then its autocovariance function is:

$$C_{\mathcal{N}}(s,t) = R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t) \quad (5)$$

Proof

$$C_{\mathcal{N}}(s,t) = cov[\mathcal{N}(s), \mathcal{N}(t)] = E\{[\mathcal{N}(s) - \mu_{\mathcal{N}}(s)][\mathcal{N}(t) - \mu_{\mathcal{N}}(t)]\}$$

$$= E\{\mathcal{N}(s)\mathcal{N}(t) - \mu_{\mathcal{N}}(t)\mathcal{N}(s) - \mu_{\mathcal{N}}(s)\mathcal{N}(t) + \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)\}$$

$$= R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(t)E[\mathcal{N}(s)] - \mu_{\mathcal{N}}(s)E[\mathcal{N}(t)] + \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)$$

$$= R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(t)\mu_{\mathcal{N}}(s) - \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t) + \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)$$

$$= R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)$$

Remark 3.2

If $s = t$ then:

$$C_{\mathcal{N}}(s\,,t) = C_{\mathcal{N}}(t\,,t) = E\{[\mathcal{N}(t) - \mu_{\mathcal{N}}(t)][\mathcal{N}(t) - \mu_{\mathcal{N}}(t)]\} = Var[\mathcal{N}(t)]$$

Definition 3.2

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process, we call $F(x_N, t) = P\{\mathcal{N}(t) \leq x_N\}$ the first order distribution of $\{\mathcal{N}(t), t \in T\}$ where $x_N = x + Iy$ and $x, y \in R$.

Definition 3.3

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process, we call $\frac{\partial}{\partial x_N} F(x_N, t) = f(x_N, t)$ the first order density of $\{\mathcal{N}(t), t \in T\}$ where $x_N = x + Iy$ and $x, y \in R$.

Definition 3.4

A literal neutrosophic stochastic process is called strongly stationary if its distribution is invariant under neutrosophic transition of time, i.e., $f(x_N, t) = f(x_N, t + h_N); h_N = h_1 + Ih_2$

Definition 3.5

A literal neutrosophic stochastic process is called weakly stationary if it satisfies the following two conditions:

1. $\mu_{\mathcal{N}}(t) = \mu_N = \mu_1 + I\mu_2$
2. $E[\mathcal{N}(t) \cdot \mathcal{N}(t - \tau_N)] = R(\tau_N)$

4. Literal Neutrosophic Stochastic Processes Using AH-Isometry:

Consider the literal neutrosophic stochastic process $\{\mathcal{N}(t), t \in T\}$ then applying AH-isometry on it yields to:

$$g[\mathcal{N}(t)] = g[\xi(t) + \eta(t)I] = \big(\xi(t), \xi(t) + \eta(t)\big)$$

Notice that using the one-dimensional AH-Isometry we transfer the literal neutrosophic stochastic process $\{\mathcal{N}(t), t \in T\}$ into two classical stochastic processes $\{\xi(t), t \in T\}$ and $\{\xi(t) + \eta(t), t \in T\}$.

So, we can study the characteristics of $\{\mathcal{N}(t), t \in T\}$ by studying the characteristics of both $\{\xi(t), t \in T\}$ and $\{\xi(t) + \eta(t), t \in T\}$.

Example 4.1

In theorem 1 we show that $\mu_{\mathcal{N}}(t) = \mu_\xi(t) + \mu_\eta(t)I$, we can reach the same result by using the one-dimensional AH-Isometry as follows:

We have:

$$\mathcal{N}(t) = \xi(t) + \eta(t)I$$

So:

$$E[\mathcal{N}(t)] = E[\xi(t) + \eta(t)I]$$

$$g(E[\mathcal{N}(t)]) = g(E[\xi(t) + I\,\eta(t)]) = E[g(\xi(t) + \eta(t)I)] = E[\xi(t), \xi(t) + \eta(t)]$$

$$= \left(\mu_\xi(t), \mu_\xi(t) + \mu_\eta(t)\right)$$

Taking the inverse isometry:

$$g^{-1}g(E[\mathcal{N}(t)]) = E[\mathcal{N}(t)] = \mu_\xi(t) + \left[\mu_\xi(t) + \mu_\eta\,(t) - \mu_\xi(t)\right]I = \mu_\xi(t) + \mu_\eta(t)I$$

Which is the same result presented in theorem 1.

Example 4.2

Let's calculate the autocorrelation function $R_{\mathcal{N}}(s,t)$ using the AH-Isometry:

$$R_{\mathcal{N}}(s,t) = E[\mathcal{N}(s)\cdot\mathcal{N}(t)]$$

$$g\left(R_{\mathcal{N}}(s,t)\right) = E\{g[\mathcal{N}(s)\cdot\mathcal{N}(t)]\} = E\{g[\xi(s) + \eta(s)I][\xi(t) + \eta(t)I]\}$$

$$= E\{g[\xi(s) + \eta(s)I]g[\xi(t) + \eta(t)I]\}$$

$$= E\{\left(\xi(s), \xi(s) + \eta(s)\right)\left(\xi(t), \xi(t) + \eta(t)\right)\}$$

$$= \{E\left(\xi(s)\xi(t)\right), E\left(\xi(s) + \eta(s)\right)\left(\xi(t) + \eta(t)\right)\}$$

$$= \left(R_\xi(s,t), R_\xi(s,t) + R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_\eta(s,t)\right)$$

Now taking g^{-1} yields:

$$R_{\mathcal{N}}(s,t) = R_\xi(s,t) + \left[R_\xi(s,t) + R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_\eta(s,t) - R_\xi(s,t)\right]I$$

$$= R_\xi(s,t) + I\{R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_\eta(s,t)\}$$

Which is the same result in theorem 2.

Theorem 4

A literal neutrosophic stochastic process $\mathcal{N}(t) = \xi(t) + \eta(t)I$ is weakly stationary if and only if $\{\xi(t), t \in T\}$ is weakly stationary and $\{\xi(t) + \eta(t), t \in T\}$ is weakly stationary.

Proof

We will first suppose that $\{\xi(t), t \in T\}$ and $\{\xi(t) + \eta(t), t \in T\}$ are weakly stationary and prove that $\mathcal{N}(t) = \xi(t) + \eta(t)I$ is also stationary:

Since $\{\xi(t), t \in T\}$ is weakly stationary then $\mu_\xi(t) = \mu_\xi = constant$ and $E[\xi(t)\cdot\xi(t-\tau)] = R_\xi(\tau)$

We also supposed that $\{\xi(t) + \eta(t), t \in T\}$ is weakly stationary so $\mu_{\xi+\eta}(t) = E[\xi(t) + \eta(t)] = \mu_{\xi+\eta} = costant,$ which means that $\mu_\eta(t) = \mu_\eta = constant.$

and $R_{\xi+\eta}(t,t-\tau) = E[\xi(t)+\eta(t)][\xi(t-\tau)+\eta(t-\tau)] = E[\xi(t)\xi(t-\tau)+\xi(t)\eta(t-\tau)+\eta(t)\xi(t-\tau)+\eta(t)\eta(t-\tau)] = R_{\xi}(t,t-\tau)+R_{\xi\eta}(t,t-\tau)+R_{\eta\xi}(t,t-\tau)+R_{\eta}(t,t-\tau)$

Since $\xi(t)+\eta(t)$ is weakly stationary then $R_{\xi+\eta}(t,t-\tau)$ must depend only on the difference τ, so the only possible form of it will be:

$$R_{\xi+\eta}(t,t-\tau) = R_{\xi}(\tau)+R_{\xi\eta}(\tau)+R_{\eta\xi}(\tau)+R_{\eta}(\tau) = R_{\xi+\eta}(\tau)$$

Which means that $R_{\xi\eta}(t,t-\tau) = R_{\xi\eta}(\tau), R_{\eta\xi}(t,t-\tau) = R_{\eta\xi}(\tau), R_{\eta}(t,t-\tau) = R_{\eta}(\tau)$

$$E(\mathcal{N}(t)) = E[\xi(t)+\eta(t)I] = \mu_{\xi}\,(t)+\mu_{\eta}(t)I = \mu_{\xi}+\mu_{\eta}I = \mu_{N} = constant$$

Using equation (4):

$$\begin{aligned} R_{\mathcal{N}}(t,t-\tau) &= E[\mathcal{N}(t)\cdot\mathcal{N}(t-\tau)] \\ &= R_{\xi}(t,t-\tau)+I\{R_{\xi\eta}(t,t-\tau)+R_{\eta\xi}(t,t-\tau)+R_{\eta}(t,t-\tau)\} \\ &= R_{\xi}(\tau)+I\{R_{\xi\eta}(\tau)+R_{\eta\xi}(\tau)+R_{\eta}(\tau)\} = R_{\mathcal{N}}(\tau) \end{aligned}$$

So, we conclude that $\{\mathcal{N}(t), t\in T\}$ is weakly stationary.

Now let's assume that $\{\mathcal{N}(t), t\in T\}$ is weakly stationary and prove that both $\{\xi(t), t\in T\}$ and $\{\xi(t)+\eta(t), t\in T\}$ are weakly stationary.

Since $\{\mathcal{N}(t), t\in T\}$ is weakly stationary then $E(\mathcal{N}(t)) = \mu_{N}(t) = \mu_{N} = constant$

but $E(\mathcal{N}(t)) = \mu_{\xi}(t)+I\mu_{\eta}(t)$ so both $\mu_{\xi}(t)$ and $\mu_{\eta}(t)$ must be dependent of time, then

$$\mu_{\xi}(t) = \mu_{\xi} \quad (6)$$

$$\mu_{\eta}(t) = \mu_{\eta} \quad (7)$$

which meant that:

$$\mu_{\xi+\eta}(t) = \mu_{\xi}+\mu_{\eta} = constant \ (8)$$

Also, we have: $R_{\mathcal{N}}(t,t-\tau) = R_{\xi}(t,t-\tau)+I\{R_{\xi\eta}(t,t-\tau)+R_{\eta\xi}(t,t-\tau)+R_{\eta}(t,t-\tau)\}$ and since $\{\mathcal{N}(t), t\in T\}$ is weakly stationary then $R_{\mathcal{N}}(t,t-\tau)$ must depend only on the difference τ so the following equations must hold:

$$R_{\xi}(t,t-\tau) = R_{\xi}(\tau) \quad (9)$$

$$R_{\xi\eta}(t,t-\tau) = R_{\xi\eta}(\tau) \ (10)$$

$$R_{\eta\xi}(t,t-\tau) = R_{\eta\xi}(\tau) \ (11)$$

$$R_{\eta}(t,t-\tau) = R_{\eta}(\tau) \quad (12)$$

From equations (6), (9) we conclude that $\{\xi(t), t\in T\}$ is weakly stationary.

And using equations (8), (9-12) we conclude that $\{\xi(t)+\eta(t), t \in T\}$ is weakly stationary.

Theorem 5

Suppose that $\{\mathcal{N}(t), t \in T\}$ is a weakly stationary literal neutrosophic stochastic process with autocorrelation function $R_{\mathcal{N}}(\tau)$, then the following holds:

1. $R_{\mathcal{N}}(\tau) = R_{\mathcal{N}}(-\tau)$
2. $|R_{\mathcal{N}}(\tau)| \leq R(0)$

Proof

1. we have:

$$R_{\mathcal{N}}(\tau) = R_{\xi}(\tau) + I\left\{R_{\xi\eta}(\tau) + R_{\eta\xi}(\tau) + R_{\eta}(\tau)\right\}$$

So:

$$R_{\mathcal{N}}(-\tau) = R_{\xi}(-\tau) + I\left\{R_{\xi\eta}(-\tau) + R_{\eta\xi}(-\tau) + R_{\eta}(-\tau)\right\}$$

And using properties of cross-correlation function in classical stationary processes we get:

$$R_{\mathcal{N}}(-\tau) = R_{\xi}(\tau) + I\left\{R_{\eta\xi}(\tau) + R_{\xi\eta}(\tau) + R_{\eta}(\tau)\right\} = R_{\mathcal{N}}(\tau)$$

2. Taking AH-Isometry:

$$g(|R_{\mathcal{N}}(\tau)|) = |E\{g[\mathcal{N}(t)\cdot\mathcal{N}(t-\tau)]\} = E\{g[\xi(t)+I\eta(t)][\xi(t-\tau)+I\eta(t-\tau)]\}|$$

$$= |E\{g[\xi(t)+I\eta(t)]g[\xi(t-\tau)+I\eta(t-\tau)]\}|$$

$$= |E\{\big(\xi(t),\xi(t)+\eta(t)\big)\big(\xi(t-\tau),\xi(t-\tau)+\eta(t-\tau)\big)\}|$$

$$= |E\{\xi(t)\xi(t-\tau),[\xi(t)+\eta(t)][\xi(t-\tau)+\eta(t-\tau)]\}|$$

$$= \big(|R_{\xi}(\tau)|, |R_{\xi+\eta}(\tau)|\big) \leq (0,0)$$

Now taking g^{-1}:

$$|R_{\mathcal{N}}(\tau)| = |R_{\xi}(\tau)| + \big(|R_{\xi+\eta}(\tau)| - |R_{\xi}(\tau)|\big)I \leq 0$$

5. Some Applications:

Example 5.1

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process defined as follows:

$$\mathcal{N}(t) = A_N\cos(t) + \sin(t)\ I$$

Where distribution of the literal neutrosophic random variable A_N is:

A_N	0	1
$Prob$	$\frac{1}{3}I$	$1-\frac{1}{3}I$

Let's find $\mu_{\mathcal{N}}(t), R_{\mathcal{N}}(s,t)$ and show whether $\{\mathcal{N}(t), t \in T\}$ is stationary or not.

Solution

$$\boldsymbol{E(A_N) = 0 \cdot \frac{1}{3}I + 1 \cdot \left(1-\frac{1}{3}I\right) = 1-\frac{1}{3}I}$$

$$\boldsymbol{E(A_N^2) = 0^2 \cdot \frac{1}{3}I + 1^2 \cdot \left(1-\frac{1}{3}I\right) = 1-\frac{1}{3}I}$$

$$\boldsymbol{\mu_{\mathcal{N}}(t) = E(\mathcal{N}(t)) = E(A_N \cos(t) + \sin(t)\ I) = \left(1-\frac{1}{3}I\right) \cdot \cos(t) + \sin(t)\, I}$$

Since $\boldsymbol{\mu_{\mathcal{N}}(t)}$ is a function of $\boldsymbol{t}$ then $\{\boldsymbol{\mathcal{N}(t), t \in T}\}$ is not stationary stochastic process.

$$\begin{aligned} R_{\mathcal{N}}(s,t) &= E[\mathcal{N}(s)\cdot\mathcal{N}(t)] = E[(A_N\cos(t)+\sin(t)\ I)(A_N\cos(s)+\sin(s)\ I)] \\ &= E[A_N^2\cos(t)\cos(s) + A_N\cos(t)\sin(s)\, I + \sin(t)\, I\, A_N\cos(s) \\ &+ \sin(t)\sin(s)\, I^2] \\ &= \left(1-\frac{1}{3}I\right)\cos(t)\cos(s) + \left(1-\frac{1}{3}I\right)\cos(t)\sin(s)\, I + \left(1-\frac{1}{3}I\right)\sin(t)\cos(s)\, I \\ &+ \sin(t)\sin(s)\, I \end{aligned}$$

Example 5.2

let $\{\mathcal{N}(t), t \in T\}$ be a neutrosophic stochastic process defined as follows:

$$\mathcal{N}(t) = \xi(t) + \xi(t)\, I$$

Where $\{\xi(t), t \in T\}$ is a classical stochastic process defined by:

$$\xi(t) = A\cos(t) + B\sin(t)$$

Where A, B are random variables both defined as by:

	-2	1
$Prob$	$\frac{1}{3}$	$\frac{2}{3}$

Let's find $\mu_{\mathcal{N}}(t), R_{\mathcal{N}}(s,t)$ and show whether $\{\mathcal{N}(t), t \in T\}$ is stationary or not.

solution

$$E(A) = E(B) = \frac{2}{3} - \frac{2}{3} = 0$$

$$E(A^2) = E(B^2) = \frac{2}{3} + \frac{4}{3} = 2$$

$$\mu_\xi(t) = \cos(t)\ E(A) + \sin(t)\ E(B) = 0$$

$$R_\xi(s,t) = E[\xi(s)\ \xi(t)] = E[(A\cos(s) + B\sin(s))(A\cos(t) + B\sin(t))]$$

$$= E(A^2\cos(s)\cos(t) + AB\cos(s)\sin(t) + BA\sin(s)\cos(t) + B^2\sin(s)\sin(t))$$

$$= 2(\cos(s)\cos(t) + \sin(s)\sin(t)) = 2\cos(t-s) = 2\cos\tau$$

So:

$$\mu_{\mathcal{N}}(t) = E[\mathcal{N}(t)] = E[\xi(t) + \xi(t)\ I] = \mu_\xi(t) + \mu_\xi(t)I = 0 = const$$

$$R_{\mathcal{N}}(s,t) = E[\mathcal{N}(s) \cdot \mathcal{N}(t)] = E[(\xi(s) + \xi(s)I)(\xi(t) + \xi(t)I)]$$

$$= E[\xi(s)\xi(t) + \xi(s)\xi(t)I + \xi(s)X(t)I + \xi(s)\xi(t)I^2]$$

$$= R_\xi(s,t) + R_\xi(s,t)I + R_\xi(s,t)I + R_\xi(s,t)I = 2\cos(\tau) + 6\cos(\tau)\ I = R_N(\tau)$$

We conclude that $\{\mathcal{N}(t), t \in T\}$ is weakly stationary process.

In fact, it is clear that $\{\mathcal{N}(t), t \in T\}$ is weakly stationary process since $\{\xi(t), t \in T\}$ and $\{2\xi(t), t \in T\}$ are both weakly stationary processes.

6. Conclusions and future research directions

Concept of literal neutrosophic stochastic process is well defined by $\boldsymbol{\mathcal{N}(t) = \xi(t) + \eta(t)I}$. We proved that a literal neutrosophic stochastic process can be presented in $\boldsymbol{R^2}$ as two classical stochastic processes, first is $\{\boldsymbol{\xi(t), t \in T}\}$ and second is the convolution $\{\boldsymbol{\xi(t) + \eta(t), t \in T}\}$. Many theorems were proved successfully especially the theorem of stationary stochastic process where we have seen that $\{\boldsymbol{\mathcal{N}(t), t \in T}\}$ is stationary if and only if $\{\boldsymbol{\xi(t), t \in T}\}$ is stationary and $\{\boldsymbol{\xi(t) + \eta(t), t \in T}\}$ is stationary. This paper can be applied in many fields related to probability theory including game theory, polling, statistical analysis, financial mathematics, etc. In future researches we are looking forward to study cross neutrosophic stochastic processes and define its characteristics and the theorems related to it. Also, we are looking forward to study applications of literal neutrosophic stochastic processes in related fields.

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Solutions of Some Kandasamy-Smarandache Open Problems About the Algebraic Structure of Neutrosophic Complex Finite Numbers

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Abstract: The aim of this paper is to study the neutrosophic complex finite rings $C(Z_n)$ *and* $C(< Z_n \cup I >)$, and to give a classification theorem of these rings. Also, this work introduces full solutions for 12 Kandasamy-Smarandache open problems concerning these structures of generalized rings modulo integers. Also, a necessary and sufficient condition of invertibility in $C(Z_n)$ *and* $C(< Z_n \cup I >)$ is presented as a partial solution of the famous group of units problem.

Keywords: Neutrosophic complex number, neutrosophic finite complex number, maximal ideal, minimal ideal

Introduction.

Neutrosophy as a new kind of generalized logic deals with indeterminacy in nature, reality, and ideas found its way into algebraic studies. A lot of neutrosophic algebraic structures were defined and studied in a wide range. See [1-11].

In the literature, many generalizations appeared such as refined neutrosophic rings, n-refined neutrosophic rings, n-refined neutrosophic groups, and n-refined neutrosophic vector spaces and modules. Recently, algebraic equations and Diophantine linear equations were solved in neutrosophic rings and refined neutrosophic rings. See [5-18].

In [20], Smarandache and Kandasamy introduced the neutrosophic complex numbers modulo integers as an interesting generalized structure. Their work suggests a new approach to the concept of classical complex numbers, and they proposed 150 open problems concerning substructures and factorization properties in these complex neutrosophic structures modulo integers (some of these problems were solved in [17]). In this paper, we aim to continue their efforts and to suggest a classification of neutrosophic complex finite rings modulo integers. Also, we suggest solutions for 12 problems of Kandasamy-Smarandache problems introduced in [20].

Main results

We start our discussion by some easy Kandasamy-Smarandache problems about finite neutrosophic complex rings.

Problem (56): Does every $C(Z_n)$ contain a zero divisor?.

The answer is no. If n is a prime and there are $a, b \in Z_n;\ a^2 + b^2 \equiv 0(mod\ n)$, then $C(Z_n)$ is a field according to Theorem , and then it has no zero divisors.

Problem (58): Is every element in C(Z_7) invertible?.

The answer is yes, since C(Z_7) is a field, thus all elements different from zero are invertible.

Problem (57): Can every C(Z_n) be a field?.

The answer is no, since C(Z_5) is just a ring but not a field.

Problem (53): Find a subring S in C(Z_n)   so that S is not an ideal.

We take $S = Z_n$ which is a subring of C(Z_n) , but it is not an ideal, that is because 1∈ Z_n and $i_F \in \mathrm{C}(Z_n)$, where1. $i_F = i_F$, which is not in S. Thus S is not an ideal.

Problem (26): Can C(⟨Z_{12} ∪ I⟩) be a S-ring? Justify.

The answer is yes. That is because the set $M = \{0,9,3\}$ is a field under multiplication with 9 acts as the identity.

Problem (25): Prove C(⟨Z_{25} ∪ I⟩) can only be a ring.

It is sufficient to prove that C(⟨Z_{25} ∪ I⟩) has zero divisors. We take $5 + 5I \in \mathrm{C}(\langle Z_{25} \cup \mathrm{I}\rangle)$, and

$(5 + 5I).(5 + 5I) = 25(1 + I)(1 + I) = 0.$

Definition:

(a) Let R be any commutative ring, m be any element (not from R) which is a root of a polynomial $p(x) \in R[x]$. Then if there is no root of $p(x)$ in R, we call R(m) an algebraic extension. For example the ring Z(i) is an algebraic extension of the ring Z, since i is a root of the polynomial $p(x) = x^2 + 1 \in Z[x]$, and p(x) has no roots in Z. (The concept of classical algebraic extension).

(b) Let R be any commutative ring, m be any element (not from R) which is a root of a polynomial $p(x) \in R[x]$. Then if there is a root of $p(x)$ in R, we call R(m) a logical extension.

For example the neutrosophic ring Z(I) is a logical extension of the ring Z, since I is a root of the polynomial $p(x) = x^2 - x \in Z[x]$, and p(x) has roots {0,1} in Z.

The following theorem realizes the algebraic structure of $C(Z_n)$.

Theorem:

Let $C(Z_n)$ be the ring of complex numbers modulo n, we have the following:

(a) If n=p is a prime and $p(x) = x^2 + 1$ is irreducible over Z_p, then $C(Z_p)$ is an algebraic extension field of the field Z_p with degree two.

(b) If n=p is a prime and $p(x) = x^2 + 1$ is reducible over Z_p, then $C(Z_p)$ is just a ring (logical extension).

(c) If n is not a prime and $p(x) = x^2 + 1$ is irreducible over Z_n, then $C(Z_n)$ is an algebraic extension ring of the ring Z_n with degree two.

(d) If n is not a prime and $p(x) = x^2 + 1$ is reducible over Z_n, then $C(Z_n)$ is a logical extension of the ring Z_n.

Proof:

(a) Suppose that $p(x) = x^2 + 1$ is irreducible over Z_p, then it has no roots in Z_p, thus i_F is an algebraic element over Z_p, and by classical algebraic result, we get that C(Z_p) is an algebraic extension field of the field Z_p with degree equal to deg(p) which is two.

(b) i_F is a root of $p(x) = x^2 + 1$, but p(x) has a root in Z_p, because it is reducible, hence $C(Z_p)$ is just a ring (logical extension). [$C(Z_p)$ is not a field because there are $a, b \in Z_p$ such that $a^2 + b^2 \equiv 0 (mod\ p)$, where $b = 1\ and\ a\ is\ the\ root\ of\ p(x) in\ Z_p$].

(c) It holds by a similar argument of section (a).

(d) It holds by a similar argument of (b).

The following theorem suggests a classification of the ring $C(< Z_n \cup I >)$.

Theorem:

Let $C(< Z_n \cup I >)$ be the neutrosophic complex modulo integers ring. Then $C(< Z_n \cup I >) \cong \mathrm{C(Z_n)} \times \mathrm{C(Z_n)}$.

Proof:

Firstly, we prove that $C(< Z_n \cup I >) = [\mathrm{C(Z_n)}](\mathrm{I})$, where $[\mathrm{C(Z_n)}](\mathrm{I})$ is the neutrosophic ring generated by I and $\mathrm{C(Z_n)}$.

Let $x = a + bi + cI + diI \in C(< Z_n \cup I >)$, then $x = (a + bi) + I(c + di) \in [\mathrm{C(Z_n)}](\mathrm{I})$, hence $C(< Z_n \cup I >) \leq [\mathrm{C(Z_n)}](\mathrm{I})$. Conversely, let $x = (a + bi) + (c + di)I \in [\mathrm{C(Z_n)}](\mathrm{I})$. It is clear that

$x \in C(< Z_n \cup I >)$. This implies that $C(< Z_n \cup I >) = [\mathrm{C(Z_n)}](\mathrm{I})$.

By the classification theorem of neutrosophic rings in [5], we find that $C(< Z_n \cup I >) = [\mathrm{C(Z_n)}](\mathrm{I}) \cong \mathrm{C(Z_n)} \times \mathrm{C(Z_n)}$.

Problem (24): Is $C(< Z_{19} \cup I >)$ a field?.

The answer is no, since I is not invertible.

The group of units problem and other open questions

In this section, we determine the necessary and sufficient condition for the invertibility of neutrosophic complex numbers modulo integers.

First of all, we characterize the algebraic structure of $C(Z_n)$ as an isomorphic image of a matrices subring of size 2×2.

Theorem:

Let $C(Z_n)$ be the ring of neutrosophic complex numbers modulo integers. Then $C(Z_n)$ is isomorphic to a sub ring of $M_{2\times 2}(Z_n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in Z_n\}$.

Proof:

Let $S = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}; a, b \in Z_n\}$ be a subring of $M_{2\times 2}(Z_n)$, we define

$f: C(Z_n) \to S; f(a + bi_F) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, it is easy to see that f is a well defined bijective map.

Let $x = a + bi_F, y = c + di_F$ be two arbitrary elements in $C(Z_n)$, we have

$f(x + y) = \begin{pmatrix} a + c & b + d \\ -b - d & a + c \end{pmatrix} = \begin{pmatrix} a & b \\ -b & b \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = f(x) + f(y)$.

$f(x.y) = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} . \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = f(x). f(y)$. Thus f is a ring isomorphism.

Now, we can find the condition of invertibility, as an easy result from Theorem.

Theorem:

Let $C(Z_n)$ be the ring of neutrosophic complex numbers modulo integers, $x = a + bi_F$ be an arbitrary elements in $C(Z_n)$. Then x is invertible if and only if $a^2 + b^2 \neq 0 \; and \; a^2 + b^2$ is invertible in Z_n.

Proof:

Since $C(Z_n) \cong S$, then x is invertible in $C(Z_n)$ if and only if $f(x) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is invertible in S.

It is well known that the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is invertible if and only if its inverse matrix is an element from S. Hence we have the following

(a) $\det\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right] = a^2 + b^2 \neq 0$.

(b) $\det\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right] = a^2 + b^2$ is invertible in Z_n, so the inverse matrix can be defined.

Thus, our proof is complete.

The condition (b) is sufficient, that is because if a^2+b^2 is invertible in Z_n, then $a^2+b^2 \neq 0$.

Example:

Consider the ring $C(Z_5)=\{a+bi_F; a,b \in Z_5\}$. The group of units in $C(Z_5)$ is equal to

$U=\{1,2,3,4,i_F,2i_F,3i_F,4i_F,1+i_F,1+4i_F,2+2i_F,2+3i_F,3+2i_F,3+3i_F,4+i_F,4+4i_F\}$.

Example :

Consider the ring $C(Z_4)=\{a+bi_F; a,b \in Z_4\}$. The group of units in $C(Z_4)$ is equal to

$U=\{1,3,i_F,3i_F,1+2i_F,2+i_F,2+3i_F,3+2i_F\}$.

Example:

Consider the ring $C(Z_6)=\{a+bi_F; a,b \in Z_6\}$. The group of units in $C(Z_6)$ is equal to

$U=\{1,5,i_F,5i_F,1+2i_F,1+4i_F,2+i_F,2+3i_F,2+5i_F,3+2i_F,3+4i_F,4+i_F,4+3i_F,4+5i_F,5+2i_F,5+4i_F\}$.

Now, we introduce the algebraic structure of the group of units in the ring $C(<Z_n \cup I>)$.

Theorem:

The group of units in the ring $C(<Z_n \cup I>)$, has the following property

$U(C(<Z_n \cup I>)) \cong U(C(Z_n)) \times U(C(Z_n))$.

The proof holds directly from the fact that $C(<Z_n \cup I>) \cong \mathrm{C}(\mathrm{Z_n}) \times \mathrm{C}(\mathrm{Z_n})$.

Remark:

A very interesting and hard problem is still open. This problem can be summarized as follows:

Describe the algebraic structure of the group of units in the ring $C(Z_n)$.

Although we have found the necessary and sufficient condition of any element in $C(Z_n)$ to be a unit, but the classification of this group as a direct product of cyclic groups is still unknown.

Remark:

As a result of Theorem 4.2, we can find zero divisors in $C(Z_n)$. Every element $x = a + bi_F \in C(Z_n)$ is a zero divisor if and only if its isomorphic image $f(x) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a zero divisor in the ring S.

Any matrix with form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a zero divisor if and only if its determinant is a zero divisor in Z_n, thus the necessary and sufficient condition for any element $x = a + bi_F \in C(Z_n)$ to be a zero divisor is $a^2 + b^2$is a zero divisor in Z_n. Now, we are able to solve another open problem.

Problem (50): Find Zero divisors and units in $C(Z_{24})$.

To solve the problem we shall determine the zero divisors in Z_{24} firstly.

We have 3,8,6,4,12,2 are zero divisors, that is because $3.8 = 6.4 = 12.2 = 0$. And $-3 = 21, -8 = 16, -6 = 18, -4 = 20, -2 = 22$ are zero divisors clearly. Also, the product of any two zero divisors is a zero divisor.

According to our discussion, zero divisors in $C(Z_{24})$ are

3,8,4,6,12,2,21,16,18,20,22, 15. The rest of zero divisors in $C(Z_{24})$ are elements with form $a + bi_F$, where $a^2 + b^2 \in \{3,8,4,6,12,2,21,16,18,20,22,15\}$.

To determine the units in $C(Z_{24})$, we shall determine units in Z_{24}. We have $U(Z_{24}) = \{1,5,7,11,13,17,19,23\}$. The other units in $C(Z_{24})$ are the elements with form $x = a + bi_F;\ a^2 + b^2 \in U(Z_{24})$.

The following theorems helps us in finding ideals of the ring $C(Z_n)$, and $C(< Z_n \cup I >)$.

Theorem:

Let $C(Z_n)$ be a neutrosophic complex modulo integers ring, $S = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}; a, b \in Z_n\}$ be its corresponding isomorphic subring. Let $I_{H_j} = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}; a, b \in H_j\}$, where $(H_j, +)$ is a subgroup of Z_n. We have

(a) Ideals of $C(Z_n)$ are exactly the isomorphic image of the sets I_{H_j}.

(b) If $(H_j, +, .)$ is a maximal ideal in $(Z_n, +, .)$, then I_{H_j} is a maximal ideal in $C(Z_n)$.

Proof:

Firstly, we shall determine the structure of additive subgroups in S. Let A,B be two subsets of Z_n, and $M = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}; a \in A, b \in B\}$. Let $x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, y = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ be two arbitrary elements in M.

(M,+) is a subgroup of S if and only if $x - y \in M$, which is equivalent to $a - b \in A, c - d \in B$, hence A,B are subgroups of Z_n.

Now, we prove that M is an ideal in S if and only if $A = B$.

Since A,B are subgroups of Z_n, we find that $(A.+,.), (B,+,.)$ are ideals in the ring $(Z_n, +, .)$.

Firstly, we assume that A=B. Let $x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M \text{ and } r = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in S$, we have

$x.r = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix}$. We have

$ac - bd \in A$, that is because $ac \in A$ (A *is an ideal in* Z_n) and $bd \in A$ (*for the same reason*). This implies that $x.r \in M$ and M is an ideal in S. Conversely, we suppose that M is an ideal in S, hence for any two elements Let $x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M \text{ and } r = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in S$, we have

$x.r = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix} \in M$, this implies that $ac - bd \in A$ and $ad + bc \in B$.

We know that A,B are ideals in Z_n, hence $ac \in A$ (*because* $a \in A, c \in Z_n$) and $bc \in B$(*because* $b \in B$ *and* $c \in Z_n$). This means that $-bd \in A$ and $ad \in B$ for all $b \in B, a \in A, d \in Z_n$, we put $d = 1$ to find that $a \in B$ *and* $b \in A$. Thus $A = B$.

According to Theorem , we have $C(Z_n) \cong S$, hence all ideals in $C(Z_n)$ are exactly the isomorphic image of the ideals in S. hence the proof is complete.

(b) Suppose that $I_{H_j} = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}; a, b \in H_j\}$ is a maximal ideal in S, hence it is easy to see that H_j is a maximal ideal in Z_n.

Remark:

Every ideal in $C(Z_n)$ has the form $f^{-1}\left(I_{H_j}\right) = \{a + bi_F; a, b \in H_j\}$, where H_j is a subgroup of Z_n.

Theorem:

Ideals in $C(< Z_n \cup I >)$ are equal to the isomorphic image of the set $J = \{I_{H_j} \times I_{H_s};\ H_j, H_s \leq Z_n\}$. Also, maximal ideals in $C(< Z_n \cup I >)$ are equal to the isomorphic image of the set $J = \{I_{H_j} \times I_{H_s};\ H_j, H_s \leq Z_n\ and\ I_{H_j}, I_{H_s}\ are\ maximal\}$.

Proof:

According to Theorem , we have $C(< Z_n \cup I >) \cong \mathrm{C(Z_n) \times C(Z_n)}$. the isomorphism between them is defined in [5] as follows:

$f: C(< Z_n \cup I >) \to \mathrm{C(Z_n) \times C(Z_n)}$; $f(a + bI) = (a, a + b); a, b \in C(Z_n)$. The inverse isomorphism is $f^{-1}: C(Z_n) \times C(Z_n) \to C(< Z_n \cup I >);\ f^{-1}(a, b) = a + (b - a)I; a, b \in C(Z_n)$.

According to Remark 4.11, ideals in $C(Z_n)$ has the form $\{a + bi_F; a, b \in H_j\}$, where H_j is a subgroup of Z_n, hence ideals in $\mathrm{C(Z_n) \times C(Z_n)}$ has the form $I = \{(a + bi_F, c + di_F); a, b \in H_j\ and\ c, d \in H_s\}$, where H_j, H_s are two subgroups of Z_n. Thus ideals in $C(< Z_n \cup I >)$ has the form

$$f^{-1}(I) = \{(a + bi_F) + [(c + di_F) - (a + bi_F)]I; a, b \in H_j\ and\ c, d \in H_s\} = \mathrm{I_{H_j}} + \left(\mathrm{I_{H_s}} - \mathrm{I_{H_j}}\right)\mathrm{I},$$ where H_j, H_s are two subgroups of Z_n.

Also, maximal ideals in $C(< Z_n \cup I >)$ has the form $f^{-1}(I) = \{(a + bi_F) + [(c + di_F) - (a + bi_F)]I; a, b \in H_j\ and\ c, d \in H_s\}$, where H_j, H_s are two maximal ideals of Z_n.

Problem (28): Find ideals in $\mathrm{C}(\langle Z_6 \cup \mathrm{I}\rangle)$.

Subgroups (Ideals) of Z_6 are $A = \{0\}, B = \{0,2,4\}, C = \{0,3\}, D = \{0,1,2,3,4,5\}$.

Ideals of C(Z_6) are $X = I_A = \{0\}, Y = I_B = \{0,2,4,2i_F, 4i_F, 2+2i_F, 2+4i_F, 4+4i_F, 4+2i_F\}$,

$Z = I_C = \{0,3,3i_F, 3+3i_F\}, T = I_D = \mathrm{C}(Z_6)$.

Ideals of C($\langle Z_6 \cup \mathrm{I}\rangle$) are the sets with form $M + (N-M)I; M, N \in \{X, Y, Z, T\}$.

Problem (29): Find maximum ideals of C($\langle Z_{18} \cup \mathrm{I}\rangle$).

First of all, we shall find maximum ideals in Z_{18}. They are $A = \{0,2,4,6,8,10,12,14,16\}$,

$B = \{0,3,6,9,12,15\}, C = Z_{18}$.

Maximal ideals in $C(Z_{18})$ are $I_A = \{a + bi_F; a, b \in A\}, I_B = \{c + di_F; c, d \in B), I_C = C(Z_{18})$.

Hence, maximal ideals in C($\langle Z_{18} \cup \mathrm{I}\rangle$) are P= $I_A + (I_B - I_A)I = I_A + I_C I, Q = I_B + (I_A - I_B)I = I_B + I_C I$,

$R = I_C + (I_A - I_C)I = I_C + (I_B - I_C)I = I_C + I_C I = \mathrm{C}(\langle Z_{18} \cup \mathrm{I}\rangle)$.

Find an ideal I in C(Z_{128}) so that C(Z_{128})/I is a field..***Problem (51):***

We have $J =< 2 >$ is a maximal ideal in Z_{128}. Hence $I_J = \{a + bi_F; a, b \in J\}$ is a maximal ideal in C(Z_{128}), thus C(Z_{128})/I_J is a field with order 4.

Problem (52): Does there exist an ideal I in C(Z_{49}) so that C(Z_{49})/I is a field?.

It is sufficient to find a maximal ideal in Z_{49}. We have $J =< 7 >$ is maximal in Z_{49}, hence $I_J = \{a + bi_F; a, b \in J\}$ is maximal in C(Z_{49}), and C(Z_{49})/ I_J is a field with order 49.

Problem (55): Find a necessary and sufficient condition for a complex modulo integers ring $S = C(Z_n)$ to have ideal I such that $C(Z_n)/_I$ is never a field.

The answer is depending on finding a non maximal ideal in $C(Z_n)$, since if I is a maximal ideal in $C(Z_n)$, we get a field $C(Z_n)/_I$.

We have the following cases:

(a) If n is a prime and $P(x) = x^2 + 1$ is irreducible over Z_n, then $C(Z_n)$ is a field and it has no proper ideals. (The only maximal ideal is I={0}). Thus the problem is not solvable in this case.

(b) If n is a prime and $P(x) = x^2 + 1$ is reducible over Z_n, then $C(Z_n)$ is a finite ring with n^2 elements. Thus every proper ideal I in $C(Z_n)$ has exactly n elements (because I is a subgroup under addition and then its order divides the order of $C(Z_n)$ by classical Lagrange's theorem).

Now, $C(Z_n)/_I$ is a ring with n elements (n is a prime), thus it is a field. Hence the problem is not solvable in this case.

(c) If n is not a prime and there is an integer s with property $s \neq gcd(s,n) = a \geq 2$, we define the following principal ideal $I =< s >$, where s is an integer with property $s \neq gcd(s,n) = a \geq 2$. It is clear that $I < J =< a >\neq C(Z_n)$, hence I is not maximal and $C(Z_n)/_I$ is never a field.

(d) If n is not a prime, but a prime power $n = p^n$. For $n = 2$, there is $< p >$ as the unique proper ideal and it is a maximal ideal in Z_n, hence $I_{<p>} = \{a + bi_F; a, b \in< p >\}$ is maximal in $C(Z_n)$, hence $C(Z_n)/_I$ is a field and the problem is not solvable in this case.

For $n \geq 3$, there is a non maximal ideal $< p^2 >$ in Z_n, hence $I_{<p^2>} = \{a + bi_F; a, b \in< p^2 >\}$ is non maximal in $C(Z_n)$, hence $C(Z_n)/_I$ is never a field.

(e) If n is not a prime and not a prime power, and there is not any integer s with property $s \neq gcd(s,n) = a \geq 2$, then $< s >$ is maximal in Z_n, , hence $I_{<s>} = \{a + bi_F; a, b \in< s >\}$ is maximal in $C(Z_n)$, hence $C(Z_n)/_I$ is a field, and the problem is not solvable in this case. (All ideals are maximal in this case).

Conclusion

In this paper, we have classified the ring of finite neutrosophic complex numbers as irect product of two rings. On the other hand, we have presented solutions for 12 open problems suggested by Smarandache and Kandasamy in [20].

As a future research direction, we aim to solve all Smarandache-Kandasamy open problems.

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On The Roots of Unity in Several Complex Neutrosophic Rings

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Abstract:

Roots of unity play a basic role in the theory of algebraic extensions of fields and rings. The aim of this paper is to obtain an algorithm to find all n-th roots of unity in five different kinds of neutrosophic complex rings, where many theorems and examples will be illustrated and suggested.

Keywords: Neutrosophic root of unity, refined neutrosophic unity, n-cyclic refined neutrosophic root of unity, complex neutrosophic number

1. Introduction and preliminaries

Neutrosophic algebraic structures are considered as generalizations of classical algebraic structures. The first defined neutrosophic algebraic structure is the neutrosophic ring which was defined and studied on a wide range by Smarandache et.al [1-11].

Laterally, many other neutrosophic algebraic structures were defined such as n-cyclic refined neutrosophic rings, neutrosophic matrices, and vector spaces [12-22].

Neutrosophic complex numbers were defined as novel generalizations of classical complex numbers, in a similar way of split-complex or weak fuzzy complex numbers [23-24].

One of the most classical interesting problems in classical algebra is the extending of fields and rings by complex roots of unity. From this point of view, we study for the first time the concept of neutrosophic roots of unity, where we obtain the classification of the roots of unity in five different neutrosophic rings. In addition, many examples will be discussed and presented.

We recall some basic concepts in neutrosophic algebra.

Definition:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i$$

$$= \sum_{i=0}^{n} (x_i + y_i) I_i \, , \sum_{i=0}^{n} x_i I_i$$

$$\times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j = \sum_{i,j=0}^{n} (x_i \times y_j) I_{(i+j \, mod n)}$$

× is the multiplication on the ring R.

Definition:

Let (R,+,×) be a ring, $R(I) = \{a + bI : a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Definition:

Let (R,+,×) be a ring, (R(I_1, I_2) , + ,×) is called a refined neutrosophic ring generated by R ,I_1, I_2.

Definition:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Main concepts

Neutrosophic roots of unity.

Let $C(I) = \{X + IY;\ I^2 = I\ ; X, Y \in C\}$be the complex neutrosophic ring. According to [21], we have:

$$(X + IY)^n = X^n + I[(X + Y)^n - X^n]$$

So that $X + IY$ is an n-th root of unity if and only if $(X + IY)^n = 1$, hence $X^n = 1$, $(X + Y)^n = 1$ which is equivalent to $X, X + Y$ are two classical roots of unity.

Theorem.

n-th roots of unity in the complex neutrosophic ring $C(I)$ are:

$$U = \left\{\alpha_j + (\alpha_t - \alpha_j)I;\ \alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}; 1 \leq j \leq n, 1 \leq t \leq n\right\}$$

Proof.

According to the previous discussion $X + IY$ is a neutrosophic n-th root of unity if and only if $X, X + Y$ are two roots of unity, thus $X = \alpha_j = e^{\frac{2\pi j}{n}i};\ 1 \leq j \leq n\,, X + Y = \alpha_t = e^{\frac{2\pi t}{n}i};\ 1 \leq t \leq n.$

This implies that $X + IY = \alpha_j + (\alpha_t - \alpha_j)I.$

Theorem.

The set of n-th roots of unity in $C(I)$ is a group under multiplication. Also $U \cong Z_n \times Z_n$.

Proof.

$$\forall\, T_1 = (\alpha_j) + (\alpha_t - \alpha_j)I, T_2 = (\alpha_k) + (\alpha_s - \alpha_k)I;\ 1 \leq k, s \leq n, 1 \leq j, t \leq n$$

Then:

$T_1.T_2 = \alpha_j\alpha_k + (\alpha_j\alpha_s - \alpha_j\alpha_k)I + (\alpha_k\alpha_t - \alpha_j\alpha_k)I + (\alpha_t\alpha_s - \alpha_t\alpha_k - \alpha_j\alpha_s + \alpha_j\alpha_k)I$

$T_1.T_2 = \alpha_j\alpha_k + (\alpha_t\alpha_s - \alpha_j\alpha_k)I = \alpha_l + (\alpha_m - \alpha_l)I \in U$

Where α_l, α_m are two roots of unity.

On the other hand, we have $\alpha_j^{-1}, \alpha_t^{-1}$ are two roots of unity.

So that we can put $T_3 = \alpha_j^{-1} + (\alpha_t^{-1} - \alpha_j^{-1})I \in U.$

$$T_1.T_3 = \alpha_j\alpha_j^{-1} + (\alpha_j\alpha_t^{-1} - \alpha_j\alpha_j^{-1} + \alpha_t\alpha_j^{-1} - \alpha_j\alpha_j^{-1} + \alpha_t\alpha_t^{-1} - \alpha_t\alpha_j^{-1} - \alpha_j\alpha_t^{-1} + \alpha_j\alpha_j^{-1})I$$

$T_1.T_3 = 1 + (0)I = 1$, thus $T_3 = T_1^{-1} \in U$. This implies that U is a group under multiplication.

We define $f: U \to Z_n \times Z_n$ such that:

$$f\left(\alpha_j + \left(\alpha_t - \alpha_j\right)I\right) = \left(\alpha_j, \alpha_t\right)$$

f is well defined, that is because:

If $T_1 = \alpha_j + \left(\alpha_t - \alpha_j\right)I = T_2 = \alpha_k + (\alpha_s - \alpha_k)I$, then $\alpha_j = \alpha_k,\ \alpha_t = \alpha_s$, hence

$f(T_1) = \left(\alpha_j, \alpha_t\right) = (\alpha_k, \alpha_s) = f(T_2)$

f is a group homomorphism, that is because:

$T_1 \times T_2 = \alpha_j\alpha_k + \left(\alpha_t\alpha_s - \alpha_j\alpha_k\right)I$

$f(T_1 \times T_2) = \left(\alpha_j\alpha_k, \alpha_t\alpha_s\right) = \left(\alpha_j, \alpha_t\right) \times (\alpha_k, \alpha_s) = f(T_1) \times f(T_2)$.

It is clear that f is surjective. Also, f is injective that is because:

$$ker(f) = \left\{\alpha_j + \left(\alpha_t - \alpha_j\right)I \in U;\ \left(\alpha_j, \alpha_t\right) = (1,1)\right\} = \{1\}$$

Thus f is a group isomorphic, hence $U \cong Z_n \times Z_n$.

Refined neutrosophic roots of unity.

Let $C(I_1, I_2) = \left\{X + YI_1 + ZI_2;\ I_1I_2 = I_2I_1 = I_1, {I_1}^2 = I_1, {I_2}^2 = I_2, X, Y, Z \in C\right\}$ be the complex ring of refined neutrosophic numbers.

According to [ ], we have:

$(X + YI_1 + ZI_2)^n = X^n + I_1[(X + Y + Z)^n - (X + Z)^n] + I_2[(X + Z)^n - X^n]$

So that $X + YI_1 + ZI_2$ is a refined neutrosophic n-th root of unity if and only if $(X + YI_1 + ZI_2)^n = 1$, thus $X^n = (X + Y + Z)^n = (X + Z)^n = 1$, i.e, $X, X + Y + Z, X + Z$ are three classical roots of unity.

Theorem.

n-th roots of unity in the complex refined neutrosophic ring $C(I_1, I_2)$ are:

$$U = \left\{\alpha_j + (\alpha_t - \alpha_k)I_1 + \left(\alpha_k - \alpha_j\right)I_2;\ \alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}, \alpha_k = e^{\frac{2\pi k}{n}i}; 1 \le j, k, t \le n\right\}$$

Proof.

According to the previous discussion $X + YI_1 + ZI_2$ is a refined neutrosophic n-th root of unity if $X, X + Y + Z, X + Z$ are three roots of unity, thus:

$X = \alpha_j, X + Z = \alpha_k, X + Y + Z = \alpha_t$ where $1 \le j, k, t \le n$ and $\alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}, \alpha_k = e^{\frac{2\pi k}{n}i}$, thus:

$$\begin{cases} X = \alpha_j \\ Y = \alpha_t - \alpha_k \\ Z = \alpha_k - \alpha_j \end{cases}, \text{ thus } X + YI_1 + ZI_2 = \alpha_j + (\alpha_t - \alpha_k)I_1 + \left(\alpha_k - \alpha_j\right)I_2.$$

Theorem.

Let U be the set of refined neutrosophic n-th roots of unity, then U is a group under multiplication with $U \cong Z_n \times Z_n \times Z_n$

Proof.

Let $T_1 = \alpha_j + (\alpha_t - \alpha_k)I_1 + (\alpha_k - \alpha_j)I_2$, $T_2 = \acute{\alpha}_j + (\acute{\alpha}_t - \acute{\alpha}_k)I_1 + (\acute{\alpha}_k - \acute{\alpha}_j)I_2$ be two element of U, then:

$$T_1 \times T_2 = \alpha_j\acute{\alpha}_j + (\alpha_j\acute{\alpha}_t - \alpha_j\acute{\alpha}_k)I_1 + (\alpha_j\acute{\alpha}_k - \alpha_j\acute{\alpha}_j)I_2 + (\acute{\alpha}_j\alpha_t - \acute{\alpha}_j\alpha_k)I_1$$
$$+ (\alpha_t\acute{\alpha}_t - \alpha_t\acute{\alpha}_k - \alpha_k\acute{\alpha}_t + \alpha_k\acute{\alpha}_k)I_1 + (\alpha_t\acute{\alpha}_k - \alpha_t\acute{\alpha}_j - \alpha_k\acute{\alpha}_k + \alpha_k\acute{\alpha}_j)I_1$$
$$+ (\acute{\alpha}_j\alpha_k - \acute{\alpha}_j\alpha_j)I_2 + (\acute{\alpha}_t\alpha_k - \acute{\alpha}_t\alpha_j - \acute{\alpha}_k\alpha_k + \acute{\alpha}_k\alpha_j)I_1$$
$$+ (\alpha_k\acute{\alpha}_k - \alpha_k\acute{\alpha}_j - \alpha_j\acute{\alpha}_k + \alpha_j\acute{\alpha}_j)I_2$$

$T_1 \times T_2 = \alpha_j\acute{\alpha}_j + (\alpha_t\acute{\alpha}_t - \alpha_k\acute{\alpha}_k)I_1 + (\alpha_k\acute{\alpha}_k - \alpha_j\acute{\alpha}_j)I_2 \in U.$

Also, ${T_1}^{-1} = {\alpha_j}^{-1} + ({\alpha_t}^{-1} - {\alpha_j}^{-1})I_1 + ({\alpha_k}^{-1} - {\alpha_j}^{-1})I_2$ is inverse of T_1, so that $(U,\times)$ is a group.

We define $f: U \to Z_n \times Z_n \times Z_n$ such that:

$$f[\alpha_j + (\alpha_t - \alpha_k)I_1 + (\alpha_k - \alpha_j)I_2] = (\alpha_j, \alpha_t, \alpha_k)$$

f is a well define one to one mapping.

f is a group homomorphism that is because:

$f(T_1 \times T_2) = (\alpha_j\acute{\alpha}_j, \alpha_t\acute{\alpha}_t, \acute{\alpha}_k\alpha_k) = (\alpha_j, \alpha_t, \alpha_k) \times (\acute{\alpha}_j, \acute{\alpha}_t, \acute{\alpha}_k) = f(T_1) \times f(T_2)$

So that $U \cong Z_n \times Z_n \times Z_n$.

2-cyclic refined neutrosophic ring.

Let $C_2(I) = \{X + YI_1 + ZI_2;\ I_1I_2 = I_2I_1 = I_1, {I_1}^2 = I_2, {I_2}^2 = I_2, X, Y, Z \in C\}$ be the 2-cyclic complex refined neutrosophic ring.

$X + YI_1 + ZI_2$ is an n-th root of unity in $C_2(I)$ if and only if $(X + YI_1 + ZI_2)^n = 1$.

Firstly, we present a formula to find the n-th power of $X + YI_1 + ZI_2$.

Theorem.

Let $X + YI_1 + ZI_2 \in C_2(I)$, then,

$$T^n = X^n + \frac{1}{2}I_1[(X + Y + Z)^n - (X - Y + Z)^n] + \frac{1}{2}I_2[(X + Y + Z)^n + (X - Y + Z)^n - 2X^n]$$

Proof.

(known befor).

Theorem.

Let $X + YI_1 + ZI_2 \in C_2(I)$, then T is n-th root of unity if and only if $X, X + Y + Z, X - Y + Z$ are three classical roots of unity.

Proof.

$T^n = 1$ is equivalent to:

$X^n + \frac{1}{2}I_1[(X+Y+Z)^n - (X-Y+Z)^n] + \frac{1}{2}I_2[(X+Y+Z)^n + (X-Y+Z)^n - 2X^n] = 1$,

thus:

$$\begin{cases} X^n = 1 \\ (X+Y+Z)^n - (X-Y+Z)^n = 0 \\ (X+Y+Z)^n + (X-Y+Z)^n - 2X^n = 0 \end{cases}$$

This implies that $X^n = (X+Y+Z)^n = (X-Y+Z)^n = 1$, this complete proof.

Theorem.

Let U be the set of all 2-cyclic n-th roots of unity, then:

1. $U = \left\{\alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j];\ \alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}, \alpha_k = e^{\frac{2\pi k}{n}i}; 1 \leq j,k,t \leq n\right\}$.
2. $(U,\times)$ is a group.
3. $U \cong Z_n \times Z_n \times Z_n$.

Proof.

1. Assume that $T = X + YI_1 + ZI_2$ is an n-th root of unity, then $X = \alpha_j, X+Y+Z = \alpha_t, X-Y+Z = \alpha_k$ with $1 \leq j,k,t \leq n$ so that $Y = \frac{1}{2}[\alpha_t - \alpha_k], Z = \frac{1}{2}[\alpha_t + \alpha_k] - \alpha_j$, hence:

$$T = \alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j]$$

2. Let $T_1 = \alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j], T_2 = \acute{\alpha}_j + \frac{1}{2}I_1[\acute{\alpha}_t - \acute{\alpha}_k] + \frac{1}{2}I_2[\acute{\alpha}_t + \acute{\alpha}_k - 2\acute{\alpha}_j]$

We have $T_1 \times T_2 = \alpha_j\acute{\alpha}_j + \frac{1}{2}I_1[\alpha_t\acute{\alpha}_t - \alpha_k\acute{\alpha}_k] + \frac{1}{2}I_2[\alpha_t\acute{\alpha}_t + \alpha_k\acute{\alpha}_k - 2\alpha_j\acute{\alpha}_j] \in U$

The inverse of T_1 is $T^{-1}{}_1 = \alpha_j{}^{-1} + \frac{1}{2}I_1[\alpha_t{}^{-1} - \alpha_k{}^{-1}] + \frac{1}{2}I_2[\alpha_t{}^{-1} + \alpha_k{}^{-1} - 2\alpha_j{}^{-1}] \in U$, so that $(U,\times)$ is a group.

3. Define $f: U \to Z_n \times Z_n \times Z_n$ such that:

$$f\left[\alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j]\right] = (\alpha_j, \alpha_t, \alpha_k)$$

By a similar discussion of previous classification theorems, we get the proof.

Examples.

We find the 3-roots of unity in the neutrosophic complex ring $C(I)$.

In the classical case, we have three roots $\alpha_1 = 1, \alpha_2 = e^{\frac{2\pi}{3}i}, \alpha_3 = e^{\frac{4\pi}{4}i}$, thus the neutrosophic roots of unity are:

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + (\alpha_2 - \alpha_1)I, \alpha_1 + (\alpha_3 - \alpha_1)I, \alpha_2 + (\alpha_1 - \alpha_2)I, \alpha_2 + (\alpha_3 - \alpha_2)I, \alpha_3 + (\alpha_1 - \alpha_3)I, \alpha_3 + (\alpha_2 - \alpha_3)I\}$.

Example.

The 2-nd roots of unity in $C(I_1, I_2)$ are:

$$U = \{\alpha_1, \alpha_2, \alpha_1 + (\alpha_2 - \alpha_1)I_1, \alpha_1 + (\alpha_1 - \alpha_2)I_1 + (\alpha_2 - \alpha_1)I_2, \alpha_2 + (\alpha_1 - \alpha_2)I_2, \alpha_2 + (\alpha_2 - \alpha_1)I_1 + (\alpha_1 - \alpha_2)I_2, \alpha_1 + (\alpha_2 - \alpha_1)I_2, \alpha_2 + (\alpha_1 - \alpha_2)I_1\}\ ;\ \alpha_1 = 1, \alpha_2 = -1$$

Thus

$$U = \{1, -1, 1 - 2I_1, 1 + 2I_1 - 2I_2, -1 + 2I_2, -1 - 2I_1 + 2I_2, 1 - 2I_2, -1 + 2I_1\}.$$

3- Refined and 4-Refined Neutrosophic roots Of Unity

Definition.

Let C be the complex field, the 3-refined neutrosophic complex ring is defined as follows:

$C_3(I) = \{a + bI_1 + cI_2 + dI_3\ ; a, b, c, d \in C\}$, with $I_i.I_j = I_{min(i,j)}, {I_i}^2 = I_i; 1 \le i \le 3$.

The 4-refined neutrosophic complex ring is defined:

$C_4(I) = \{a + bI_1 + cI_2 + dI_3 + eI_4\ ; a, b, c, d, e \in C\}$, with $I_i.I_j = I_{min(i,j)}, {I_i}^2 = I_i; 1 \le i \le 4$.

Definition.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in C_3(I)$, then X is called the n-th root of unity if and only if $X^n = 1$.

X is called the 3-refined neutrosophic root of unity.

Definition.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in C_4(I)$, then X is called the n-th root of unity if and only if $X^n = 1$.

X is called the 4-refined neutrosophic root of unity.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in C_3(I), n \in N$, then:

$$X^n = {x_0}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - {x_0}^n]I_3$$

For the proof see [ ].

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in C_3(I)$, then X is a 3-refined neutrosophic root of unity if and only if $x_0, x_0 + x_3, x_0 + x_2 + x_3x_0 + x_1 + x_2 + x_3$ are roots of unity.

Proof.

$X^n = 1 \Leftrightarrow {x_0}^n = 1, (x_0 + x_3)^n = 1, (x_0 + x_2 + x_3)^n = 1, (x_0 + x_1 + x_2 + x_3)^n = 1$, thus the proof is complete.

Now, we find the 3-refined neutrosophic roots of unity.

Let $U = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the set of classical n-th roots of unity.

If X is 3-refined neutrosophic roots of unity, then $x_0 \in U, x_0 + x_3 \in U, x_0 + x_2 + x_3 \in U, x_0 + x_1 + x_2 + x_3 \in U$.

If $x_0 = \alpha_i, x_0 + x_3 = \alpha_j, x_0 + x_2 + x_3 = \alpha_t, x_0 + x_1 + x_2 + x_3 = \alpha_s$, where $i, j, t, s \in \{1, \dots, n\}$, thus

$x_0 = \alpha_i, x_3 = \alpha_j - \alpha_i, x_2 = \alpha_t - \alpha_j, x_1 = \alpha_s - \alpha_t$.

Remark.

For n, there exists n^4 root of unity in $C_3(I)$.

Example.

For $n = 3$, we have $U = \{1, \alpha_1, \alpha_2\}$, with $\alpha_1 = e^{i\frac{2\pi}{3}}, \alpha_2 = e^{i\frac{4\pi}{3}}$, hence the 3-refined neutrosophic cubic roots of unity are:

$X = t_0 + (t_1 - t_2)I_1 + (t_2 - t_3)I_2 + (t_3 - t_0)I_3$, where $t_i \in U$.

We show some of them:

$X = 1 + (\alpha_1 - \alpha_2)I_1 + (\alpha_2 - 1)I_2 + (1 - \alpha_2)I_3$, $(t_0 = 1, t_1 = \alpha_1, t_2 = \alpha_2, t_3 = \alpha_1)$.

$Y = \alpha_2 + (\alpha_2 - \alpha_1)I_1 + (\alpha_2 - 1)I_2 + (1 - \alpha_2)I_3$, $(t_0 = \alpha_2, t_1 = \alpha_1, t_2 = \alpha_2, t_3 = 1)$.

And so on.

Remark.

Since $(U,\times)$ is a group with order n(cyclic group), the corresponding set of 3-refined neutrosophic roots of unity is an abelian group with order n^4.

Also, it is isomorphic to $U_n \times U_n \times U_n \times U_n$.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3$, $Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3 \in C_3(I)$, then:

$X^Y = {x_0}^{y_0} + [(x_0 + x_1 + x_2 + x_3)^{y_0+y_1+y_2+y_3} - (x_0 + x_2 + x_3)^{y_0+y_2+y_3}]I_1 + [(x_0 + x_2 + x_3)^{y_0+y_2+y_3} - (x_0 + x_3)^{y_0+y_3}]I_2 + [(x_0 + x_3)^{y_0+y_3} - {x_0}^{y_0}]I_3$.

Check [ ].

Definition.

We define the unity duplet (X, Y) as follows:

(X, Y) is a unity duplet if and only if $X^Y = 1$, where $X \in C_3(I), Y \in C_3(I)$.

Theorem.

Let (X, Y) be a unity duplet, this equivalents:

$x_0^{y_0} = (x_0 + x_3)^{y_0+y_3} = (x_0 + x_2 + x_3)^{y_0+y_2+y_3} = (x_0 + x_1 + x_2 + x_3)^{y_0+y_1+y_2+y_3} = 1$.

The proof is clear.

Example.

Take $X = 1 + \left(e^{i\frac{2\pi}{3}} - i\right) I_1 + \left(i - e^{i\frac{\pi}{4}}\right) I_2 + \left(e^{i\frac{\pi}{4}} - 1\right) I_3, Y = 2 - I_1 - 4I_2 + 6I_3$.

We have $X^Y = 1$, hence (X, Y) is a unity duplet.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in C_4(I), n \in N$, then:

$$X^n = x_0{}^n + [(x_0 + x_1 + x_2 + x_3 + x_4)^n - (x_0 + x_2 + x_3 + x_4)^n]I_1$$
$$+ [(x_0 + x_2 + x_3 + x_4)^n - (x_0 + x_3 + x_4)^n]I_2$$
$$+ [(x_0 + x_3 + x_4)^n - (x_0 + x_4)^n]I_3 + [(x_0 + x_4)^n - x_0{}^n]I_4$$

Proof.

The proof can be checked easily by induction.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in C_4(I)$, then X is an n-th root of unity if and only if: $x_0, x_0 + x_4, x_0 + x_3 + x_4, x_0 + x_2 + x_3 + x_4, x_0 + x_1 + x_2 + x_3 + x_4$ are classical n-th roots of unity.

The proof is clear.

Remark.

If $U = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of n-th roots of unity, the corresponding 4-refined neutrosophic roots of unity are:

$\{t_0 + t_1I_1 + t_2I_2 + t_3I_3 + t_4I_4; t_0 = \alpha_i, t_4 = \alpha_j - \alpha_i, t_3 = \alpha_k - \alpha_j, t_2 = \alpha_s - \alpha_k, t_1 = \alpha_l - \alpha_s\}$

where $k, j, i, s, l \in \{1, \ldots, n\}$.

Example.

For $n = 4$, we have $U = \{1, -1, i, -i\}$.

The 4-refined neutrosophic roots of unity for $n = 4$ are:

$\{X = t_0 + t_1I_1 + t_2I_2 + t_3I_3 + t_4I_4\}$, with $t_0 \in U, t_0 + t_4 \in U, t_0 + t_3 + t_4 \in U, t_0 + t_2 + t_3 + t_4 \in U, t_0 + t_1 + t_2 + t_3 + t_4 \in U$.

For example $X = i + (-2i)I_2 + (-1 + i)I_3 + (1 - i)I_4$.

$(t_0 = i, t_4 = 1 - i, t_3 = -1 + i, t_2 = -2i, t_1 = 0)$.

Conclusion

In this paper, we have studied the roots of unity of five neutrosophic different kinds of rings, where the roots of unity in neutrosophic rings, refined neutrosophic rings, 3-refined, 4-refined neutrosophic rings, and 2-cyclic refined neutrosophic rings are obtained and classified as direct products of well known classical finite groups. Many related examples were presented and discussed to clarify the validity of our work.

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A Study of Algebraic Curves in Neutrosophic Real Ring $R(I)$ by Using the One-Dimensional Geometric AH-Isometry

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Abstract: The objective of this paper is to study and define some algebraic curves with neutrosophic variables in neutrosophic real field $R(I)$, where we study what are the relationships between classical algebraic curves and neutrosophic algebraic curves depending on the geometric isometry (AH-Isometry).

Keywords: Neutrosophic real ring $R(I)$, AH-isometry, Neutrosophic algebraic curves.

Introduction

Algebraic Geometry is one of the branches of algebra that deals with the study of geometric shapes through familiar algebraic concepts and theories [1]. There were several approaches to geometry, all of which are usually classified as algebraic geometry, at the end of the nineteenth century. Lazare Carnot (1753-1823) attributed to algebraic geometry which is about algebraic curves and their intersection with the sides of a triangle [2], but this concept developed a lot in the second half of the nineteenth century.

Neutrosophy is a new branch of philosophy concerns with the indeterminacy in all areas of life and science. It has become a useful tool in generalizing many classical systems such as equations [30], number theory [3], and linear spaces [4,5], and ring of matrices [19-31].

Recently, Abobala, and Hatip have presented the concept of one-dimensional AH-isometry to study the correspondence between neutrosophic plane $\boldsymbol{R}(\boldsymbol{I})$ and the classical module $\boldsymbol{R} \times \boldsymbol{R}$.

In this work, we use the one-dimensional AH-isometry to turn the general case of algebraic curves in real ring $\boldsymbol{R}(\boldsymbol{I})$ with one variable into two classical algebraic curves so we will go from $\boldsymbol{R}(\boldsymbol{I})$ space into $\boldsymbol{R} \times \boldsymbol{R}$ space, we study the properties of our algebraic curves then we go back to $\boldsymbol{R}(\boldsymbol{I})$ space using AH-isometry.

Neutrosophic Functions on $\boldsymbol{R}(\boldsymbol{I})$.

Definition:

Let $R(I) = \{a + bI\,; a, b \in R\}$ where $I^2 = I$ be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows: [49]

$$T: R(I) \to R \times R$$
$$T(a + bI) = (a, a + b)$$

Definition:

Let $f: R(I) \to R(I); f = f(X)$ and $X = x + yI \in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable.

Example:

Take $f: R(I) \to R(I); f(X) = X^2 + IX + 2I = (x + yI)^2 + I(x + yI) + 2I$

$$= x^2 + I(y^2 + 2xy + x + y + 2)$$

Theorem:

Let $f: R(I) \to R(I)$ be a neutrosophic real function with one variable,$X = x + yI \in R(I)$ then f can be turned into two classical real functions.

Computing Powers in $\boldsymbol{R}(\boldsymbol{I})$.

To compute such equation: $(a + bI)^{c+dI}\,; a, b, c, d \in R$ we need the one-dimensional isometry again:

$T[(a + bI)^{c+dI}] = (a, a + b)^{(c,c+d)} = \left(a^c, (a + b)^{c+d}\right)$,

Which means

$(a + bI)^{c+dI} = T^{-1}\left(a^c, (a + b)^{c+d}\right)$,

$= a^c + I[(a + b)^{c+d} - a^c]$.

Theorem:

Let $R(I)$ be the neutrosophic field of reals, we have:

1. $\sin(a+bI)=\sin a+I[\sin(a+b)-\sin a\ ]$
2. $\cos(a+bI)=\cos a+I[\cos(a+b)-\cos a]$
3. $e^{x+Iy}=e^{x}+I(e^{x+y}-e^{x})$

Algebraic Curves In Neutrosophic Real Ring $\boldsymbol{R(I)}$:

Definition : Neutrosophic Strophoide.

Let $\boldsymbol{Y=y_1+y_2I, X=x_1+x_2I, A=a_1+a_2I\in R(I), a_1, a_2, x_1, x_2, y_1, y_2\in R}$, then we define a neutroophic strophoide as follows:

$$\boldsymbol{Y^2=X^2.\frac{A+X}{A-X}\ ;A>0}$$

This equation can be written as follows:

$$\boldsymbol{(y_1+y_2I)^2=(x_1+x_2I)^2.\frac{(a_1+a_2I)+(x_1+x_2I)}{(a_1+a_2I)-(x_1+x_2I)}\ ;a_1+a_2I>0}$$

Theorem :

Let $Y=y_1+y_2I, X=x_1+x_2I, A=a_1+a_2I\in R(I)$, then if $A=a_1+a_2I$ is invertible, the neutrosophic strophoide $(y_1+y_2I)^2=(x_1+x_2I)^2.\frac{(a_1+a_2I)+(x_1+x_2I)}{(a_1+a_2I)-(x_1+x_2I)}$ is equivalent to the direct product of two classical strophoide.

Proof. Consider the equation $(y_1+y_2I)^2=(x_1+x_2I)^2.\frac{(a_1+a_2I)+(x_1+x_2I)}{(a_1+a_2I)-(x_1+x_2I)}$

Now, we have:

$$y_1{}^2+(y_2{}^2+2y_1y_2)I=[x_1{}^2+(x_2{}^2+2x_1x_2)I].\frac{(a_1+x_1)+(a_2+x_2)I}{(a_1-x_1)+(a_2-x_2)I}$$

$$\begin{aligned}y_1{}^2&+(y_2{}^2+2y_1y_2)I\\&=[x_1{}^2+(x_2{}^2+2x_1x_2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)}\right.\\&\left.+\frac{(a_1-x_1)(a_2+x_2)-(a_1+x_1)(a_2-x_2)}{(a_1-x_1)[(a_1+a_2)-(x_1+x_2)]}\ I\right]\end{aligned}$$

by computing its direct image with AH-isometry, we get:

$$T({y_1}^2 + ({y_2}^2 + 2y_1y_2)I) = T({x_1}^2 + ({x_2}^2 + 2x_1x_2)I)T\left(\frac{(a_1 + x_1)}{(a_1 - x_1)} + \frac{(a_1 - x_1)(a_2 + x_2) - (a_1 + x_1)(a_2 - x_2)}{(a_1 - x_1)[(a_1 + a_2) - (x_1 + x_2)]}\ I\right)$$

$$({y_1}^2, {y_1}^2 + {y_2}^2 + 2y_1y_2) = ({x_1}^2, {x_1}^2 + {x_2}^2 + 2x_1x_2).\left(\frac{(a_1 + x_1)}{(a_1 - x_1)}, \frac{(a_1 + x_1)}{(a_1 - x_1)} + \frac{(a_1 - x_1)(a_2 + x_2) - (a_1 + x_1)(a_2 - x_2)}{(a_1 - x_1)[(a_1 + a_2) - (x_1 + x_2)]}\right)$$

Then.

$$({y_1}^2, (y_1 + y_2)^2) = ({x_1}^2, (x_1 + x_2)^2)$$

$$.\left(\frac{(a_1 + x_1)}{(a_1 - x_1)}, \frac{(a_1 + x_1)[(a_1 + a_2) - (x_1 + x_2)] + (a_1 - x_1)(a_2 + x_2) - (a_1 + x_1)(a_2 - x_2)}{(a_1 - x_1)[(a_1 + a_2) - (x_1 + x_2)]}\right)$$

$$({y_1}^2, (y_1 + y_2)^2) = ({x_1}^2, (x_1 + x_2)^2).\left(\frac{(a_1 + x_1)}{(a_1 - x_1)}, \frac{(a_1 + x_1)[(a_1 + a_2) - (x_1 + x_2) - (a_2 - x_2)] + (a_1 - x_1)(a_2 + x_2)}{(a_1 - x_1)[(a_1 + a_2) - (x_1 + x_2)]}\right)$$

$$({y_1}^2, (y_1 + y_2)^2) = ({x_1}^2, (x_1 + x_2)^2).\left(\frac{(a_1 + x_1)}{(a_1 - x_1)}, \frac{(a_1 + x_1)[(a_1 - x_1)] + (a_1 - x_1)(a_2 + x_2)}{(a_1 - x_1)[(a_1 + a_2) - (x_1 + x_2)]}\right)$$

$$({y_1}^2, (y_1 + y_2)^2) = ({x_1}^2, (x_1 + x_2)^2).\left(\frac{(a_1 + x_1)}{(a_1 - x_1)}, \frac{(a_1 - x_1)(a_1 + x_1 + a_2 + x_2)}{(a_1 + x_1)[(a_1 + a_2) - (x_1 + x_2)]}\right)$$

$$({y_1}^2, (y_1 + y_2)^2) = ({x_1}^2, (x_1 + x_2)^2).\left(\frac{(a_1 + x_1)}{(a_1 - x_1)}, \frac{(a_1 + a_2) + (x_1 + x_2)}{(a_1 + a_2) - (x_1 + x_2)}\right)$$

$$({y_1}^2, (y_1 + y_2)^2) = \left({x_1}^2 \frac{(a_1 + x_1)}{(a_1 - x_1)}, (x_1 + x_2)^2 \frac{(a_1 + a_2) + (x_1 + x_2)}{(a_1 + a_2) - (x_1 + x_2)}\right)$$

So that we have:

$$\begin{cases} \boldsymbol{\Gamma_1: y_1{}^2 = x_1{}^2 \dfrac{(a_1 + x_1)}{(a_1 - x_1)}; a_1 > 0} \\ \boldsymbol{\Gamma_2: (y_1 + y_2)^2 = (x_1 + x_2)^2 \dfrac{(a_1 + a_2) + (x_1 + x_2)}{(a_1 + a_2) - (x_1 + x_2)}; (a_1 + a_2) > 0} \end{cases}$$

Remark :

If $a_1 + a_2 I$ is invertible, we can write the equation of neutrosophic strophoide as follows:

$(y_1 + y_2 I)^2 = (x_1 + x_2 I)^2 . \frac{(a_1 + a_2 I) + (x_1 + x_2 I)}{(a_1 + a_2 I) - (x_1 + x_2 I)}$.

Now, we should discuss the cases of non-invertible of $a_1 + a_2 I$.

$a_1 + a_2 I$ is not invertible, then we have cases:

1- $a_1 = 0,\ a_1 + a_2 \neq 0$, this means that the neutrosophic strophoide will be equivalent to direct product of classical strophoide $(y_1 + y_2)^2 = (x_1 + x_2)^2 \frac{(a_1 + a_2) + (x_1 + x_2)}{(a_1 + a_2) - (x_1 + x_2)}; (a_1 + a_2) > 0$ with classical image two line $\begin{cases} y_1 = i\, x_1 \\ y_1 = -i\, x_1 \end{cases}$.

2- $a_1 \neq 0,\ a_1 + a_2 = 0$, this implies that the neutrosophic strophoide will be equivalent to direct product of classical strophoide $y_1{}^2 = x_1{}^2 \frac{(a_1 + x_1)}{(a_1 - x_1)}; a_1 > 0$ with classical image two line $\begin{cases} y_1 = i\, (x_1 + x_2) \\ y_1 = -i\, (x_1 + x_2) \end{cases}$.

3- If $a_1 = 0,\ a_1 + a_2 = 0$, this implies that the neutrosophic strophoide will be equivalent to direct product of classical image two line $\begin{cases} y_1 = i\, x_1 \\ y_1 = -i\, x_1 \end{cases}$ with classical image two line $\begin{cases} y_1 = i\, (x_1 + x_2) \\ y_1 = -i\, (x_1 + x_2) \end{cases}$.

Theorem:

Let Γ_1, Γ_2 are two classical strophoide, then the direct product of Γ_1, Γ_2 is equivalent to the neutrosophic strophoide Γ.

Proof.

Let Γ_1, Γ_2 are two classical strophoide, where:

$$\begin{cases} \Gamma_1: y_1{}^2 = x_1{}^2 . \dfrac{(a_1 + x_1)}{(a_1 - x_1)}; a_1 > 0 \\ \Gamma_2: y_2{}^2 = x_2{}^2 . \dfrac{(a_2 + x_2)}{(a_2 - x_2)}; a_2 > 0 \end{cases}$$

Now, we take the inverse image of the AH-isometry, we have:

$$T^{-1}(y_1{}^2, y_2{}^2) = T^{-1}(x_1{}^2, x_2{}^2).T^{-1}\left(\frac{(a_1+x_1)}{(a_1-x_1)}, \frac{(a_2+x_2)}{(a_2-x_2)}\right)$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I = [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)} + \left(\frac{(a_2+x_2)}{(a_2-x_2)} - \frac{(a_1+x_1)}{(a_1-x_1)}\right)I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I$$
$$= [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)} + \left(\frac{(a_2+x_2)(a_1-x_1) - (a_2-x_2)(a_1+x_1)}{(a_1-x_1)(a_2-x_2)}\right)I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I = [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)} + \left(\frac{(a_2+x_2)}{(a_2-x_2)} - \frac{(a_1+x_1)}{(a_1-x_1)}\right)I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I$$
$$= [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)} + \left(\frac{(a_2+x_2)}{(a_2-x_2)} - \frac{(a_1+x_1)}{(a_1-x_1)} + \frac{(a_1+x_1)}{(a_2-x_2)} - \frac{(a_1+x_1)}{(a_2-x_2)}\right)I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I$$
$$= [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)} + \left(\left\{\frac{(a_2+x_2)}{(a_2-x_2)} - \frac{(a_1+x_1)}{(a_2-x_2)}\right\} + \left\{-\frac{(a_1+x_1)}{(a_1-x_1)} + \frac{(a_1+x_1)}{(a_2-x_2)}\right\}\right)I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I$$
$$= [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1+x_1)}{(a_1-x_1)} + \left\{\frac{(a_2+x_2) - (a_1+x_1)}{(a_2-x_2)} - \frac{(a_1+x_1)(a_2-x_2) - (a_1+x_1)(a_1-x_1)}{(a_1-x_1)(a_2-x_2)}\right\}I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I$$
$$= [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1 + x_1)}{(a_1 - x_1)} + \frac{(a_1 - x_1)[(a_2 + x_2) - (a_1 + x_1)] - (a_1 + x_1)(a_2 - x_2) - (a_1 + x_1)(a_1 - x_1)}{(a_1 - x_1)(a_2 - x_2)}I\right]$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I = [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].$$

$$\left[\frac{(a_1 + x_1)}{(a_1 - x_1)} + \frac{(a_1 - x_1)[(a_2 + x_2) - (a_1 + x_1)] - (a_1 + x_1)[(a_2 - x_2) - (a_1 - x_1)]}{(a_1 - x_1)(a_2 - x_2)}I\right]\frac{(a_1 + x_1)}{(a_1 - x_1)} + \frac{(a_1 - x_1)[(a_2 + x_2) - (a_1 + x_1)] - (a_1 + x_1)[(a_2 - x_2) - (a_1 - x_1)]}{(a_1 - x_1)(a_2 - x_2)}I = \frac{(a_1 + x_1) + [(a_2 + x_2) - (a_1 + x_1)]I}{(a_1 - x_1) + [(a_2 - x_2) - (a_1 - x_1)]I}$$

$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I = [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{(a_1 + x_1) + [(a_2 + x_2) - (a_1 + x_1)]I}{(a_1 - x_1) + [(a_2 - x_2) - (a_1 - x_1)]I}\right]$$
$$y_1{}^2 + (y_2{}^2 - y_1{}^2)I = [x_1{}^2 + (x_2{}^2 - x_1{}^2)I].\left[\frac{a_1 + (a_2 - a_1)I + [x_1 + (x_2 - x_1)I]}{a_1 + (a_2 - a_1)I - [x_1 + (x_2 - x_1)I]}\right] \dots (*)$$

We let $X = x_1 + (x_2 - x_1)I, Y = y_1 + (y_2 - y_1)I, A = a_1 + (a_2 - a_1)$, then we can prove that:

$$Y^2 = y_1{}^2 + (y_2{}^2 - y_1{}^2)I, X^2 = x_1{}^2 + (x_2{}^2 - x_1{}^2)$$

Then the equation $(*)$ can be written as follows:

$$\Gamma: Y^2 = X^2.\frac{A + X}{A - X} \quad ; A > 0$$

This equation is a neutrosophic strophoide Γ.

Example:

Let the equation by a neutrosophic strophoide:

$$\Gamma: (y_1 + y_2 I)^2 = (x_1 + x_2 I)^2.\frac{(4 - 2I) + (x_1 + x_2 I)}{(4 - 2I) - (x_1 + x_2 I)}$$

Then, its equation be equivalent to direct product of two classical strophoide:

$$\begin{cases} \Gamma_1: y_1{}^2 = x_1{}^2\left(\frac{4 + x_1}{4 - x_1}\right) \\ \Gamma_2: (y_1 + y_2)^2 = (x_1 + x_2)^2\left(\frac{2 + x_1}{2 - x_1}\right) \end{cases}$$

Example:

Let Γ_1, Γ_2 are two classical strophoide, where:

$$\begin{cases} \Gamma_1 : {y_1}^2 = {x_1}^2\left(\dfrac{\frac{1}{2}+x_1}{\frac{1}{2}-x_1}\right) \\ \Gamma_2 : {y_2}^2 = {x_2}^2\left(\dfrac{2+x_2}{2-x_2}\right) \end{cases}$$

Then by theorem 6.4 we have.

$$\Gamma : Y^2 = X^2.\frac{\left(\frac{1}{2}+\frac{3}{2}I\right)+X}{\left(\frac{1}{2}+\frac{3}{2}I\right)-X}$$

.**Definition: Neutrosophic Cycloide.**

Let $\boldsymbol{Y = y_1 + y_2 I, X = x_1 + x_2 I, R = r_1 + r_2 I\ , t = t_1 + t_2 I \in}$ $\boldsymbol{R(I), r_1, r_2, t_1, t_2, x_1, x_2, y_1, y_2 \in R}$, then we define a neutroophic Cycloide as follows:

$$\boldsymbol{X = \mathrm{R}(1 - sint)\ , Y = \mathrm{R}(1 - cost)}$$

This equation can be written as follows:

$$\boldsymbol{x_1 + x_2 I = (r_1 + r_2 I)\big(1 - sin(t_1 + t_2 I)\big)\ , y_1 + y_2 I = (r_1 + r_2 I)\big(1 - cos(t_1 + t_2 I)\big)}$$

Theorem:

Let $\boldsymbol{Y = y_1 + y_2 I, X = x_1 + x_2 I}, R = r_1 + r_2 I\ , t = t_1 + t_2 I \in \boldsymbol{R(I)}$, then if $r_1 + r_2 I$ is invertible, the neutrosophic Cycloide $X = \mathrm{R}(1 - sint)\ , Y = \mathrm{R}(1 - cost)$ is equivalent to the direct product of two classical Cycloide.

Proof. Consider the equation $X = \mathrm{R}(1 - sint)\ , Y = \mathrm{R}(1 - cost)$

Now, we have:

$$x_1 + x_2 I = (r_1 + r_2 I)\Big(1 - sin(t_1) - I\big(sin(t_1 + t_2) - sin(t_1)\big)\Big)$$

$$y_1 + y_2 I = (r_1 + r_2 I)\Big(1 - cos(t_1) - I\big(cos(t_1 + t_2) - cos(t_1)\big)\Big)$$

by computing its direct image with AH-isometry, we get:

$$T(x_1 + x_2 I) = T(r_1 + r_2 I).T(1 - sin(t_1) - I[sin(t_1 + t_2) - sin(t_1)])$$

$(x_1, x_1 + x_2) = (r_1, r_1 + r_2).\left(1 - sin(t_1), 1 - sin(t_1 + t_2)\right)$

$$(x_1, x_1 + x_2) = \left(r_1\left(1 - sin(t_1)\right), (r_1 + r_2)\left(1 - sin(t_1 + t_2)\right)\right)$$

Then.

$$\begin{cases} x_1 = r_1\left(1 - sin(t_1)\right) \\ x_1 + x_2 = (r_1 + r_2)\left(1 - sin(t_1 + t_2)\right) \end{cases}$$

By a similar, we have.

$$\begin{cases} y_1 = r_1\left(1 - cos(t_1)\right) \\ y_1 + y_2 = (r_1 + r_2)\left(1 - cos(t_1 + t_2)\right) \end{cases}$$

So that we have:

$$\begin{cases} \boldsymbol{\Gamma_1: x_1 = r_1\left(1 - sin(t_1)\right), y_1 = r_1\left(1 - cos(t_1)\right)} \\ \boldsymbol{\Gamma_2: x_1 + x_2 = (r_1 + r_2)\left(1 - sin(t_1 + t_2)\right), y_1 + y_2 = (r_1 + r_2)\left(1 - cos(t_1 + t_2)\right)} \end{cases}$$

Remark:

If $r_1 + r_2 I$ is invertible, we can write the equation of neutrosophic cycloide as follows:

$X = \mathrm{R}(1 - sint)$, $Y = \mathrm{R}(1 - cost)$.

Now, we should discuss the cases of non-invertible of $r_1 + r_2 I$.

The $r_1 + r_2 I$ is not invertible, then we have two cases:

1- $r_1 = 0$, $r_1 + r_2 \neq 0$, this means that the neutrosophic cycloide will be equivalent to direct product of classical cycloide $x_1 + x_2 = (r_1 + r_2)\left(1 - sin(t_1 + t_2)\right), y_1 + y_2 = (r_1 + r_2)\left(1 - cos(t_1 + t_2)\right)$ with the origin point (0,0).

2- $r_1 \neq 0$, $r_1 + r_2 = 0$, this means that the neutrosophic cycloide will be equivalent to direct product of classical cycloide $x_1 = r_1\left(1 - sin(t_1)\right), y_1 = r_1\left(1 - cos(t_1)\right)$ with the origin point (0,0).

3- If $r_1 = 0$, $r_1 + r_2 = 0$, this implies that the neutrosophic cycloide will be equivalent to the origin point (0,0).

Theorem:

Let Γ_1, Γ_2 are two classical cycloide, then the direct product of Γ_1, Γ_2 is equivalent to the neutrosophic cycloide Γ.

Proof.

Let Γ_1, Γ_2 are two classical cycloide, where:

$$\begin{cases} \Gamma_1: x_1 = r_1(1 - sint_1)\,, y_1 = r_1(1 - cost_1) \\ \Gamma_2: x_2 = r_2(1 - sint_2)\,, y_2 = r_2(1 - cost_2) \end{cases}$$

Now, we take the inverse image of the AH-isometry, we have:

$$T^{-1}(x_1, x_2) = T^{-1}(r_1, r_2).T^{-1}\big((1 - sint_1), (1 - sint_2)\big)$$

$$x_1 + (x_2 - x_1)I = [r_1 + (r_2 - r_1)I].\big[1 - sint_1 + \big((1 - sint_2) - (1 - sint_1)\big)I\big]$$

$$x_1 + (x_2 - x_1)I = [r_1 + (r_2 - r_1)I].[1 - sint_1 + (1 - sint_2 - 1 + sint_1)I]$$

$$x_1 + (x_2 - x_1)I = [r_1 + (r_2 - r_1)I].[1 - sint_1 - (sint_2 - sint_1)I]$$

$$x_1 + (x_2 - x_1)I = [r_1 + (r_2 - r_1)I].\big[1 - \big(sint_1 + (sint_2 - sint_1)\big)I\big]$$

$$x_1 + (x_2 - x_1)I = [r_1 + (r_2 - r_1)I].\big[1 - \big(sint_1 + (sin(t_2 - t_1 + t_1) - sint_1)\big)I\big]$$

We let $X = x_1 + (x_2 - x_1)I, R = r_1 + (r_2 - r_1)I, t = t_1 + (t_2 - t_1)I$, then we can prove that:

$$sint = sin[t_1 + (t_2 - t_1)I] = sint_1 + (sin(t_2 - t_1 + t_1) - sint_1)I$$

Then, we have.

$$X = R.(1 - sint)$$

Now, by the same argument, we have.

$$Y = R.(1 - cost)$$

So.

$$\Gamma: \begin{cases} X = R.(1 - sint) \\ Y = R.(1 - cost) \end{cases}$$

This equation is a neutrosophic cycloid Γ.

Example:

Let the equation by a neutrosophic cycloide:

$$\begin{cases} x_1 + x_2 I = (3 - 2I).[1 - sin(t_1 + t_2 I)] \\ y_1 + y_2 I = (3 - 2I).[1 - cos(t_1 + t_2 I)] \end{cases}$$

Then, its equation be equivalent to direct product of two classical cycloide:

$$\begin{cases} \boldsymbol{\Gamma_1: x_1 = 3(1 - sint_1)\,, y_1 = 3(1 - cost_1)} \\ \boldsymbol{\Gamma_2: x_1 + x_2 = 1 - sin(t_1 + t_2)\,, y_1 + y_2 = 1 - cos(t_1 + t_2)} \end{cases}$$

Example:

Let Γ_1, Γ_2 are two classical cycloide, where:

$$\begin{cases} \Gamma_1: x_1 = 2(1 - sint_1)\,, y_1 = 2(1 - cost_1) \\ \Gamma_2: x_2 = 5(1 - sint_2)\,, y_2 = 5(1 - cost_2) \end{cases}$$

$$\Gamma: \begin{Bmatrix} X = (2 + 3I)(1 - sint) \\ Y = (2 + 3I)(1 - cost) \end{Bmatrix}$$

.**Definition: Neutrosophic Cardioide.**

Let $\boldsymbol{\rho = \rho_1 + \rho_2 I, \theta = \theta_1 + \theta_2 I \in R(I), \rho_1, \rho_2, \theta_1, \theta_2 \in R}$, then we define a neutroophic Cardoide as follows:

$$\boldsymbol{\rho = (1 + cos\theta)}$$

This equation can be written as follows:

$$\boldsymbol{\rho_1 + \rho_2 I = (1 + cos\theta_1) + [cos(\theta_1 + \theta_2) - cos\theta_1]I}$$

Theorem:

Let $= \boldsymbol{\rho_1 + \rho_2 I, \theta = \theta_1 + \theta_2 I \in R(I)}$, then if $\boldsymbol{\theta_1 + \theta_2 I}$ is invertible, the neutrosophic Cardioide

$\boldsymbol{\rho = (1 + cos\theta)}$ is equivalent to the direct product of two classical Cardioide.

Proof. Consider the equation $\rho = (1 + cos\theta)$

Now, we have:

$\rho_1 + \rho_2 I = (1 + cos\theta_1) + [cos(\theta_1 + \theta_2) - cos\theta_1]I$

by computing its direct image with AH-isometry, we get:

$T(\rho_1 + \rho_2 I) = T((1 + cos\theta_1) + [cos(\theta_1 + \theta_2) - cos\theta_1]I)$

$(\rho_1, \rho_1 + \rho_2) = (1 + cos\theta_1, 1 + cos(\theta_1 + \theta_2))$

Then.

$$\begin{cases} \rho_1 = 1 + cos\theta_1 \\ \rho_1 + \rho_2 = 1 + cos(\theta_1 + \theta_2) \end{cases}$$

So that we have:

$$\begin{cases} \boldsymbol{\Gamma_1: \rho_1 = 1 + cos\theta_1} \\ \boldsymbol{\Gamma_2: \rho_1 + \rho_2 = 1 + cos(\theta_1 + \theta_2)} \end{cases}$$

Remark:

If $\theta_1 + \theta_2 I$ is invertible, we can write the equation of neutrosophic Cardioide as follows:

$\rho = (1 + cos\theta)$.

Now, we should discuss the cases of non-invertible of $\theta_1 + \theta_2 I$.

The $\theta_1 + \theta_2 I$ is not invertible, then we have two cases:

1- $\theta_1 = 0,\ \theta_1 + \theta_2 \neq 0$, this means that the neutrosophic Cardioide will be equivalent to direct product of classical Cardioide $(\rho_1 + \rho_2) = 1 + cos(\theta_1 + \theta_2)$ with the classical circle $\rho_1 = 2$.

2- $\theta_1 \neq 0,\ \theta_1 + \theta_2 = 0$, this means that the neutrosophic Cardioide will be equivalent to direct product of classical Cardioide $\rho_1 = 1 + cos(\theta_1)$ with the classical circle $(\rho_1 + \rho_2) = 2$.

3- If $\theta_1 = 0,\ \theta_1 + \theta_2 = 0$ this means that the neutrosophic Cardioide will be equivalent to direct product of classical circle $(\rho_1 + \rho_2) = 2$ with the classical circle $\rho_1 = 2$.

Theorem:

Let Γ_1, Γ_2 are two classical Cardioide, then the direct product of Γ_1, Γ_2 is equivalent to the neutrosophic Cardioide Γ.

Proof.

Let Γ_1, Γ_2 are two classical Cardioide, where:

$$\begin{cases} \Gamma_1: \rho_1 = 1 + cos\theta_1 \\ \Gamma_2: \rho_2 = 1 + cos\theta_2 \end{cases}$$

Now, we take the inverse image of the AH-isometry, we have:

$$T^{-1}(\rho_1, \rho_2) = T^{-1}(1 + cos\theta_1, 1 + cos\theta_2)$$

$$\rho_1 + (\rho_2 - \rho_1)I = [1 + cos\theta_1 + (1 + cos\theta_2 - (1 + cos\theta_1))I]$$

$\rho_1 + (\rho_2 - \rho_1)I = [1 + cos\theta_1 + (cos\theta_2 - cos\theta_1)I]$

$\rho_1 + (\rho_2 - \rho_1)I = 1 + [cos\theta_1 + (cos(\theta_1 + [\theta_2 - \theta_1]) - cos\theta_1)I]$

We let $\rho = \rho_1 + (\rho_2 - \rho_1)I, \theta = \theta_1 + (\theta_2 - \theta_1)I$, then we can prove that:

$$cos\theta = cos(\theta_1 + [\theta_2 - \theta_1]I) = cos\theta_1 + (cos(\theta_1 + [\theta_2 - \theta_1]) - cos\theta_1)I$$

Then, we have.

$$\rho_1 + (\rho_2 - \rho_1)I = 1 + cos(\theta_1 + [\theta_2 - \theta_1]I)$$

So.

$$\Gamma: \rho = 1 + cos\theta$$

This equation is a neutrosophic Cardioide Γ.

Example:

Let the equation by a neutrosophic Cardioide:

$$\rho_1 + \rho_2 I = 1 + cos\left(\frac{\pi}{3} + \frac{\pi}{4}I\right)$$

Then, its equation be equivalent to direct product of two classical Cardioide:

$$\begin{cases} \boldsymbol{\Gamma_1: \rho_1 = 1 + cos\left(\frac{\pi}{3}\right)} \\ \boldsymbol{\Gamma_2: \rho_1 + \rho_2 = 1 + cos\left(\frac{7\pi}{12}\right)} \end{cases}$$

Example:

Let Γ_1, Γ_2 are two classical Cardioide, where:

$$\begin{cases} \Gamma_1: \rho_1 = 1 + cos\left(\frac{\pi}{4}\right) \\ \Gamma_2: \rho_2 = 1 + cos\left(\frac{\pi}{2}\right) \end{cases}$$

$$\Gamma: \left\{\rho = 1 + cos\left(\frac{\pi}{4} + \frac{\pi}{4}I\right)\right.$$

Conclusions

In this paper we have studied some concepts of neutrosophic real analysis depending on the one-dimensional AH-isometry. We have provided a strict definition of some algebraic curves in neutrosophic real ring $\boldsymbol{R(I)}$, and we study the properties of this curves, and we proved some theorems for this curves, also, we find relationships between a classical algerbraic curves and neutrosophic algebraic curves.

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Separation Axioms for Intuitionistic Neutrosophic Crisp supra and Infra Topological Spaces

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Abstract

The objective of this paper is to introduce a new intuitionistic neutrosophic crisp points in intuitionistic neutrosophic crisp topological space, where the intuitionistic neutrosophic crisp limit point was defined using intuitionistic neutrosophic crisp points with some of its properties. Also, a generalized form of intuitionistic neutrosophic crisp topological space as intuitionistic neutrosophic crisp supra topological space and intuitionistic neutrosophic crisp infra topological space were defined. Moreover, the separation axioms were constructed in these new spaces and the relationship between them will be examined in details.

Keywords:

Intuitionistic neutrosophic crisp topological space, intuitionistic neutrosophic crisp supra topological space, intuitionistic neutrosophic crisp infra topological space, intuitionistic neutrosophic crisp point, intuitionistic neutrosophic crisp separation axioms.

Introduction

For the first time in the world, F. Smarandache [1,2,3] introduced the notions of neutrosophic theory as a generalization of the fuzzy and intuitionistic fuzzy theories. Also, D. Cocer [4] introduced the concept of intuitionistic sets and studied its applications in algebraic and topological structures.

As the generalization of classical sets, Salama et al. in 2014 proposed the concept of neutrosophic crisp sets [5]. Neutrosophic crisp sets is a special case of neutrosophic sets.

Recently, J .Kim et al. [6] introduced the concept of intuitionistic neutrosophic crisp sets by combined intuitionistic set and neutrosophic crisp set.

They applied it to topology by defined intuitionistic neutrosophic crisp topological space and studied some concepts related to intuitionistic neutrosophic crisp sets as intuitionistic neutrosophic crisp interior and closure.

In 2015, Adel. M. AL-Odhari [7] have discussed the concept of infra-Topological spaces as an extension of topological space.

Also, G.Jayaparthasarathy et al. presented a more general study, where he created the concept of neutrosophic supra topological spaces [8] in 2019.

A. B.AL-Nafee et al. In 2015, have been discussed the concept of neutrosophic points and separation axioms in neutrosophic crisp topological spaces [9].

In fact, the concept of neutrosophic sets represents an important idea to open the door in front of many researchers especially in pure and applied mathematics [10].

In this paper, we give some important spaces via intuitionistic neutrosophic crisp sets, where we define intuitionistic neutrosophic crisp supra topological space and intuitionistic neutrosophic crisp infra topological space, as well as new sets in these new spaces as intuitionistic neutrosophic crisp supra open (closed) sets and intuitionistic neutrosophic crisp infra open (closed) sets. On other hand we define, for the first time, the intuitionistic neutrosophic crisp points and we use these points to define separation axioms in all of this new spaces (intuitionistic neutrosophic crisp topological space , intuitionistic neutrosophic crisp supra topological space and intuitionistic neutrosophic crisp infra topological space).

1. Basic Concepts

Definition:[4]

Let $X \neq \phi$ be a set. Then A is called an intuitionistic set (IS) of X, if it is an object having the form $A = (A_{\in} , A_{\notin})$; such that $A_{\in} \cap A_{\notin} = \phi$, in this case $A_{\in}$ ($A_{\notin}$) represents the set of memberships (non-memberships) of each element in X.

- The intuitionistic empty set of X, is defined by $\overline{\phi} = (\phi , X)$.

- The intuitionistic whole set of X, is defined by $\overline{X} = (X , \phi)$

- all ISs in X as IS(X).

Definition :[6]

Let X≠ϕ be a set. Then the form $<\tilde{A}_T, \tilde{A}_I, \tilde{A}_F>$;

($\tilde{A}_T = (A_{1,1}, A_{1,2}), \tilde{A}_I = (A_{2,1}, A_{2,2}), \tilde{A}_F = (A_{3,1}, A_{3,2})$ ∈ IS(X)).

is called an intuitionistic neutrosophic crisp set in X (INCS), if $A_{1,1} \cap A_{3,1} = \phi$.

- $\tilde{A}_T = (A_{1,1}, A_{1,2}), \tilde{A}_I = (A_{2,1}, A_{2,2}), \tilde{A}_F = (A_{3,1}, A_{3,2})$ represent the IS of memberships, indeterminacies and non-memberships respectively of each element x ∈ X to A.
- We will denote the set of all INCS by INCS(X).

Definition: [6]

Types of INCS $\overline{\phi}_{IN}$ & $\overline{X}_{IN}$ as follows:

1. $\overline{\phi}_{IN,i}$ may be defined in many ways as a INCS as follows: (i=1,2,3,4)
   1. $\overline{\phi}_{IN,1} = < \overline{\phi}, \overline{\phi}, \overline{X} >$
   2. $\overline{\phi}_{IN,2} = < \overline{\phi}, \overline{X}, \overline{X} >$
   3. $\overline{\phi}_{IN,3} = < \overline{\phi}, \overline{X}, \overline{\phi} >$
   4. $\overline{\phi}_{IN,4} = < \overline{\phi}, \overline{\phi}, \overline{\phi} >$.
2. $\overline{X}_{IN,i}$ may be defined in many ways as a INCS as follows: (i=1,2,3,4)
   1. $\overline{X}_{IN,1} = <\overline{X}, \overline{\phi}, \overline{\phi} >$
   2. $\overline{X}_{IN,2} = < \overline{X}, \overline{X}, \overline{\phi} >$
   3. $\overline{X}_{IN,3} = < \overline{X}, \overline{\phi}, \overline{X} >$
   4. $\overline{X}_{IN,4} = < \overline{X}, \overline{X}, \overline{X} >$.

Definition: [6]

A Intuitionistic neutrosophic crisp topology (INCT) on a non-empty set X is a family T of intuitionistic neutrosophic crisp subsets in X satisfying the following axioms:

1. $\overline{\phi}_{IN,i}$ & $\overline{X}_{IN,i} \in T$. (i=1,2,3,4)
2. $C \cap D \in T$, for any $C, D \in T$.
3. T is closed under arbitrary union.

The pair (X, T) is said to be a intuitionistic neutrosophic crisp topological space (INCTS)in X. Moreover, The elements in T are said to be intuitionistic neutrosophic crisp open sets

(INCOS), a intuitionistic neutrosophic crisp set F is intuitionistic neutrosophic crisp closed (INCCS) if and only if its complement F^c is an intuitionistic neutrosophic crisp open set.

2. Intuitionistic Neutrosophic crisp point

In this part, we will introduce the intuitionistic neutrosophic crisp point and intuitionistic neutrosophic crisp limit points with some of its properties.

Definition 2.1.

For all x, y, z belonging to a non-empty set X. Then the intuitionistic neutrosophic crisp points related to x, y, z are defined as follows:

- x_{I_1}<({x}, X − {x}), $\bar{\phi}$, $\bar{\phi}$ > is called an intuitionistic neutrosophic crisp point ($INCP_{I_1}$) in X .
- y_{I_2}<$\bar{\phi}$,({y}, X − {y}),$\bar{\phi}$ > is called an intuitionistic neutrosophic crisp point ($INCP_{I_2}$) in X.
- z_{I_3}<$\bar{\phi}$,$\bar{\phi}$, ({z}, X − {z}) > is called an intuitionistic neutrosophic crisp point ($INCP_{N_3}$) in X .

The set of all intuitionistic neutrosophic crisp points ($INCP_{I_1}$, $INCP_{I_2}$, $INCP_{I_3}$) is denoted by $INCP_I$

Definition 2.2.

Let X be a non-empty set and x ,y, z ∈X. Then the intuitionistic neutrosophic crisp point:

- x_{I_1} is belonging to the intuitionistic neutrosophic crisp set B⊚⊚($B_{1,1}$, $B_{1,2}$), ($B_{2,1}$, $B_{2,2}$), ($B_{3,1}$, $B_{3,2}$)>, denoted by $x_{I_1} \in B$, if $x \in B_{1,1}$,wherein x_{I_1} not belongs to the intuitionistic neutrosophic crisp set B denoted by $x_{I_1} \notin B$,if $x \notin B_{1,1}$.
- y_{I_2} is belonging to the intuitionistic neutrosophic crisp set B⊚⊚($B_{1,1}$, $B_{1,2}$), ($B_{2,1}$, $B_{2,2}$), ($B_{3,1}$, $B_{3,2}$)>, denoted by $y_{I_2} \in B$, if $y \in B_{2,1}$,wherein y_{I_2} not belongs to the intuitionistic neutrosophic crisp set B denoted by $y_{I_2} \notin B$,if $y \notin B_{2,1}$.
- z_{I_3} is belonging to the intuitionistic neutrosophic crisp set B⊚⊚($B_{1,1}$, $B_{1,2}$), ($B_{2,1}$, $B_{2,2}$), ($B_{3,1}$, $B_{3,2}$)>, denoted by $z_{I_3} \in B$, if $z \in B_{3,1}$,wherein z_{I_3} not belongs to the intuitionistic neutrosophic crisp set B denoted by $z_{I_3} \notin B$,if $z \notin B_{3,1}$.

Definition 2.3.

Let (X, T) be an INCTS , $P \in INCP_N$ in X , an intuitionistic neutrosophic crisp set B<(B1,1 , B1,2), (B2,1 , B2,2), (B3,1 , B3,2)> $\in T$ is said to be intuitionistic neutrosophic crisp open nhd of P in (X, T) if $P \in B$.

Definition 2.4.

Let (X, T) be an INCTS , $P \in INCP_N$ in X , an intuitionistic neutrosophic crisp set B<(B1,1 , B1,2), (B2,1 , B2,2), (B3,1 , B3,2)> is said to be intuitionistic neutrosophic crisp nhd of P in (X, T), if there is an intuitionistic neutrosophic crisp open set A<(A1,1 , A1,2), (A2,1 , A2,2), (A3,1 , A3,2)> containing P such that $A \subseteq B$

Note 2.5.

Every intuitionistic neutrosophic crisp open nhd of any point $P \in INCP_N$ in X is intuitionistic neutrosophic crisp nhd of P.

3 .Separation Axioms In an intuitionistic neutrosophic Crisp Topological Space

Definition 3.1.

An intuitionistic neutrosophic. crisp topological space (X, T) is called:

- I_1-T_o-space if $\forall\ x_{I_1} \neq y_{I_1} \in X$ $\exists$ an intuitionistic neutrosophic crisp open set G in X containing one of them but not the other.
- I_2-T_o-space if $\forall\ x_{I_2} \neq y_{I_2} \in X$ $\exists$ an intuitionistic neutrosophic crisp open set G in X containing one of them but not the other .
- I_3-T_o-space if $\forall\ x_{I_3} \neq y_{I_3} \in X$ $\exists$ an intuitionistic neutrosophic crisp open set G in X containing one of them but not the other.
- I_1-T_1-space if $\forall\ x_{N_1} \neq y_{N_1} \in X$ $\exists$ an intuitionistic neutrosophic crisp open sets G_1, G_2 in X such that $x_{I_1} \in G_1$, $y_{I_1} \notin G_1$ and $x_{I_1} \notin G_2$, $y_{I_1} \in G_2$.
- I_2-T_1-space if $\forall\ x_{N_2} \neq y_{N_2} \in X$ $\exists$ an intuitionistic neutrosophic crisp open sets G_1, G_2 in X such that $x_{I_2} \in G_1$, $y_{I_2} \notin G_1$ and $x_{I_2} \notin G_2$, $y_{I_2} \in G_2$.
- I_3-T_1-space if $\forall\ x_{I_3} \neq y_{I_3} \in X$ $\exists$ an intuitionistic neutrosophic crisp open sets G_1, G_2 in X such that $x_{I_3} \in G_1$, $y_{I_3} \notin G_1$ and $x_{I_3} \notin G_2$, $y_{I_3} \in G_2$.
- I_1-T_2-space if $\forall\ x_{I_1} \neq y_{I_1} \in X$ $\exists$ an intuitionistic neutrosophic crisp open sets G_1,G_2 in X such that $x_{I_1} \in G_1$, $y_{I_1} \notin G_1$ and $x_{I_1} \notin G_2$, $y_{I_1} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.

- I_2-T_2-space if $\forall\ x_{I_2} \neq y_{I_2} \in X\ \exists$ an intuitionistic neutrosophic crisp open sets G_1, G_2 in X such that $x_{I_2} \in G_1$, $y_{I_2} \notin G_1$ and $x_{I_2} \notin G_2$, $y_{I_2} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.
- I_3-T_2-space if $\forall\ x_{I_3} \neq y_{I_3} \in X\ \exists$ an intuitionistic neutrosophic crisp open sets G_1, G_2 in X such that $x_{I_3} \in G_1$, $y_{I_3} \notin G_1$ and $x_{I_3} \notin G_2$, $y_{I_3} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.

Example 3.2.

If $X = \{x, y\}$, $T_1 = \{\overline{\phi}_{IN}\ \&\ \overline{X}_{IN}, A\}$,$T_2 = \{\overline{\phi}_{IN}\ \&\ \overline{X}_{IN}, B\}$, $T_3 = \{X_N, \emptyset_N, G\}$, A <({x},{y}),$\overline{\emptyset}$,$\overline{\emptyset}$>, B <$\overline{\emptyset}$,({y},{x}),$\overline{\emptyset}$> , G <$\overline{\emptyset}$,$\overline{\emptyset}$,({x},{y})>, Then (X,$T_1$) is I_1-T_o-space, (X,T_2) is I_2-T_o-space, (X,T_3) is I_3-T_o-space.

Remark 3.3.

For an intuitionistic neutrosophic crisp topological space (X, T)

- EveryI_i-T_1-space is I_i-T_0-space (i=1,2,3).
- Every I_i-T_2-space is I_i-T_1-space (i=1,2,3).

Proof : the proof holds directly.

Remark 3.4.

The inverse of remark (3.3) is not true as it is shown in the following example :

Example 3.5.

If $= \{x, y\}$, A <({x},{y}),$\overline{\emptyset}$,$\overline{\emptyset}$>, B <$\overline{\emptyset}$,({y},{x}),$\overline{\emptyset}$> , G <$\overline{\emptyset}$,$\overline{\emptyset}$,({x},{y})>, Then:

- When $T = \{\overline{\phi}_{IN}\ \&\ \overline{X}_{IN}, A\}$, then (X,$T$) is I_1-T_o-space but not I_1-T_1-space.
- When $T = \{\overline{\phi}_{IN}\ \&\ \overline{X}_{IN}, B\}$, then (X,$T$) is I_2-T_o-space but not I_2-T_1-space.
- When $T = \{\overline{\phi}_{IN}\ \&\ \overline{X}_{IN}, G\}$, then (X,$T$) is I_3-T_o-space but not I_3-T_1-space.

4. Intuitionistic neutrosophic Crisp Supra Topological Space

Definition 4.1.

An intuitionistic neutrosophic crisp supra topology (INCST) on a non-empty set X is a family T of intuitionistic neutrosophic crisp subsets in X satisfying the following axioms:

1. $\overline{\phi}_{IN,i}$, $\overline{X}_{IN,i} \in T$.
2. T is closed under arbitrary union.

The pair (X,T) is said to be a intuitionistic neutrosophic crisp supra topological space (INCSTS)in X. Moreover, The elements in T are said to be intuitionistic neutrosophic crisp supra open sets (INCSOS), a neutrosophic crisp supra set F is intuitionistic neutrosophic crisp supra closed set (INCSCS) if and only if its complement F^c is an intuitionistic neutrosophic crisp supra open set.

Remark 4.2.

Every (INCTS) is (INCSTS), But the converse not true as it is shown in the following example.

Example 4.3.

Let X={a,b,c,d,e,f,g,i} and $T = \{ \overline{\phi}_{IN}, \overline{X}_{IN}, A_1, A_2, A_3\}$;

$A_1 =< (\{a,b,c\},\{d,e\}),(\{e,f\},\{g\}),(\{g,h\},\{b,i\}) >$

$A_2 =< (\{a,c,d\},\{e,i\}),(\{e,g\},\{h\}),(\{h,i\},\{a\}) >$

$A_3 =< (\{a,b,c,d\},\{e\}),(\{e,f,g\},\phi),(\{g,h,i\},\phi) >$

(X,T) is (INCSTS) , but (X ,T) is not (INCTS). because $A_1, A_2 \in T$ but $A_1 \cap A_2 =< (\{a,c\},\{d,i,e\}),(\{e\},\{g,h\}),(\{h\},\{a,b,i\}) > \notin T$.

Separation Axioms In an intuitionistic neutrosophic Crisp supra Topological Space

Definition 4.5.

An intuitionistic neutrosophic crisp supra topological space (X,T) is called:

- I_1NS-T₀-space if $\forall\ x_{I_1} \neq y_{I_1} \in X\ \exists$ an intuitionistic neutrosophic crisp supra open set G in X containing one of them but not the other.
- I_2NS-T₀-space if $\forall\ x_{I_2} \neq y_{I_2} \in X\ \exists$ an intuitionistic neutrosophic crisp supra open set G in X containing one of them but not the other.
- I_3NS-T₀-space if $\forall\ x_{I_3} \neq y_{I_3} \in X\ \exists$ an intuitionistic neutrosophic crisp open set G in X containing one of them but not the other.
- I_1NS-T₁-space if $\forall\ x_{N_1} \neq y_{N_1} \in X\ \exists$ an intuitionistic neutrosophic crisp supra open sets G_1, G_2 in X such that $x_{I_1} \in G_1$, $y_{I_1} \notin G_1$ and $x_{I_1} \notin G_2$, $y_{I_1} \in G_2$.
- I_2NS-T₁-space if $\forall\ x_{N_2} \neq y_{N_2} \in X\ \exists$ an intuitionistic neutrosophic crisp supra open sets G_1, G_2 in X such that $x_{I_2} \in G_1$, $y_{I_2} \notin G_1$ and $x_{I_2} \notin G_2$, $y_{I_2} \in G_2$.

- □ I_3NS-T_1-space if $\forall$ $x_{I_3} \neq y_{I_3} \in X$ $\exists$ an intuitionistic neutrosophic crisp supra open sets G_1, G_2 in X such that $x_{I_3} \in G_1$, $y_{I_3} \notin G_1$ and $x_{I_3} \notin G_2$, $y_{I_3} \in G_2$
- □ I_1NS-T_2-space if $\forall$ $x_{I_1} \neq y_{I_1} \in X$ $\exists$ an intuitionistic neutrosophic crisp supra open sets G_1,G_2 in X such that $x_{I_1} \in G_1$, $y_{I_1} \notin G_1$ and $x_{I_1} \notin G_2$, $y_{I_1} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$
- □ I_2NS-T_2-space if $\forall$ $x_{I_2} \neq y_{I_2} \in X$ $\exists$ an intuitionistic neutrosophic crisp open supra sets G_1,G_2 in X such that $x_{I_2} \in G_1$, $y_{I_2} \notin G_1$ and $x_{I_2} \notin G_2$, $y_{I_2} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.
- □ I_3NS-T_2-space if $\forall$ $x_{I_3} \neq y_{I_3} \in X$ $\exists$ an intuitionistic neutrosophic crisp open supra sets G_1,G_2 in X such that $x_{I_3} \in G_1$, $y_{I_3} \notin G_1$ and $x_{I_3} \notin G_2$, $y_{I_3} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.

Example 4.6.

If $X = \{x, y\}$, $T = \{\overline{\phi}_{IN}$ & $\overline{X}_{IN}, A, B, C\}$, A <({x},{y}), $\overline{\emptyset}$, $\overline{\emptyset}$ >, B M $\overline{\emptyset}$,({y},{x}), $\overline{\emptyset}$ >, C ◎◎({x},∅),({y},∅) ,$\overline{\emptyset}$>, Then (X,T) is I_1NS-T_o-space, and I_2NS-T_o-space.

Example 4.7.

If $X = \{x, y\}$, $T = \{\overline{\phi}_{IN}$ & $\overline{X}_{IN}, G, A, C\}$, A <({x},{y}), $\overline{\emptyset}$, $\overline{\emptyset}$ >, G ◎◎ $\overline{\emptyset}$, $\overline{\emptyset}$,({x},{y})>, C <({x},∅),$\overline{\emptyset}$,({y},{ x}) >, Then (X,T) is I_3NS-T_o-space.

Remark 4.8.

For an intuitionistic neutrosophic crisp supra topological space (X, T)

- □ Every I_iNS-T_1-space is I_iNS-T_0-space (i=1,2,3).
- □ Every I_iNS-T_2-space is I_iNS-T_1-space (i=1,2,3).

Proof : the proof holds directly.

Remark 4.9.

The inverse of remark (4.8) is not true as it is shown in the following example :

Example 4.10.

In example 4.6, (X,T) is I_iNS-T_o-space, but not I_iNS-T_1-space(i=1,2).

In example 4.7, (X,T) is I_iNS-T_o-space, but not I_iNS-T_1-space(i=3).

5. Intuitionistic neutrosophic Crisp Infra Topological Space

Definition 5.1.

An Intuitionistic neutrosophic crisp topology infra (INCIT) on a non-empty set X is a family T of intuitionistic neutrosophic crisp subsets in X, satisfying the following axioms:

1. $\overline{\phi}_{IN,i}, \overline{X}_{IN,i} \in T$.
2. T is closed under finite intersection.

The pair (X, T) is said to be a intuitionistic neutrosophic crisp infra topological space (INCITS)in X. Moreover, The elements in T are said to be intuitionistic neutrosophic crisp infra open sets (INCIOS), a neutrosophic crisp infra set F is neutrosophic crisp infra closed (INCICS) if and only if its complement F^c is an intuitionistic neutrosophic crisp infra open set.

Remark 5.2.

Every (INCTS) is (INCITS), But the converse not true as it is shown in the following example.

Example 5.3.

Let X={a,b,c,d,e,f,g,i} and $T = \{\overline{\phi}_{IN}, \overline{X}_{IN}, A_1, A_2, A_3\}$;

$A_1 =< (\{a,b,c\},\{d,e\}),(\{e,f\},\{g\}),(\{g,h\},\{b,i\}) >$

$A_2 =< (\{a,c,d\},\{e,i\}),(\{e,g\},\{h\}),(\{h,i\},\{a\}) >$

$A_3 =< (\{a,c\},\{d,e,i\}),(\{e\},\{g,h\}),(\{h\},\{a,b,i\}) >$

(X,T) is (INCITS), but (X ,T) is not (INCTS). because $A_1, A_2 \in$T.

But $A_1 \cup A_2 =< (\{a,b,c,d\},\{e\}),(\{e,f,g\},\phi),(\{g,h,i\},\phi) > \notin$T.

Remark 5.4.

Let (X ,T) be a (INCITS), then :

The union of two intuitionistic neutrosophic crisp infra open sets is not necessary intuitionistic neutrosophic crisp infra open set.

Proof:

In example 5.3, A_1, A_2 are intuitionistic neutrosophic crisp infra open sets but $A_1 \cup A_2 =< (\{a,b,c,d\},\{e\}),(\{e,f,g\},\phi),(\{g,h,i\},\phi) >$ is not intuitionistic neutrosophic crisp infra open set.

Remark 5.6.

(INCITS) is not necessary (INCSTS).

Example 5.7.

In example 5.3, (X ,T) is (INCITS) , but (X ,T) is not (INCSTS). because $A_1, A_2 \in$T but $A_1 \cup A_2 =< (\{a,b,c,d\},\{e\}),(\{e,f,g\},\phi),(\{g,h,i\},\phi) > \notin$T.

Remark 5.8.

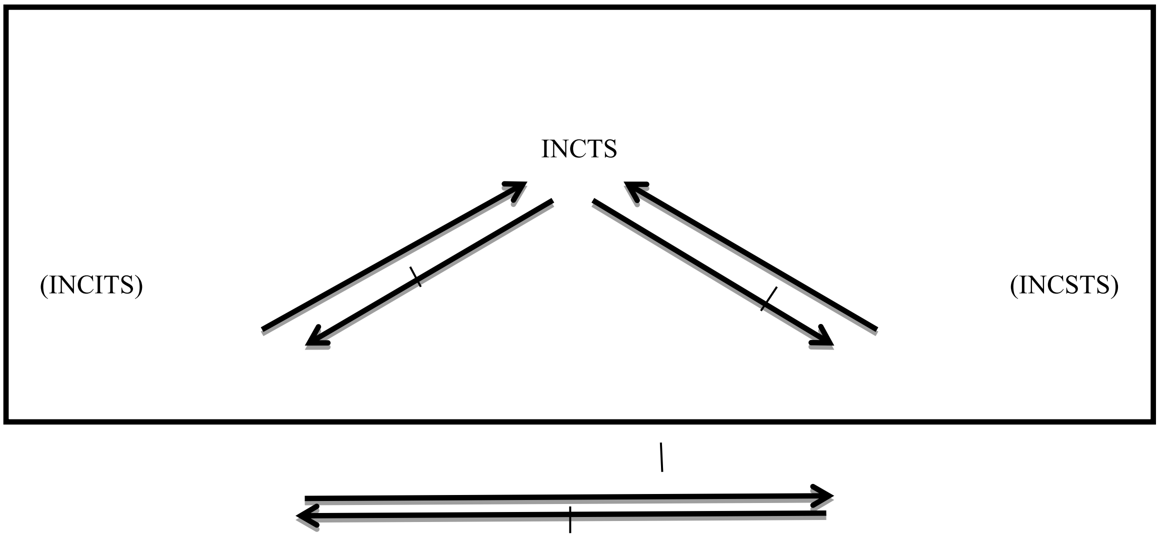
(INCSTS) is not necessary (INCITS).

Example 5.9.

In example 4.3, (X ,T) is (INCSTS), but (X ,T) is not (INCITS). Because $A_1, A_2 \in$T but $A_1 \cap A_2 =< (\{a,c\},\{d,i,e\}),(\{e\},\{g,h\}),(\{h\},\{a,b,i\}) > \notin$T.

Remark 5.10.

The relations between (INCITS) , (INCSTS) and (INCTS) in the following diagram :



6 .Separation Axioms In an intuitionistic neutrosophic Crisp infra Topological Space

Definition 6.1.

An intuitionistic neutrosophic crisp infra topological space (X, T) is called:

- I_1NI-T_o-space if $\forall\ x_{I_1} \neq y_{I_1} \in X\ \exists$ an intuitionistic neutrosophic crisp infra open set G in X containing one of them but not the other.
- I_2NI-T_o-space if $\forall\ x_{I_2} \neq y_{I_2} \in X\ \exists$ an intuitionistic neutrosophic crisp infra open set G in X containing one of them but not the other .
- I_3NI-T_o-space if $\forall\ x_{I_3} \neq y_{I_3} \in X\ \exists$ an intuitionistic neutrosophic crisp open set G in X containing one of them but not the other .
- I_1NI-T_1-space if $\forall\ x_{N_1} \neq y_{N_1} \in X\ \exists$ an intuitionistic neutrosophic crisp infra open sets G_1, G_2 in X such that $x_{I_1} \in G_1$, $y_{I_1} \notin G_1$ and $x_{I_1} \notin G_2$, $y_{I_1} \in G_2$
- I_2NI-T_1-space if $\forall\ x_{N_2} \neq y_{N_2} \in X\ \exists$ an intuitionistic neutrosophic crisp infra open sets G_1, G_2 in X such that $x_{I_2} \in G_1$, $y_{I_2} \notin G_1$ and $x_{I_2} \notin G_2$, $y_{I_2} \in G_2$
- I_3NI-T_1-space if $\forall\ x_{I_3} \neq y_{I_3} \in X\ \exists$ an intuitionistic neutrosophic crisp infra open sets G_1, G_2 in X such that $x_{I_3} \in G_1$, $y_{I_3} \notin G_1$ and $x_{I_3} \notin G_2$, $y_{I_3} \in G_2$
- I_1NI-T_2-space if $\forall\ x_{I_1} \neq y_{I_1} \in X\ \exists$ an intuitionistic neutrosophic crisp infra open sets G_1,G_2 in X such that $x_{I_1} \in G_1$, $y_{I_1} \notin G_1$ and $x_{I_1} \notin G_2$, $y_{I_1} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.
- I_2NI-T_2-space if $\forall\ x_{I_2} \neq y_{I_2} \in X\ \exists$ an intuitionistic neutrosophic crisp open infra sets G_1,G_2 in X such that $x_{I_2} \in G_1$, $y_{I_2} \notin G_1$ and $x_{I_2} \notin G_2$, $y_{I_2} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.
- I_3NI-T_2-space if $\forall\ x_{I_3} \neq y_{I_3} \in X\ \exists$ an intuitionistic neutrosophic crisp open infra sets G_1,G_2 in X such that $x_{I_3} \in G_1$, $y_{I_3} \notin G_1$ and $x_{I_3} \notin G_2$, $y_{I_3} \in G_2$ with $G_1 \cap G_2 = \overline{\phi}_{IN,i}$.

Example 6.2.

If $X = \{x, y\}$, $T = \{\overline{\phi}_{IN}$ & $\overline{X}_{IN}, A, B, C\}$, A ⊜⊜({x},{y}), $\overline{\emptyset}$, $\overline{\emptyset}$ >, B ⊜⊜ $\overline{\emptyset}$,({y},{x}), $\overline{\emptyset}$ > , C ⊜⊜(∅,{y}),(∅,{x}) ,$\overline{\emptyset}$>, Then (X,T) is I_1NI-T_o-space, and I_2NI-T_o-space.

Example 6.3.

If $X = \{x, y\}$, $T = \{\overline{\phi}_{IN}$ & $\overline{X}_{IN}, G, A, C\}$, A ⊜⊜({x},{y}), $\overline{\emptyset}$, $\overline{\emptyset}$ >, G ⊜⊜ $\overline{\emptyset}$, $\overline{\emptyset}$,({x},{y})>, C ⊜⊜(∅,{y}),$\overline{\emptyset}$,(∅,{ y}) >, Then (X,T) is I_3NI-T_o-space.

Example 6.4.

If $X = \{x, y\}$, $T = \{\overline{\phi}_{IN}$ & $\overline{X}_{IN}, A, B, C\}$, A ⊜⊜({x},{y}),$\overline{\emptyset}$,$\overline{\emptyset}$>, B ⊜⊜({y},{x}),$\overline{\emptyset}$,$\overline{\emptyset}$>, C ⊜⊜(∅,{x,y}),$\overline{\emptyset}$,$\overline{\emptyset}$>, Then (X,T) is I_1NI-T_1-space, but(X,T) in not I_1NI-T_2-space.

Example 6.5.

If $X = \{x, y\}$, $T = \{\bar{\phi}_{IN} \,\&\, \bar{X}_{IN}, G, A, C\}$, A <({x},{y}), $\bar{\emptyset}$, $\bar{\emptyset}$ >, G ◎◎ $\bar{\emptyset}$, $\bar{\emptyset}$,({x},{y})>, C ◎◎(∅,{y}),$\bar{\emptyset}$,(∅,{ y}) >, Then (X,T) is I_3NI-T_o-space, but (X,T) in not I_3NI-T_2-space.

Remark 6.6.

For an intuitionistic neutrosophic crisp infra topological space (X, T)

- EveryI_iNI-T_1-space is I_iNI-T_0-space (i=1,2,3).
- Every I_iNI-T_2-space is I_iNI-T_1-space (i=1,2,3).

Proof the proof holds directly.

The inverse of remark (3.8) is not true as it is shown in the following example:

Remark 6.7.

- In example 6.2, (X,T) is I_1NI-T_o-space, but(X,T) is I_1NI-T_1-space, and (X,T) I_2NI-T_o-space, but(X,T) is not I_2NI-T_1-space.
- In example 6.4, (X,T) is I_1NI-T_1-space, but(X,T) is I_1NI-T_2-space.

7. Conclusion

In this paper, we have defined new topological spaces by using intuitionistic neutrosophic crisp sets. This new space is called intuitionistic neutrosophic crisp supra space and intuitionistic neutrosophic crisp infra space. Then we have introduced new intuitionistic neutrosophic crisp supra open (closed) sets and intuitionistic neutrosophic crisp infra open (closed) sets in this new spaces. Also we studied some of their basic properties and their relationship with each other. Also we defined intuitionistic neutrosophic crisp points, using these notions, various classes of separation axioms were defined. In the future, many researchers can study the intuitionistic neutrosophic crisp supra space and intuitionistic neutrosophic crisp infra space.

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